Measuring Risk and Return for Portfolios

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Though the concept of utility as a way to model degrees of satisfaction can be traced back at least two hundred years, it was von Neumann and Morgenstern’s clever axiomatization of the concept that sparked a heightened level of activity with it. Amplified, explained, and popularized by Howard Raiffa’s landmark book Decision Analysis, the new discipline took hold in universities and consulting firms across the world. Great excitement and anticipation permeated the field. Many of us believed that this methodology was the universal solution to difficult decision problems in general, and financial decisions under uncertainty in particular.

Decision analysis has been a big success. It has enlightened economic analyses in diverse areas of application and had notable success in matters of health, safety, and the environment. One area where it has enjoyed less success than might have been expected has been in the financial community.

While there is nothing inherently inappropriate about the application of decision analysis in finance, the two areas have developed separately, and with their own conventions, making technology transfer problematic. For example, decision analysts think in terms of holistic alternatives (as if this was to be the last financial decision you’ll ever make), whereas in finance decisions are viewed as dynamic and inherently reversible. Decision analysts have historically viewed payoffs as being incremental to existing wealth (plus $10) rather than as returns (plus 10 percent). Finally, decision analysts have viewed with satisfaction that the concept of utility subsumes the labile concept
of risk, whereas in finance measures of risk are in the forefront of theories.

In recent papers I have tried to close the gap between these two fields by exploring the degree to which utility may be reinterpreted in terms of risk and return. In an early paper, I showed that there is a surprisingly rich set of utility functions that are compatible with a measure of risk. More recently, I have identified all utility functions that may be represented in terms of risk and return, and provided a behavioral axiom that, when added to the original set prescribed by von Neumann and Morgenstern, leads to a theory of decision making that is entirely compatible with risk and return.

All of these papers were written from the traditional decision analysis perspective that assumes payoffs are additive with respect to current wealth. In this paper, I present the same theory but assume that payoffs are multiplicative. One unexpected by-product of this research concerns the role of the mean as the benchmark measure of a distribution's value "were it not for the risk." In the world of multiplicative returns, its preeminent role seems suspect.

In the next section I review the necessary background in a little more detail. In the succeeding sections I describe the main results. Proofs are sketched in the Appendix.

Background

Investors are constantly seeking attractive prospects for their capital, many evaluating the attractiveness of prospects in terms of two measures, risk and return. However, since von Neumann and Morgenstern first axiomatized expected utility, it has been accepted that the rational way to make decisions of this kind involves constructing a utility function defined over the domain of potential outcomes, in this case levels of net worth. The best choice is the one with the highest expected utility. This approach ignores the intuitive concepts of risk and return.

For example, suppose an investor with current wealth \( w \) is considering two possible investments. One investment, \( \tilde{x} \), is equally likely to give a rate of return on invested capital of either 10% or -5%. The second investment, \( \tilde{y} \), has a probability .10 of doubling an initial investment, and a probability .90 of losing 20%. If the investor has a utility function \( u \), then the first investment is to be preferred over the second if

\[
Eu(w\tilde{x}) = 0.5u(1.1w) + 0.5u(0.95w) \\
> 0.1u(2w) + 0.9u(0.80w) = Eu(w\tilde{y}),
\]

where \( E \) symbolizes the taking of a probabilistic average.
This evaluation process is theoretically sound, but it is very difficult to specify an appropriate utility function for a particular investor. It is generally agreed that an appropriate utility function for wealth will usually be "smooth" (there is no critical or target level of wealth), increasing (more money is better than less), and risk averse (a "fair" gamble, say a coin toss to win or lose $10, is unattractive). Most people also agree with John Pratt, Raiffa, and Robert Schlaifer, who all argue that a utility function should be decreasingly risk averse. But these conditions provide only limited guidance for an investor, and no satisfactory process for selecting a specific function has been found. In part, this stems from inevitable inconsistencies in investors' answers to both hypothetical and real choices.

Even if the selection process were to be performed satisfactorily, the very nature of an expected utility analysis seems unsatisfactory to many people: the output of such an analysis is a ranking of the alternatives, yet the value of an analysis often stems from the insight it yields into why one alternative is better than another. Might it be possible to explain these decisions in terms of risk and return? "Return" is a measure of the attractiveness of an alternative, were it possible to average across the possible outcomes. "Risk" concerns the level of disparity among possible outcomes. Utility analysis should be more intuitively helpful if its results could be explained in terms of risk and return. This paper determines the extent to which utility analysis may be interpreted in terms of risk and return, thus combining the theoretical power of utility with the intuitive appeal of risk and return.

Ideally there would exist, for alternatives \( \bar{x} \), a function \( r(\bar{x}) \) measuring return, a function \( R(\bar{x}) \) measuring risk, and a risk-return tradeoff function \( f(r, R, w) \) that evaluates the overall attractiveness of an alternative \( \bar{x} \) assuming the investor currently has wealth \( w \). The functions \( r, R, \) and \( f \) might vary across decision makers.

In earlier papers, I reported results along these lines for investments that provide outcomes that are incremental to current wealth \( (w + \bar{x}) \) rather than \( w\bar{x} \). The multiplicative case is worthy of separate development because investments are an important potential area of application for utility analysis, and also because of some interesting implications for the measurement of risk and return.

**Utility Functions Compatible with Risk and Return**

The most common use of a risk-return framework for investments is one in which return is measured by the mean \( \bar{x} \), and risk is measured by the variance \( E(\bar{x} - \bar{x})^2 \). As Harry Markowitz showed, these measures lead to a mathematically elegant, and pragmatic, evaluation of portfo-
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A mean-variance evaluation system can be compatible with the axioms of utility, since a quadratic utility function, \( u(w) = aw^2 + bw^3 \), leads to a portfolio measure of \( E\text{u}(w\bar{x}) = aw\bar{x} + bw^3E(\bar{x})^2 = aw\bar{x} + bw^3(\bar{x} - \bar{x})^2 + bw^3\bar{x}^2 \), which is of the form \( f(\bar{x}, E(\bar{x} - \bar{x})^2, w) \).

However, the quadratic utility function has some disturbing properties; in particular, it is increasingly risk averse for all \( w \); that is, a risky portfolio becomes relatively less desirable as a person gets richer. In this paper, we investigate whether there are sensible risk-return frameworks that are compatible with sensible utility functions.

We assert that a sensible utility function satisfies these properties:

\[ U_0: (\text{Smooth}) \ u(w) \text{ is continuous and infinitely differentiable.} \]

\[ U_1: (\text{More Money is Better}) \ u(w) \text{ is increasing in } w. \]

\[ U_2: (\text{Risk Aversion}) \ \bar{x} < \bar{x} \text{ for all } \bar{x} \text{ other than } \bar{x} = \bar{x}. \]

\[ U_3: (\text{Decreasing Risk Aversion}) \text{ If } \bar{x} \text{ is indifferent to a fixed payoff } y \text{ at wealth } w, \text{ then } \bar{x} \text{ is preferred to } y \text{ at higher levels of wealth.} \]

We also suggest the following properties of a risk-return function \( f(r, R, w) \):

\[ R_0: (\text{Smooth}) f \text{ is continuous and infinitely differentiable.} \]

\[ R_1: (\text{More Money is Better}) f \text{ is increasing in } r, \text{ and in } w. \]

\[ R_2: (\text{Risk Aversion}) f \text{ is decreasing in } R. \]

\[ R_3: (\text{Decreasing Risk Aversion}) \text{ If } \bar{x} \text{ is indifferent to } \bar{y} \text{ at wealth } w, \text{ then } \bar{x} \text{ is preferred to } \bar{y} \text{ at higher levels of wealth if and only if } R(\bar{x}) > R(\bar{y}). \]

We can show the following:

**Theorem 1.** The ordering of portfolios \( \bar{x} \) by an expected utility measure \( E\text{u}(w\bar{x}) \), satisfying \( U_0, U_1, U_2, \) and \( U_3 \) is consistent with a risk-return function \( f(r(\bar{x}), R(\bar{x}), w) \) satisfying \( R_0, R_1, R_2, \) and \( R_3 \) if and only if \( u \) belongs to one of the following two families:

\[ (i) \quad u(w) = aw^d + bw^c \quad ad > 0, bc > 0, c < 1, \text{ or} \]

\[ (ii) \quad u(w) = a\log w + bw^c \quad a > 0, bc > 0, c \leq 1. \]

A proof of this result is in the Appendix of this paper. It is easy to see that these functions do satisfy the various conditions of the Theorem. As a simple example, consider \( u(w) = w - w^{-1} \), which is of type (i) above with \( a = d = 1 \) and \( b = c = -1 \). It is continuous and differentiable.
(for \(w > 0\)), it is increasing \((u'(w) = 1 + w^{-2} > 0)\), and it is risk averse \((u''(w) = -2w^{-3} < 0)\). It is also decreasingly risk averse. Pratt defines risk aversion by the quantity \(-u''(w)/u'(w)\) or \(2w^{-3}/(1 + w^{-2})\). This quantity decreases with \(w\).

Now consider the risk-return representation of this utility function. We have \(E(wX) = w\bar{x} - w^{-1}E(1/\bar{x})\). If we say \(r(\bar{x}) = \bar{x}\) and \(R(\bar{x}) = E(1/\bar{x})\), then \(f(r, \bar{r}, w) = wr - R/w\). This is continuous and differentiable in \(r, \bar{r}\), and \(w\) (for \(w > 0\)). It is increasing in \(r\) and decreasing in \(\bar{r}\). Finally, it satisfies the decreasing risk-aversion condition because \(r\) becomes increasingly weighted as \(w\) increases. Note that a given utility function does not have a unique representation in terms of risk and return. For example, an alternative definition of \(r(\bar{x}) = \bar{x} - \bar{x}\) and \(R(\bar{x}) = E(\bar{x}/\bar{x})\) yields \(f(r, \bar{r}, w) = w \bar{r} - R/(\bar{r}w)\), which also satisfies \(R_{\bar{r}}, R_{\bar{r}}\), \(R_{\bar{r}}\), and \(R_{\bar{r}}\).

It is easy to generate measures of risk and return that are compatible with each of the utility families of Theorem 1 and with conditions \(R_{\bar{r}}, R_{\bar{r}}\), and \(R_{\bar{r}}\). Each utility function of Theorem 1 may be expressed, in the obvious way, as the sum of two utility functions, \(u(w) = u_1(w) + u_2(w)\), and in such a way that \(u_1\) is always less risk averse than \(u_2\) for any particular level of \(w\). For example, if \(u(w) = aw^d + bw^e\), and if \(d > e\), we may set \(u_1(w) = aw^d\) and \(u_2(w) = bw^e\). If we define \(r(\bar{x}) = E(\bar{x})\) and \(R(\bar{x}) = -(1/c)E(\bar{x})\), then \(f(r, \bar{r}, w) = aw^d - bcw^eR\), and this satisfies \(R_{\bar{r}}, R_{\bar{r}}\), and \(R_{\bar{r}}\).

If \(u(w) = \log w + bw^e\) then there are two cases; \(c\) can be positive or negative. If \(c\) is positive then \(w^e\) is less risk averse than \(\log w\) and so we set \(u_1(w) = bw^e\) and \(u_2(w) = \log w\). In this case \(r(\bar{x}) = E(\bar{x})\), \(R(\bar{x}) = -\log \bar{x}\), and \(f(r, \bar{r}, w) = bw^eR + \log w - ar\). If \(c\) is negative, then \(\log w\) is less risk averse than \(w^e\) and so we set \(u_1(w) = \log w\) and \(u_2(w) = bw^e\). In this case \(r(\bar{x}) = \log \bar{x}\), \(R(\bar{x}) = -(1/c)E(\bar{x})\), and \(f(r, \bar{r}, w) = \log w + ar - bcR\).

Alternative Measures of Risk and Return

The risk-return functions we derived in the last section were each equivalent to the general form \(r(\hat{x}) - k(\hat{w})R(\hat{x})\), where \(k(\hat{w})\) represents the tradeoff factor between \(r\) and \(R\) (which declines as \(w\) increases). This form is appealing due to its simple linear structure. However, do the risk and return measures that we have been using come close to representing a decision maker's intuitive thinking about risk and return? While there is considerable debate about how people wish to measure risk,\(^{11}\) it is reasonable to suppose that most people measure return by the mean, \(\bar{x}\), the statistic used in all elementary analyses with decision trees.
Theorem 2. If we add the requirement that \( r(\bar{x}) = \bar{x} \) to the list of conditions in Theorem 1, then the compatible utility functions are:

(i) \( u(w) = w - bw^{-c} \quad b > 0, \ c > 0, \ \text{and} \)
(ii) \( u(w) = w + a \log w \quad a > 0. \)

While a direct proof of this result is fairly easy,\(^{12}\) it can also be understood in the context of the representation \( u(w) = u_1(w) + u_2(w) \) described in the previous section of this paper. The functions in Theorem 2 are those from Theorem 1 for which \( u_1(w) = w. \)

What are some desirable properties of a risk measure? Although each investor will have a unique risk measure, for our purpose the general form of the measure can be defined by an investor’s answers to three questions:

1. Is a guaranteed payoff of 5% less risky, more risky, or equally risky than a guaranteed payoff of 10%?
2. If an investment is equally likely to pay off 5% or 10%, is this risky?
3. If any investment is combined with a second (independent) alternative offering a 50-50 payoff of either +1% or −1%, does the investment necessarily become riskier?

I believe that an investment with a guaranteed fixed payoff has no risk. Therefore, a guaranteed payoff of 5 percent has the same risk as a guaranteed payoff of 10 percent—none. On the other hand, I believe that alternatives that are uncertain, but guaranteed to leave the decision maker better off (as in question 2 above), are risky, since the payoff is uncertain. Finally, I believe that any alternative is made riskier (and less attractive) by the manipulation described in question 3. Indeed, someone who believes that all guaranteed payoffs are riskless, and agrees with the premise in question 3, cannot logically also believe that the alternative described in question 2 is riskless.

Fortunately, a wide range of risk and return measures are compatible with the utility functions of Theorems 1 and 2. The following conditions summarize my own predispositions.

\( R_1: \) All sure things have zero risk.

\( R_2: \) Assuming the decision maker begins with a known fixed wealth, the riskiness of an alternative depends only upon the probability distribution of final wealth.
To explain $R_s$, consider a person with initial wealth $w$ investing in $\bar{x}$, and another person with initial wealth $w/2$ investing in $2\bar{x}$. How much risk does each face? It seems to me that they face the same risk since their distribution of final wealth is the same. This view emphasizes the concept of risk as an indication of variability in outcome rather than as a measure of "decision correctness" represented by concepts such as probability of loss or regret.\textsuperscript{13} Finally, for reasons explained in the next section of the paper, to assume that $r(\bar{x}) = \bar{x}$ seems hasty, yet a measure of return should be some form of average across possible outcomes. I suggest the following condition:

$R_s$: A measure of return should be a utility function.

The mean averages the raw percentage payoffs across the possible scenarios. $R_s$ simply allows more flexibility in the definition of return; what we average may be any function of the percentage payoff associated with a particular outcome. Of course, $R_s$ also permits the mean.

\textbf{Theorem 3.} Given the conditions of Theorem 1, and further assuming $R_p$, $R_s$, and $R_e$, three cases are possible:

(i) $u(w) = aw^d + bw^c$ with $1 \geq d > c$, $bc \geq 0$, $ad \geq 0$, $r(\bar{x}) = (1/d)E\bar{x}^d$, and

$R(\bar{x}) = (1/d)\log E\bar{x}^d - (1/c)\log E\bar{x}^c$.

(ii) $u(w) = a\log w + bw^c$ $a, b, c > 0$ $c \leq 1$, $r(\bar{x}) = (1/c)E\bar{x}^c$, and

$R(\bar{x}) = (1/c)\log E\bar{x}^c - E\log \bar{x}$.

(iii) $u(w) = a\log w + bw^c$ $a > 0$, $b < 0$, $c < 0$, $r(\bar{x}) = E\log \bar{x}$, and

$R(\bar{x}) = E\log \bar{x} - (1/c)\log E\bar{x}^c$.

A sample proof is given in the appendix.

\textbf{Numerical Examples}

This section graphs some numerical examples, using, for illustrative purposes, the case $r(\bar{x}) = \bar{x}$, $u(w) = w - bw^{-c}$ ($b > 0$, $c > 0$), $R(\bar{x}) = \log \bar{x} + (1/c)\log E(\bar{x}^{-c})$. The risk measure has only one parameter to be assessed, $c$. As $c$ increases from zero, the measure of risk becomes more and more influenced by the size of the worse outcome; it approaches
a "maximin" mentality. Consider the four alternatives A, B, C, and D in Table 6-1. They are discrete for expositional purposes; the risk-return framework works just as well for continuous distributions.

Table 6-2 shows the relative riskiness of each alternative for each of seven values of c, ranging from 0.01 to 10. The four alternatives are ranked, for each value of c, from 1 (most risky) to 4 (least risky); for example, the risk of alternative A would be calculated as:

$$R(A) = \log(1.12) + (1/c)\log(0.2 \times 2^{-c} + 0.8 \times 0.9^{-c}).$$

Table 6-2 highlights the reversal in riskiness of the investments A, B, C, and D as we move from $c = 0.01$ to $c = 10$. The seven values of $c$ were selected to show the sequence in which each pair of alternatives switches rank. (For example, A and B switch between $c = 0.01$ and $c = 0.3$.) Figure 6-1 shows risk-return graphs of these alternatives for four of the $c$ values.
Figure 6-1
Risk-Return Graphs
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Is the Mean the Right Measure of Return?

Why is the mean so favored as a measure of the "non-risk" attractiveness of an alternative? Clearly, it is a convenient statistic, it is the first moment approximation, and it represents a "base case" of linear utility. But what guide to action does it represent? In the context of additive gambles \(w + x\), rather than \(wx\), I think the following is the behavioral implication most people have in mind:

\[M_1:\] Consider repeating the same investment (independently) an infinite number of times. You are guaranteed to make money if and only if the mean of the investment is positive.

Or, for those who dislike the thought of infinite repetition:

\[M_2:\] Suppose you repeat the same investment many times and it happens that the outcomes occur in exact proportion to their probability. You will make money if and only if the investment has a positive mean.

To me, those embody exactly the right notions for a measure of return. Return should measure the average performance of an alternative. (Risk, on the other hand, is a recognition that a long-term view is not always realistic.) But in the context of multiplicative investment returns, \(M_1\) and \(M_2\) are no longer true. For example, consider an investment that, with equal probability, increases wealth by 60 percent or decreases it by 50 percent. The mean is positive. Yet in two trials, one good, one bad, the investor’s wealth goes from \(w\) to \(.8w\). If this investment is taken independently, many times, the investor’s capital will plunge rapidly.

The appropriate modifications of \(M_1\) and \(M_2\) for the multiplicative investment scenario are:

\[M'_1:\] Consider repeating the same investment (independently) an infinite number of times. You are guaranteed to make money if and only if the expected logarithm of the investment is positive.

\[M'_2:\] Suppose you repeat the same investment many times and it happens that the outcomes occur in exact proportion to their probability. You will make money if and only if the expected logarithm of the investment is positive.
Thus, it may be inappropriate to label as "risk averse" an investor who rejects a 50-50 gamble between +60% return and a −50% return. The following definition of risk aversion seems more natural in the investment context.

**Portfolio Risk Aversion.** An investor is risk averse only if he or she judges some investment with a positive expected logarithm unacceptable.

Using this definition as a basis for measuring portfolio risk aversion, we may deduce that \( u(w) = \log w \) is the "risk neutral" utility function and \( u(w) = w^{-c} \) (\( c > 0 \)) is the family of utility functions exhibiting constant portfolio risk aversion.

Note that in the additive context, \( u(w) = w \) is the natural risk neutral utility function, since \( Eu(w + \bar{x}) = w + \bar{x} \), and \( \bar{x} \) is the benchmark measure of return. In the multiplicative context, \( u(w) = \log w \) is the risk neutral utility function, since \( Eu(wx) = \log w + E \log \bar{x} \), and \( E \log \bar{x} \) is the benchmark measure of return. This does not mean that the best portfolio is the one that maximizes \( E \log \bar{x} \). Though this criterion has received much attention in the finance literature,\(^{14}\) such a criterion would only be appropriate for an investor who is not concerned about risk. If \( r(\bar{x}) = E \log \bar{x} \) is the appropriate definition of return, then Theorem 3 suggests that risk should be measured as \( R(\bar{x}) = E \log \bar{x} - (1/c) \log E \bar{x}^c \), for some parameter \( c \) that varies with the individual.

**Uncertainty of Initial Wealth**

This paper assumes that a risk-return interpretation of expected utility is a desirable objective in itself, and most utility functions are ruled out of consideration simply because they do not permit such an interpretation, not because they stand accused of making aberrant recommendations. However, these ruled-out utility functions do make aberrant recommendations when initial wealth \( w \) is uncertain. This can occur if, for example, an investor rolls over one asset into another without knowing precisely what the asset to be sold is currently worth.

Suppose that an investor must choose between two investments, but her wealth is uncertain due to two prior investments. One of the two prior investments was much more risky (in a way we will define shortly). Now clearly it cannot hurt, and would probably help, if she could know the outcome of either or both of these prior investments before having to make the pending choice. But suppose she is allowed
to know the outcome of only one of the two prior investments. Which should she choose? The following condition asserts she should elect to hear the resolution of the more risky prior investment.

**Contextual Uncertainty Condition.** Suppose that an investor must commit all available wealth to one of two investments, $\tilde{x}$ or $\tilde{y}$. Suppose further that this decision must be made before learning of the outcomes of two earlier (independent) investment decisions $\tilde{z}_1$ and $\tilde{z}_2$. The investor is, in effect, selecting either $\tilde{z}_1\tilde{z}_2\tilde{x}$ or $\tilde{z}_1\tilde{z}_2\tilde{y}$. Now suppose that, while $\tilde{z}_1$ and $\tilde{z}_2$ are independent, it is the case that for some constant $k > 1$ we have $\tilde{z}_2 = k\tilde{z}_1$. That is, $\tilde{z}_2$ has the same distribution as $\tilde{z}_1$ but with more extreme outcomes. Suppose we now offer the investor the possibility of knowing the outcome of either $\tilde{z}_1$ or $\tilde{z}_2$ before the need to select between $\tilde{x}$ and $\tilde{y}$. We assume that the investor will never find it advantageous to select the resolution of $\tilde{z}_1$ over the resolution of $\tilde{z}_2$.

**Theorem 4.** The only utility functions that satisfy the Contextual Uncertainty Condition and $U_1$, $U_2$, and $U_3$ are:

1. $u(w) = aw^d - bw^{-c}$, 
   \(a, b, c, d > 0 \quad d \leq 1\), and
2. $u(w) = a\log w - bw^{-c}$, 
   \(a, b, c > 0\).

The proof is by consideration of the function $v(w) = Eu(w\tilde{x}) - Eu(w\tilde{y})$ for arbitrary alternatives $\tilde{x}$ and $\tilde{y}$. The Contextual Uncertainty Condition is equivalent to the condition that, for all $\tilde{x}$ and $\tilde{y}$ for which $v(w)$ can be zero, $v$ is monotonic in $w$. The intuition is that $v$ should have no local optima that might lead an investor to prefer localized information of investment outcomes.\(^{15}\)

While this theorem makes no reference to measures of risk and return, any utility function satisfying the condition is representable in terms of risk and return.

**Conclusions**

We have shown that it is possible to interpret expected utility analysis for investments in terms of risk and return tradeoffs. And, while we have argued for consideration of the measure $E\log \tilde{x}$ as a more appropriate measure of return than $\tilde{x}$, both are consistent with well-behaved utility functions. While academic opinions on the appropriate functional form for return are quite firm, there is very little consensus on an appropriate functional form for risk. This paper shows that if con-
sistency with expected utility is to be required, the definition of return severely constrains the selection of \( R(\xi) \).

Finally, there is good reason to believe that the popular view of investment decisions as a tradeoff between risk and return has some sound basis. The Contextual Uncertainty Condition is, to my mind, a sensible prescriptive assumption. If the condition deserves a normative status on a par with risk aversion and decreasing risk aversion, then a risk-return view of decisions under uncertainty is correct; if the condition is regarded only as a convenient approximation, it nevertheless demonstrates that any subtleties of evaluation lost by assuming a risk-return framework are exceedingly fine.
Appendix

Proof of Theorem 1

Since we require that the equation $Eu(w\bar{x}) = f(r(\bar{x}), R(\bar{x}), w)$ be an identity for all $w$ and $\bar{x}$, we may take any two values for $w$, say $w_1$ and $w_2$, to obtain: $Eu(w_1, \bar{x}) = f(r(\bar{x}), R(\bar{x}), w_1)$, and $Eu(w_2, \bar{x}) = f(r(\bar{x}), R(\bar{x}), w_2)$. These equations may be inverted to solve for $r(\bar{x})$ and $R(\bar{x})$. The inverse function theorem requires only that the tradeoff between $r$ and $R$ be different at $w_1$ and $w_2$, and this is so because of $R_x$.

Hence, $r(\bar{x}) = g_1(Eu(w_1, \bar{x}), Eu(w_2, \bar{x}), w)$, and $R(\bar{x}) = g_2(Eu(w_1, \bar{x}), Eu(w_2, \bar{x}), w)$ for some functions $g_1$ and $g_2$. Substituting $r$ and $R$ back into $f$, we conclude that $Eu(w\bar{x}) = g(Eu(w_1, \bar{x}), Eu(w_2, \bar{x}), w)$ for some function $g$. This representation is at the heart of a risk-return relationship: it says that only two “statistics” of $\bar{x}$ (in this case $Eu(w_1, \bar{x}), Eu(w_2, \bar{x})$) are needed to specify $Eu(w\bar{x})$.

We can show that $g$ must be linear in its first two arguments. Consider a case in which $\bar{x}$ has only $n$ possible outcomes, $x_i$ with probability $p_i$, $i = 1, \ldots, n$. Then

$$g(Eu(w_1, \bar{x}), Eu(w_2, \bar{x}), w) = Eu(w\bar{x}) = \sum_{i=1}^{n} p_i Eu(w_i x_i)$$

$$= \sum_{i=1}^{n} p_i g(u(w_1 x_i), u(w_2 x_i), w) = Eg(u(w_1 \bar{x}), u(w_2 \bar{x}), w).$$

We conclude that the expectation operator, $E$, may be exchanged with the function $g$. From this, we may conclude that $g$ is linear in $Eu(w_1, \bar{x})$ and $Eu(w_2, \bar{x})$. So $Eu(w\bar{x}) = a_0(w) + a_1(w) Eu(w_1, \bar{x}) + a_2(w) Eu(w_2, \bar{x})$ for some functions $a_0$, $a_1$, and $a_2$. More simply, $u(w) = a_0(w) + a_1(w) b_1(x) + a_2(w) b_2(x)$, where $b_1(x) = u(w_1 x)$ and $b_2(x) = u(w_2 x)$. This functional relationship has only five solutions in $u$:

(i) $u(w) = aw + bw$,  
(ii) $u(w) = a \log w + bw$,  
(iii) $u(w) = a(\log w)^2 + b \log w$,  
(iv) $u(w) = (a \log w + b)w$, and  
(v) $u(w) = w^c \log w$,

for arbitrary coefficients $a$, $b$, $c$, and $d$. Note that the quadratic function is in family (i) where $d = 1$, $c = 2$. While these are the only utility
families to have a risk-return representation, only families (i) and (ii) satisfy conditions \(U_1, U_2, \) and \(U_3.\) Family (v) violates \(U_i\) if \(b \neq 0.\) Families (iii) and (iv) are increasingly risk averse if \(a \neq 0.\)

It remains to show that these families have risk-return interpretations satisfying \(R_1, R_2,\) and \(R_3.\) If \(u(w) = aw^d + bw,\) then \(Eu(w, x) = aw^d E_2(x)^d + bw E_2(x),\) if \(a\) and \(b\) are both positive and if \(d > c,\) let \(r(x) = E_2(x)^d\) and \(R(x) = -E_2(x),\) which satisfies \(R_1\) and \(R_2.\) Since \(d > c,\) as \(w\) increases, \(r(x)\) becomes relatively more weighted, thus satisfying \(R_3.\) If \(d < c,\) let \(r(x) = E_2(x)^d\) and \(R(x) = -E_2(x).\) Other variations are similar.

Note that these choices of \(r\) and \(R\) may not match our intuition about sensible measures for \(r\) and \(R;\) we are merely establishing that at least one possible definition exists for each.

**Proof of Theorem 3**

Consider the important case \(u(w) = a \log w + bw,\) where \(Eu(w, x) = a \log w + a \log x + b \log x.\) For some functions \(g\) and \(h\) we must have \(r(x) = g(\log x, x)\) and \(R(x) = h(\log x, x).\) If \(r\) is a utility function then \(g(\log x, x) = E_g(\log x, x),\) which is possible only if \(g\) is linear in its arguments. That is, for some constants \(\alpha, \beta,\) we have \(r(x) = \alpha \log x + \beta x.\) Consider the case where \(x\) is a constant, say \(k,\) so \(r(x) = a \log k + \beta k.\) Because of assumption \(R,\) we know that \(r\) is increasing in \(k,\) so \(\alpha \geq 0, \beta \geq 0,\) and \(\alpha + \beta > 0.\)

But now consider some nonconstant alternative \(\tilde{y}\) for which \(r(\tilde{y}) = r(k).\) We have \(\alpha \log \tilde{y} + \beta \tilde{y} = a \log k + \beta k.\) Since \(r(\tilde{y}) = r(k),\) the preference ordering of \(\tilde{x} (= k)\) and \(\tilde{y}\) must be the same for all \(w;\) it depends only on the ordering of \(R(\tilde{x})\) and \(R(\tilde{y}).\) That is, \((a \log k + b \log x) - (a \log \tilde{y} + b \log y)\) has the same sign for all \(w.\) This means that the signs of \(\log k - E_{\log \tilde{y}}\) and \(k - \tilde{y}\) must be equal. But this is inconsistent with our earlier conclusion (based on the relation \(r(\tilde{y}) = r(k)\)), that \(a \log \tilde{y} + \beta \tilde{y} = a \log k + \beta k.\) We conclude that either \(\alpha = 0\) or \(\beta = 0.\)

Suppose \(r(\tilde{x}) = E_{\log \tilde{x}}.\) Now consider arbitrary alternatives \(\tilde{x}\) and \(\tilde{y}\) that happen to be indifferent at \(w.\) That is, \((a \log \tilde{x} + b \tilde{x}) - (a \log \tilde{y} + b \tilde{y}) = 0.\) As \(w\) increases, \(\tilde{x}\) will be preferred over \(\tilde{y}\) if and only if \(\tilde{x} > \tilde{y}\) (since \(b > 0\)). But by condition \(R, \tilde{x}\) is to be preferred to \(\tilde{y}\) as \(w\) increases if and only if \(r(\tilde{x}) > r(\tilde{y})\) or \(E_{\log \tilde{x}} > E_{\log \tilde{y}}.\) But for the equation in this paragraph to hold, it must be that \(E_{\log \tilde{x}} < E_{\log \tilde{y}}.\) This is inconsistent. But if \(r(\tilde{x}) = \tilde{x},\) then the above argument is consistent.

Now consider \(R(\tilde{x}) = h(E_{\log \tilde{x}}, \tilde{x}).\) Since \(R_3\) tells us that \(R(k) = R(\tilde{x}),\) we may arbitrarily standardize \(R(\tilde{x})\) as \(R(\tilde{x} / \tilde{x}).\) Hence \(R(\tilde{x}) = h(E_{\log \tilde{x}} - E_{\log \tilde{x}}, 1),\) or simply \(h(E_{\log \tilde{x}} / \tilde{x}).\) We know that \(Eu(w, x) = \)
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\[ a \log w + a \log (\bar{x}/\bar{\bar{x}}) + a \log \bar{x} + b \nu \bar{x}, \] and by \( R \) that this expression is decreasing in \( R(\bar{x}) \). Hence \( h \) is decreasing in \( E \log(\bar{x}/\bar{\bar{x}}) \). Since the definition of \( R(\bar{x}) \) is arbitrary up to monotonic transformations, we may assume that \( R(\bar{x}) = E \log(\bar{x}/\bar{\bar{x}}) \).

Arguments for the other cases are similar.

### Portfolio Risk Aversion

Following Pratt, we may determine a suitable measure of portfolio risk aversion for an investor with utility function for wealth \( u(w) \). We may write \( EU(\bar{w}, \bar{x}) = E u \exp(\log w + \log \bar{x}) \), where \( u \exp \) is the convolution of \( u \) with the exponential function. Changing notation, let \( v = u \exp \). Assume \( E \log \bar{x} = 0 \) and that \( \bar{x} \) is "small." Then

\[ E v(\log w + \log \bar{x}) = v(\log w) + \frac{[E \log \bar{x}]^2}{2} v'(\log w). \]

Let \( c(w, \bar{x}) \) be the certainty equivalent of \( \bar{x} \) at wealth \( w \) so that \( EU(wc) = EU(w \bar{x}) \) and, approximately,

\[ E v(\log w + \log c) = v(\log w) + \log c v'(\log w). \]

Hence

\[ \log c = \frac{E \log \bar{x}^2}{2} \frac{v'(\log w)}{v'(\log w)}. \]

Translating back into the \( u \) notation, since \( u(w) = v(\log w) \), we have

\[ u'(w) = \frac{1}{w} v'(\log w) \] and

\[ u''(w) = -\frac{1}{w^2} v'(\log w) + \frac{1}{w^2} v''(\log w), \]

so that \( v'(\log w) = w u'(w) \) and \( v''(\log w) = w^2 u''(w) + w u'(w) \). Hence

\[ \log c = \frac{E \log \bar{x}^2}{2} \left( \frac{u''(w)}{u'(w)} + 1 \right). \]

Now \( \bar{x} \) is undesirable only if \( \log c < 0 \), so a suitable measure of portfolio risk aversion is

\[ -\left( \frac{w}{w u'(w)} + 1 \right). \]

This is equivalent to Pratt's relative risk aversion \( -\left( \frac{wu'(w)}{u'(w)} \right) \) less one.
NOTES
6. Investors will typically divide their capital among many prospects. Measuring the risk of an individual prospect is made complicated by the potential correlation of its rate of return with other constituents of the investor’s portfolio; see Robert C. Merton, Continuous-Time Finance (Oxford: Blackwell, 1990), ch. 2. We will be concerned here with finding risk and return measures for portfolios viewed in their entirety.
15. A detailed proof may be derived from that in Bell, “A Contextual Uncertainty Condition.”
17. Pratt, “Risk Aversion.”