Structural GARCH: The Volatility-Leverage Connection

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Crisis highlighted how leverage and equity volatility are tightly linked.

“Leverage Effect” has been around - e.g. Black (1976), Christie (1982) - but...

A dynamic volatility model that incorporates leverage directly has remained elusive.
BAC Leverage and Realized Volatility
This Paper

- GARCH-type model where equity volatility is amplified by leverage as in structural models of credit
- Statistical test of how leverage affects volatility
- Asset return series from observed equity series
- Assets have time-varying volatility at high frequencies
- Two applications: the leverage effect puzzle and SRISK
Theoretical Foundation
Under relatively weak assumptions on the vol process, structural models say \( E_t = f (A_t, D_t, \sigma_{A,t}, \tau, r_t) \)

- \( A_t \) = market value of assets
- \( D_t \) = book value of debt
- \( \sigma_{A,t} \) = stochastic asset volatility

Generic dynamics for assets and asset variance:

\[
\frac{dA_t}{A_t} = \mu_A(t) dt + \sigma_{A,t} dB_A(t)
\]

\[
d\sigma_{A,t}^2 = \mu_v(t, \sigma_{A,t}) dt + \sigma_v(t, \sigma_{A,t}) dB_v(t)
\]

- \( B_A(t) \) and \( B_v(t) \) potentially correlated
Equity Return Dynamics

Apply Itô’s Lemma

- Ignore $\mathcal{O}(dt)$ terms, since daily equity returns $\approx 0$

- Equity returns:

$$\frac{dE_t}{E_t} = \Delta_t \frac{A_t}{E_t} \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t)$$

- $\Delta_t = \partial f / \partial A_t$ is just our familiar $\Delta$ in option pricing

- $\nu_t = \partial f (A_t, D_t, \sigma_{A,t}, \tau, r) / \partial \sigma_{A,t}$ is “vega” of the option
The Leverage Multiplier
Invert Call Option Formula

Assume $f(\cdot)$ is homogenous degree 1 in first two arguments, and invertible:

\[
\frac{A_t}{D_t} = g \left(\frac{E_t}{D_t}, 1, \sigma_{A,t}, \tau, r_t\right)
\]

\[
\equiv f^{-1} \left(\frac{E_t}{D_t}, 1, \sigma_{A,t}, \tau, r_t\right)
\]

Define the “leverage multiplier” as:

\[
LM \left(\frac{E_t}{D_t}, 1, \sigma_{A,t}, \tau, r_t\right) \equiv \Delta_t \cdot g \left(\frac{E_t}{D_t}, 1, \sigma_{A,t}^f, \tau, r_t\right) \cdot \frac{D_t}{E_t}
\]

Just the %-Delta of the option
Equity Returns
The Leverage Multiplier

- Rewrite equity returns as:

\[
\frac{dE_t}{E_t} = LM_t \sigma_{A,t} dB_A(t) + \frac{v_t \sigma_v(t, \sigma_{A,t})}{E_t} \frac{1}{2\sigma_{A,t}} dB_v(t)
\]

so, \( LM_t \) amplifies asset shocks and volatility

- Two questions:
  1. What does \( LM_t \) look like?
  2. How much does the higher order term contribute?
1. What does $LM_t$ look like?
   - Similar across many different option pricing models
2. How much do the higher order terms contribute?
   - Not much. Simple intuition...
   - Volatility mean reversion speed $\ll$ typical debt maturities, so total volatility over option is basically constant
   - We verify in paper for variety of option pricing models
Equity Volatility as a Function of Leverage

- “Vol of asset vol” contributes little to equity dynamics
- Thus, straightforward expressions for equity returns and instantaneous volatility:

\[
\frac{dE_t}{E_t} \approx LM_t \times \sigma_{A,t} dB_A(t)
\]

\[
vol_t \left( \frac{dE_t}{E_t} \right) \approx LM_t \times \sigma_{A,t}
\]

- \(LM_t\) describes how leverage interacts with equity volatility
What Does the Leverage Multiplier Look Like?
Simple Case: Black-Scholes-Merton World \((\sigma = 0.15, r = 0.03, \tau = 5)\)
What Does the Leverage Multiplier Look Like?
Simple Case: Black-Scholes-Merton World $r = 0.03$; Varying $\sigma, \tau$
What Does the Leverage Multiplier Look Like?

Other Option Pricing Models

![Graphs showing different models](image-url)
Our Specification

- The challenge is choosing the right functional form for $LM_t$
  - Need a flexible function of leverage and long-run asset volatility
- We use simple transformations of Black-Scholes-Merton (BSM) functions:

$$LM_t(D_t/E_t, \sigma_{A,t}^f, \tau) = \left[ \Delta_{BSM}^t \times g^{BSM} \left( E_t/D_t, 1, \sigma_{A,t}^f, \tau \right) \times \frac{D_t}{E_t} \right]^{\phi}$$

$g^{BSM}(\cdot)$ is inverse BSM call function. $\Delta_{BSM}^t$ is BSM delta

- $\phi \neq 1$ is the departure from the Merton model
- Dividing equity returns by $LM_t$ gives us asset returns
What Does the Leverage Multiplier Look Like?

Our Specification ($\sigma = 0.15, r = 0.03, \tau = 5$)
What's the right leverage multiplier under GARCH and/or non-normality?

Simulate risk-neutral asset returns as highly asymmetric GARCH (i.e. for risk-aversion) and symmetric GARCH

Try two types of shocks: conditional normal and conditional $t$

Calculate simulated leverage multiplier as function of leverage
Leverage Multiplier with GARCH/Non-Normality

GARCH Parameters s.t. Unconditional Asset Volatility = 0.15. $\tau = 2, r = 0$
In symmetric setting, making the tails longer via GARCH decreases LM for larger levels of debt

- Moving from normal to $t$-dist. errors amplifies this effect

Volatility asymmetry makes asset returns negatively skewed $\Rightarrow$ shorter right tails $\Rightarrow$ increases LM

- $t$-dist. errors shortens right tail further, so increases LM

- Letting $\phi$ vary in our model captures all of these cases well

$\therefore$ Our LM specification is appropriate SV/GARCH/non-normal environments
Structural GARCH
The Full Recursive Model

Structural GARCH

\[ r_{E,t} = LM_{t-1} \times \sqrt{h_{A,t}} \times \varepsilon_{A,t} \]

\[ h_{A,t} \sim GJR(\omega, \alpha, \gamma, \beta) \]

\[ LM_{t-1} = \left[ \triangle^{BSM}_{t-1} \times g^{BSM}(E_{t-1}/D_{t-1}, 1, \sigma^f_{A,t-1}, \tau) \times \frac{D_{t-1}}{E_{t-1}} \right]^\phi \]

\[ \sigma^f_{A,t-1} = \sqrt{\mathbb{E}_{t-1} \left[ \sum_{s=t}^{t+\tau} h_{A,s} \right]} \]

So parameter set is \( \Theta = (\omega, \alpha, \gamma, \beta, \phi) \)
Observations

- $\phi$ tunes our leverage multiplier
  - $\phi = 0$ is a vanilla GARCH. Leverage doesn’t affect equity vol
  - $\phi = 1$ is the classic Merton model

- Half-life of GARCH process means $\sigma_{A,t-1}^f$ is basically constant

- We re-estimate the model using a constant $\sigma_{A,t-1}^f$ ... results essentially unchanged

- The $\gamma$ parameter governs asymmetry in asset volatility
Estimation Details

- QMLE
- Iterate over $\tau \in [1, 30]$
- 88 financial firms
- Equity returns and balance sheet information from Bloomberg
- $D_t$ is exponentially smoothed book value of debt
  - smoothing parameter $= 0.01$, so half-life of weights $\approx 70$ days
Estimation Results
Estimate two types of models:

1. Using a dynamic forecast for asset volatility over life of the option
2. Using unconditional volatility of GJR process

Then take the model that delivers the highest likelihood

- 46/88 firms had higher LLF using constant forecast model

A few outliers where \( \phi \) hits lower bound (exclude from subsequent analysis):

- SCHW, LM, NTRS, CME, CINF, NYB, UNH
Parameter Values
Example: Citibank

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GJR Value</th>
<th>SGARCH Value</th>
<th>SGARCH t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>2e-6</td>
<td>3.5e-06</td>
<td>1.43</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.0391</td>
<td>0.0511</td>
<td>4.03</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.0854</td>
<td>0.0461</td>
<td>2.10</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.9181</td>
<td>0.9208</td>
<td>94.74</td>
</tr>
<tr>
<td>( \phi )</td>
<td>-</td>
<td>1.2195</td>
<td>9.15</td>
</tr>
<tr>
<td>( \tau )</td>
<td>-</td>
<td>8</td>
<td>-</td>
</tr>
</tbody>
</table>

- Optimal model: constant forecast
## Parameter Values

### Cross-Sectional Summary of Estimated Parameters

| Parameter | Median   | Median t-stat | % with $|t| > 1.64$ |
|-----------|----------|---------------|-------------|
| $\omega$  | 1.0e-06  | 1.43          | 30.9        |
| $\alpha$  | 0.0442   | 3.16          | 85.2        |
| $\gamma$  | 0.0674   | 2.50          | 72.8        |
| $\beta$   | 0.9094   | 71.21         | 98.8        |
| $\phi$    | 0.9876   | 2.87          | 75.3        |

- Average $\tau = 8.28$
- Leverage matters
Bank of America Results

$\phi = 1.4 \quad (t = 11.4)$
Lehman Results

$\phi = 1.96 \quad (t = 20.9)$
American Express Results

\[ \phi = 0.4 \quad (t = 3.9) \]
Application: The Leverage Effect
Restating the Leverage Effect

- Equity volatility is negatively correlated with equity returns (i.e. volatility asymmetry)
- One explanation: financial leverage, e.g. Black (1976), Christie (1982)
- Second explanation: risk-premium effect, e.g. Schwert (1989)
- Which one is it? e.g. Bekaert and Wu (2000)
Structural GARCH and the Leverage Effect

- $\gamma$ parameter in GJR model is a measure of volatility asymmetry
- Structural GARCH models asset returns as GJR - effectively unlevers the firm
- Median $\gamma$ for asset returns is 0.0674
- Median $\gamma$ for equity returns is 0.0846

$\approx 23\%$ of so-called leverage effect comes from leverage
Higher Leverage and Higher Asymmetry?

- Definitions:
  - $\gamma_{A,i}$ from Structural GARCH
  - $\gamma_{E,i}$ is same parameter from GJR on equity returns
  - $i$ indexes firm

- Firms with more leverage should have larger ($\gamma_{E,i} - \gamma_{A,i}$)

- So, run the following regression:

$$\gamma_{E,i} - \gamma_{A,i} = a + b \times \frac{D}{E_i} + error_i$$
Equity Asymmetry versus Asset Asymmetry

Regressions results for: $\gamma_{E,i} - \gamma_{A,i} = a + b \times \frac{D}{E_i} + \text{error}_i$

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient Value</th>
<th>t-stat</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>0.0029</td>
<td>4.0471</td>
<td>17.8%</td>
</tr>
</tbody>
</table>

- Higher leverage reduces $\gamma_{E,i}$ by larger amount ✔
Remaining asset asymmetry due to the risk-premium story

Risk-premium story (mkt. index):
- positive shock to current vol $\Rightarrow$ future vol rises $\Rightarrow$
  risk-premium rises $\Rightarrow$ current price falls

Higher market betas should mean higher asset asymmetry

So, run the following two-stage regression:

Stage 1:  
$$r_{i,t}^A = c + \beta_{mkt,i}^A r_{mkt,t}^E + e_{i,t}$$

Stage 2:  
$$\gamma_{A,i} = e + f \times \beta_{mkt,i}^A + \varepsilon_i$$
Asset Asymmetry
Risk-Premium Story?

Regression results for:

Stage 1: \[ r_{i,t}^A = c + \beta_{mkt,i} r_{mkt,t}^E + e_{i,t} \]
Stage 2: \[ \gamma_{A,i} = e + f \times \beta_{mkt,i}^A + \epsilon_i \]

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<td>( f )</td>
<td>0.0287</td>
<td>1.98</td>
<td>4.95%</td>
</tr>
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Higher \( \beta_{mkt,i}^A \) means higher asset vol asymmetry ✓
Application: SRISK
Acharya et. al (2012) and Brownlees and Engle (2012)

Three steps

1. GJR-DCC model using firm equity and market index returns
2. Expected firm equity return if market falls by 40% over 6 months $\equiv$ LRMES
3. Combine LRMES with book value of debt to determine capital shortfall in a crisis

The crisis in this case is a 40% drop in the stock market index over 6 months
The Role of Leverage?
Thought Experiment with Structural GARCH

- Firm experiences sequence of negative equity (asset) shocks
- Level of leverage goes up rapidly
- Leverage multiplier increases, equity vol amplification higher
- Painfully obvious in the crisis, so build into SRISK
Asset Volatility or Leverage?

The Financial Crisis

![Graph showing annualized volatility and aggregated leverage multiplier over years 2007 to 2010. The graph compares EVW Equity Vol Index and EVW Asset Vol Index, with significant peaks during 2008-2009. The aggregated leverage multiplier shows a steady increase over the same period.](image-url)
Asset Based Systemic Risk: Preliminary Numbers
Bank of America
LRMES: Full Sample
Bank of America
LRMES: 2006-2011
Bank of America
Capital Shortfall: 2006-2011
Citigroup
LRMES: 2006-2011
Citigroup
Capital Shortfall: 2006-2011
Figure 4: Bankruptcy Paths Under Both Models

So it appears that the bankruptcy profile of paths in the Structural GARCH model appears much different compared to the standard GARCH model. Let's quantitatively analyze this further:

Table 1: Bankruptcy Statistics Under Both Models

<table>
<thead>
<tr>
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<th>Structural GARCH</th>
<th>Regular GARCH</th>
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<tbody>
<tr>
<td># of Bankruptcies</td>
<td>45</td>
<td>9</td>
</tr>
<tr>
<td>Average Time to Bankruptcy</td>
<td>89.4</td>
<td>91.2</td>
</tr>
<tr>
<td>Min Time to Bankruptcy</td>
<td>42</td>
<td>57</td>
</tr>
<tr>
<td>Max Time to Bankruptcy</td>
<td>126</td>
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What do you think? I'm not sure what to make of it...
# Structural GARCH vs Regular GARCH

Simulation for BAC on 8/29/2008: Bankruptcy Summary Statistics

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What’s Next
Other Applications

- Endogenous Crisis Probability with Structural GARCH
- Estimation of Distance to Crisis
- Endogenous Capital Structure and Leverage Cycles
- Counter-cyclical Capital Regulation