Internet Appendix for:

Asset Price Dynamics in Partially Segmented Markets

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A Campbell-Shiller return approximation

In this section, we derive the Campbell-Shiller (1988) approximation for asset B:

$$r_{B,t+1} \approx \frac{1}{1-\theta_B} y_{B,t} - \frac{\theta_B}{1-\theta_B} y_{B,t+1} - z_{t+1}. \tag{1}$$

The derivation of the return approximation for asset A is similar.

To derive this approximation note that the Campbell-Shiller (1988) approximation of the 1-period log return on the B portfolio of defaultable perpetuities is

$$\begin{aligned} r_{B,t+1} &\equiv \ln\left(1+R_{B,t+1}\right) \\ &= \log\left(\frac{\left(1-Z_{t+1}\right)\left(P_{B,t+1}+C_B\right)}{P_{B,t}}\right) \\ &= \ln\left(1+\exp\left(c_B - p_{B,t+1}\right)\right) + p_{B,t+1} - p_{B.t} - z_{t+1} \\ &\approx \ln\left(1+\exp\left(c_B-\overline{p}_B\right)\right) - \frac{\exp\left(c_B-\overline{p}_B\right)}{1+\exp\left(c_B-\overline{p}_B\right)}\left(c_B-\overline{p}_B\right) + \frac{\exp\left(c_B-\overline{p}_B\right)}{1+\exp\left(c_B-\overline{p}_B\right)}\left(c_B-p_{B,t+1}\right) \\ &\quad + p_{B,t+1} - p_{B.t} - z_{t+1} \\ &= \kappa_B + \theta_B p_{B,t+1} + \left(1-\theta_B\right) c_B - p_{B,t} - z_{t+1} \end{aligned} \tag{2}$$

where $z_t = -\ln\left(1-Z_t\right)$ is the log default loss at time t and $\theta_B = 1/\left(1+\exp\left(c_B-\overline{p}_B\right)\right)$ and $\kappa_B = -\ln\left(\theta_B\right) - \left(1-\theta_B\right)\ln\left(\theta_B^{-1}-1\right)$ are parameters of the log-linearization. This log-linearization is obtained by taking a first-order Taylor series expansion of the function $\ln\left(1+\exp\left(c_B-p_{B,t+1}\right)\right)$ about the point $\left(c_B-\overline{p}_B\right)$ where $\overline{p}_B$ is the steady-state bond price.

Iterating equation (2) forward, we find that the log bond price is

$$p_{B,t} = \left(1-\theta_B\right)^{-1}\kappa_B + c_B - \textstyle\sum_{i=0}^{\infty}\theta_B^i E_t\left[r_{B,t+i+1}+z_{t+i+1}\right]. \tag{3}$$

Applying this approximation to *promised cash flows* (i.e., $z_{t+i+1}\equiv 0$ for all $i\geq 0$) and the *yield-to-maturity*, defined as the *constant return* that equates bond price and the discounted value of *promised* cash flows, we obtain

$$p_{B,t} = \left(1-\theta_B\right)^{-1}\kappa_B + c_B - \left(1-\theta_B\right)^{-1} y_{B,t}. \tag{4}$$

Equation (1) then follows by substituting the expression for $p_{B,t}$ in equation (4) into the Campbell-Shiller return approximation in equation (2).

Assuming the steady-state price of the bonds is par ($\overline{p}_B = 0$), we have $\theta_B = 1/\left(1+C_B\right)$. Thus, bond duration is $D_B = -\partial p_{B,t}/\partial y_{B,t} = \left(1-\theta_B\right)^{-1} = \left(1+C_B\right)/C_B$. Since $-\partial p_{B,t}/\partial y_{B,t} = -\left(\partial P_{B,t}/\partial Y_{B,t}\right)\times\left(1+Y_{B,t}\right)/P_{B,t} = (Y_{B,t}+1)/Y_{B,t}$, this corresponds to Macaulay duration when the bonds are trading at par ($Y_{B,t}=C_B$).

B Solution to the baseline model

B.1 Primitives and equilibrium conjecture

For the sake of concreteness, suppose that $k=4$ so slow-moving generalists rebalance their portfolios every four periods. We conjecture that the equilibrium yields in market A and B at time t take the form

$$\begin{aligned} y_{A,t} &= \alpha_{A0} + \boldsymbol{\alpha}_{A1}'\mathbf{x}_t, \qquad (5) \\ y_{B,t} &= \alpha_{B0} + \boldsymbol{\alpha}_{B1}'\mathbf{x}_t, \qquad (6) \end{aligned}$$

and that the demands of slow-moving generalists who rebalance at time t are of the form

$$\begin{aligned} d_{A,t} &= \delta_{A0} + \boldsymbol{\delta}'_{A1}\mathbf{x}_t, & (7) \\ d_{B,t} &= \delta_{B0} + \boldsymbol{\delta}'_{B1}\mathbf{x}_t, & (8) \end{aligned}$$

where the $2(k+1)$ dimensional state vector is

$$\mathbf{x}_t = \begin{bmatrix} r_t - \overline{r} \\ z_t - \overline{z} \\ s_{A,t} - \overline{s}_A \\ s_{B,t} - \overline{s}_B \\ d_{A,t-1} - \delta_{A0} \\ d_{A,t-2} - \delta_{A0} \\ d_{A,t-3} - \delta_{A0} \\ d_{B,t-1} - \delta_{B0} \\ d_{B,t-2} - \delta_{B0} \\ d_{B,t-3} - \delta_{B0} \end{bmatrix}. \tag{9}$$

These assumptions imply that the state vector follows an AR(1) process. Critically, the transition matrix Γ is a function of slow-moving generalist demand so we write $\Gamma = \Gamma(\boldsymbol{\delta})$. Specifically, we have

$$\mathbf{x}_{t+1} = \Gamma(\boldsymbol{\delta})\,\mathbf{x}_t + \boldsymbol{\epsilon}_{t+1} \tag{10}$$
$$= \begin{bmatrix} \rho_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{s_A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{s_B} & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_{A1} & \delta_{A2} & \delta_{A3} & \delta_{A4} & \delta_{A5} & \delta_{A6} & \delta_{A7} & \delta_{A8} & \delta_{A9} & \delta_{A10} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \delta_{B1} & \delta_{B2} & \delta_{B3} & \delta_{B4} & \delta_{B5} & \delta_{B6} & \delta_{B7} & \delta_{B8} & \delta_{B9} & \delta_{B10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_t - \overline{r} \\ z_t - \overline{z} \\ s_{A,t} - \overline{s}_A \\ s_{B,t} - \overline{s}_B \\ d_{A,t-1} - \delta_{A0} \\ d_{A,t-2} - \delta_{A0} \\ d_{A,t-3} - \delta_{A0} \\ d_{B,t-1} - \delta_{B0} \\ d_{B,t-2} - \delta_{B0} \\ d_{B,t-3} - \delta_{B0} \end{bmatrix} + \begin{bmatrix} \varepsilon_{r,t+1} \\ \varepsilon_{z,t+1} \\ \varepsilon_{s_A,t+1} \\ \varepsilon_{s_B,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In our numerical solutions we assume that $\varepsilon_{r,t+1}$, $\varepsilon_{z,t+1}$, $\varepsilon_{s_A,t+1}$, and $\varepsilon_{s_B,t+1}$ are mutually orthogonal, so we have

$$\boldsymbol{\Sigma} \equiv Var_t[\boldsymbol{\epsilon}_{t+1}] = \begin{bmatrix} \sigma_r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{s_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{s_B}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

However, as shown below, the equilibrium conditions take the same basic form if this is not the case—i.e., if the upper-left block of $\boldsymbol{\Sigma}$ is not diagonal. The intuition to come out of allowing off-diagonal terms is straightforward. For example, if $\varepsilon_{r,t+1}$ and $\varepsilon_{s_A,t+1}$ were positively correlated, then Aspecialists would bear more risk, steepening their demand curve for default-free long-term bonds.

We adopt the convention that $\mathbf{e}_r$ is the basis vector with a 1 corresponding to $r_t - \overline{r}$ and 0s elsewhere; $\mathbf{e}_z$ is the basis vector with a 1 corresponding to $z_t - \overline{z}$ and 0s elsewhere; and so on. Finally, we denote

$\mathbf{C}_{[t+i,t+j]} = Cov\left[\mathbf{x}_{t+i}, \mathbf{x}_{t+j} | \mathbf{x}_t\right]$ and note that

$$\mathbf{C}_{[t+i,t+j]} = \sum_{s=1}^{\min\{i,j\}} [\mathbf{\Gamma}^{i-s}]\mathbf{\Sigma}[\mathbf{\Gamma}^{j-s}]', \tag{12}$$

so $\mathbf{C}_{[t+j,t+i]} = \mathbf{C}'_{[t+i,t+j]}$.

B.2 Specialist demands

Given the conjectured form of equilibrium yields, we have

$$\begin{aligned} rx_{A,t+1} &= \frac{1}{1-\theta_A} y_{A,t} - \frac{\theta_A}{1-\theta_A} y_{A,t+1} - r_t \\ &= (\alpha_{A0} - \overline{r}) + \left(\frac{1}{1-\theta_A}\boldsymbol{\alpha}_{A1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \mathbf{x}_{t+1}. \end{aligned} \tag{13}$$

Thus, since

$$E_t\left[rx_{A,t+1}\right] = (\alpha_{A0} - \overline{r}) + \left(\frac{1}{1-\theta_A}\boldsymbol{\alpha}_{A1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \mathbf{\Gamma}\mathbf{x}_t \tag{14}$$

and

$$\begin{aligned} Var_t\left[rx_{A,t+1}\right] &= \left(\frac{\theta_A}{1-\theta_A}\right)^2 \boldsymbol{\alpha}'_{A1}\mathbf{\Sigma}\boldsymbol{\alpha}_{A1} \\ &= \left(\frac{\theta_A}{1-\rho_r\theta_A}\right)^2 \sigma_r^2 + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_B}\right)^2 \sigma_{s_B}^2. \end{aligned} \tag{15}$$

A-specialist demand for asset A is

$$b_{A,t} = \tau \frac{E_t\left[rx_{A,t+1}\right]}{Var_t\left[rx_{A,t+1}\right]} = \left[\tau \frac{\alpha_{A0} - \overline{r}}{\left(\frac{\theta_A}{1-\theta_A}\right)^2 \boldsymbol{\alpha}'_{A1}\mathbf{\Sigma}\boldsymbol{\alpha}_{A1}}\right] + \left[\tau \frac{\left(\frac{1}{1-\theta_A}\boldsymbol{\alpha}_{A1} - \mathbf{e}_r\right)' - \frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}'_{A1}\mathbf{\Gamma}}{\left(\frac{\theta_A}{1-\theta_A}\right)^2 \boldsymbol{\alpha}'_{A1}\mathbf{\Sigma}\boldsymbol{\alpha}_{A1}}\right] \mathbf{x}_t. \tag{16}$$

Proceeding similarly for asset B, we have

$$\begin{aligned} rx_{B,t+1} &= \frac{1}{1-\theta_B} y_{B,t} - \frac{\theta_B}{1-\theta_B} y_{B,t+1} - z_{t+1} - r_t \\ &= (\alpha_{B0} - \overline{r} - \overline{z}) + \left(\frac{1}{1-\theta_B}\boldsymbol{\alpha}_{B1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \mathbf{x}_{t+1} \end{aligned} \tag{17}$$

where $\mathbf{e}'_z \mathbf{x}_{t+1} = (z_{t+1} - \overline{z})$. Thus, since

$$E_t\left[rx_{B,t+1}\right] = (\alpha_{B0} - \overline{r} - \overline{z}) + \left(\frac{1}{1-\theta_B}\boldsymbol{\alpha}_{B1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \mathbf{\Gamma}\mathbf{x}_t \tag{18}$$

and

$$\begin{aligned} Var_t\left[rx_{B,t+1}\right] &= \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \mathbf{\Sigma} \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right) \\ &= \left(\frac{\theta_B}{1-\rho_r\theta_B}\right)^2 \sigma_r^2 + \left(\frac{1}{1-\rho_z\theta_B}\right)^2 \sigma_z^2 + \left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_B}\right)^2 \sigma_{s_B}^2 \end{aligned} \tag{19}$$

B-specialist demand for asset B is

$$\begin{aligned} b_{B,t} &= \tau \frac{E_t\left[rx_{B,t+1}\right]}{Var_t\left[rx_{B,t+1}\right]} \\ &= \left[\tau \frac{\alpha_{B0}-\overline{r}-\overline{z}}{\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\boldsymbol{\Sigma}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)}\right] + \left[\tau \frac{\left(\frac{1}{1-\theta_B}\boldsymbol{\alpha}_{B1}-\mathbf{e}_r\right)' - \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\boldsymbol{\Gamma}}{\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\boldsymbol{\Sigma}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)}\right]\mathbf{x}_t \end{aligned} \tag{20}$$

B.3 Slow-moving generalist demand

Slow-moving generalist investors solve

$$\max_{d_{A,t},d_{B,t}} \left\{ \begin{array}{c} d_{A,t}E_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right] + d_{B,t}E_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right] \\ -\frac{1}{2\tau}\left(\begin{array}{c} \left(d_{A,t}\right)^2 Var_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right] + \left(d_{B,t}\right)^2 Var_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right] \\ +2d_{A,t}d_{B,t}Cov_t\left[\sum_{i=1}^{k} rx_{A,t+i}, \sum_{i=1}^{k} rx_{B,t+i}\right] \end{array} \right) \end{array} \right\}, \tag{21}$$

which implies

$$\begin{aligned} \begin{bmatrix} d_{A,t} \\ d_{B,t} \end{bmatrix} &= \tau \left[\begin{array}{cc} V_A^{(k)} & C_{AB}^{(k)} \\ C_{AB,k} & V_B^{(k)} \end{array} \right]^{-1} \left[\begin{array}{c} E_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right] \\ E_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right] \end{array} \right] \\ &= \frac{\tau}{V_A^{(k)}V_B^{(k)} - \left(C_{AB}^{(k)}\right)^2} \left[\begin{array}{c} V_B^{(k)}E_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right] - C_{AB}^{(k)}E_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right] \\ V_A^{(k)}E_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right] - C_{AB}^{(k)}E_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right] \end{array} \right] \end{aligned} \tag{22}$$

where $V_A^{(k)} \equiv Var_t\left[\sum_{i=1}^{k} rx_{A,t+i}\right]$, $V_B^{(k)} \equiv Var_t\left[\sum_{i=1}^{k} rx_{B,t+i}\right]$, and $C_{AB}^{(k)} \equiv Cov_t\left[\sum_{i=1}^{k} rx_{A,t+i}, \sum_{i=1}^{k} rx_{B,t+i}\right]$. Realized k-period cumulative excess returns are

$$\begin{aligned} \textstyle\sum_{i=1}^{k} rx_{A,t+i} &= \textstyle\sum_{i=0}^{k-1}\left(y_{A,t+i}-r_{t+i}\right) - \frac{\theta_A}{1-\theta_A}\left(y_{A,t+k}-y_{A,t}\right) \\ &= k\left(\alpha_{A0}-\overline{r}\right) + \left(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r\right)'\left(\textstyle\sum_{j=0}^{k-1}\mathbf{x}_{t+j}\right) - \frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}'\left(\mathbf{x}_{t+k}-\mathbf{x}_t\right) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \textstyle\sum_{i=1}^{k} rx_{B,t+i} &= \textstyle\sum_{i=0}^{k-1}\left(y_{B,t+i}-r_{t+i}-z_{t+i+1}\right) - \frac{\theta_B}{1-\theta_B}\left(y_{B,t+k}-y_{B,t}\right) \\ &= k\left(\alpha_{B0}-\overline{r}-\overline{z}\right) + \left(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r\right)'\left(\textstyle\sum_{j=0}^{k-1}\mathbf{x}_{t+j}\right) - \mathbf{e}_z'\left(\textstyle\sum_{j=1}^{k}\mathbf{x}_{t+j}\right) - \frac{\theta_B}{1-\theta_B}\left(\boldsymbol{\alpha}_{B1}'\mathbf{x}_{t+k}-\boldsymbol{\alpha}_{B1}'\mathbf{x}_t\right). \end{aligned} \tag{24}$$

Thus, expected k-period cumulative returns are

$$E_t\left[\textstyle\sum_{i=1}^{k} rx_{A,t+i}\right] = k\left(\alpha_{A0}-\overline{r}\right) + \left(\left(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r\right)'\left(\mathbf{I}-\boldsymbol{\Gamma}\right)^{-1} + \frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}'\right)\left(\mathbf{I}-\boldsymbol{\Gamma}^k\right)\mathbf{x}_t \tag{25}$$

and

$$E_t\left[\textstyle\sum_{i=1}^{k} rx_{B,t+i}\right] = k\left(\alpha_{B0}-\overline{r}-\overline{z}\right) + \left(\left(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r\right)'\left(\mathbf{I}-\boldsymbol{\Gamma}\right)^{-1} + \frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}' - \mathbf{e}_z'\left(\mathbf{I}-\boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\Gamma}\right)\left(\mathbf{I}-\boldsymbol{\Gamma}^k\right)\mathbf{x}_t. \tag{26}$$

The variance of A market k-period cumulative excess returns is

$$V_A^{(k)} = Var_t\left[(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\left(\sum_{j=1}^{k-1}\mathbf{x}_{t+j}\right)-\left(\frac{\theta_A}{1-\theta_A}\right)\boldsymbol{\alpha}'_{A1}\mathbf{x}_{t+k}\right] \tag{27}$$

$$= (\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\left(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathbf{C}_{[t+i,t+j]}\right)(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)+\left(\frac{\theta_A}{1-\theta_A}\right)^2\boldsymbol{\alpha}'_{A1}\mathbf{C}_{[t+k,t+k]}\boldsymbol{\alpha}_{A1}$$

$$-2\left(\frac{\theta_A}{1-\theta_A}\right)(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\sum_{i=1}^{k-1}\mathbf{C}_{[t+i,t+k]}\boldsymbol{\alpha}_{A1}.$$

Similarly, the variance of B market k-period cumulative returns is

$$V_B^{(k)} = Var_t\left[(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)'\left(\sum_{j=1}^{k-1}\mathbf{x}_{t+j}\right)-\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\mathbf{x}_{t+k}\right] \tag{28}$$

$$= (\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)'\left(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathbf{C}_{[t+i,t+j]}\right)(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)$$

$$+\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\mathbf{C}_{[t+k,t+k]}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)$$

$$-2(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)'\sum_{i=1}^{k-1}\mathbf{C}_{[t+i,t+k]}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)$$

The covariance between A and B market k-period cumulative returns is

$$C_{AB}^{(k)} = Cov_t\left[\begin{array}{c}(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\left(\sum_{j=1}^{k-1}\mathbf{x}_{t+j}\right)-\left(\frac{\theta_A}{1-\theta_A}\right)\boldsymbol{\alpha}'_{A1}\mathbf{x}_{t+k},\\(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)'\left(\sum_{j=1}^{k-1}\mathbf{x}_{t+j}\right)-\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\mathbf{x}_{t+k}\end{array}\right] \tag{29}$$

$$= (\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\left(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathbf{C}_{[t+i,t+j]}\right)(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)$$

$$-(\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'\sum_{i=1}^{k-1}\mathbf{C}_{[t+i,t+k]}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)$$

$$-(\boldsymbol{\alpha}_{B1}-\mathbf{e}_r-\mathbf{e}_z)'\sum_{i=1}^{k-1}\mathbf{C}_{[t+i,t+k]}\left(\frac{\theta_A}{1-\theta_A}\right)\boldsymbol{\alpha}_{A1}$$

$$+\left(\frac{\theta_A}{1-\theta_A}\right)\boldsymbol{\alpha}'_{A1}\mathbf{C}_{[t+k,t+k]}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)$$

Thus, given our conjectures, slow-moving generalists demands will indeed take a linear form. Specifically, we have

$$\delta_{A0} = \tau\frac{V_B^{(k)}k(\alpha_{A0}-\overline{r})-C_{AB}^{(k)}k(\alpha_{B0}-\overline{r}-\overline{z})}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}, \tag{30}$$

$$\delta_{B0} = \tau\frac{V_A^{(k)}k(\alpha_{B0}-\overline{r}-\overline{z})-C_{AB}^{(k)}k(\alpha_{A0}-\overline{r})}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}, \tag{31}$$

$$\boldsymbol{\delta}'_{A1} = \tau\frac{\left(\begin{array}{c}V_B^{(k)}\left((\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma})^{-1}+\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}'_{A1}\right)\\-C_{AB}^{(k)}\left((\boldsymbol{\alpha}_{B1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma})^{-1}+\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}'_{B1}-\mathbf{e}'_z(\mathbf{I}-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right)\end{array}\right)}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}\left(\mathbf{I}-\boldsymbol{\Gamma}^k\right) \tag{32}$$

and

$$\boldsymbol{\delta}'_{B1} = \tau \frac{\left(\begin{array}{c} V_A^{(k)} \left((\boldsymbol{\alpha}_{B1} - \mathbf{e}_r)' (\mathbf{I} - \boldsymbol{\Gamma})^{-1} + \frac{\theta_B}{1-\theta_B} \boldsymbol{\alpha}'_{B1} - \mathbf{e}'_z (\mathbf{I} - \boldsymbol{\Gamma})^{-1} \boldsymbol{\Gamma} \right) \\ -C_{AB}^{(k)} \left((\boldsymbol{\alpha}_{A1} - \mathbf{e}_r)' (\mathbf{I} - \boldsymbol{\Gamma})^{-1} + \frac{\theta_A}{1-\theta_A} \boldsymbol{\alpha}'_{A1} \right) \end{array} \right)}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2} \left(\mathbf{I} - \boldsymbol{\Gamma}^k \right). \tag{33}$$

B.4 Rational expectations equilibrium

To solve for a rational expectations equilibrium of our model, we need to clear the market for both the A and B assets in a way that is consistent with (i) optimization on the part of Aspecialists, Bspecialists, and generalists and (ii) where all agents correctly perceive the laws of motions for all exogenous and endogenous variables.

B.4.1 Clearing market A

The market-clearing condition for market A is

$$\overbrace{(1 - q_A - q_B)k^{-1} d_{A,t} + q_A b_{A,t}}^{\text{Active demand}} = \overbrace{s_{A,t} - (1 - q_A - q_B)(k^{-1} \textstyle\sum_{i=1}^{k-1} d_{A,t-i})}^{\text{Active supply}}. \tag{34}$$

Letting $V_A^{(1)} = Var_t \left[rx_{A,t+1} \right] = \left(\frac{\theta_A}{1-\theta_A} \right)^2 \boldsymbol{\alpha}'_{A1} \boldsymbol{\Sigma} \boldsymbol{\alpha}_{A1}$, denote the variance of 1-period A-market excess returns, active demand is

$$\begin{aligned} & (1 - q_A - q_B)k^{-1} d_{A,t} + q_A b_{A,t} \\ & = \left[(1 - q_A - q_B)k^{-1} \delta_{A0} + q_A \tau \frac{(\alpha_{A0} - \overline{r})}{V_A^{(1)}} \right] \\ & + \left[(1 - q_A - q_B)k^{-1} \boldsymbol{\delta}'_{A1} + q_A \tau \frac{\left(\frac{1}{1-\theta_A} \boldsymbol{\alpha}_{A1} - \mathbf{e}_r \right)' - \frac{\theta_A}{1-\theta_A} \boldsymbol{\alpha}'_{A1} \boldsymbol{\Gamma}}{V_A^{(1)}} \right] \mathbf{x}_t \end{aligned} \tag{35}$$

Active supply is

$$\begin{aligned} & s_{A,t} - (1 - q_A - q_B)(k^{-1} \textstyle\sum_{i=1}^{k-1} d_{A,t-i}) \\ & = s_{A,t} - (1 - q_A - q_B) \frac{(k-1)}{k} \delta_{A0} - (1 - q_A - q_B) k^{-1} \textstyle\sum_{i=1}^{k-1} (d_{A,t-i} - \delta_{A0}) \\ & = \left[\overline{s}_A - (1 - q_A - q_B) \frac{(k-1)}{k} \delta_{A0} \right] + \left[\left(\mathbf{e}_{s_A} - (1 - q_A - q_B) k^{-1} \mathbf{1}_{(A)} \right)' \right] \mathbf{x}_t \end{aligned} \tag{36}$$

where $\mathbf{1}_{(A)} = \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7$.

Matching constants terms in (35) and (36), we obtain

$$\alpha_{A0} = \overline{r} + \frac{V_A^{(1)}}{\tau q_A} \left(\overline{s}_A - (1 - q_A - q_B) \delta_{A0} \right) \tag{37}$$

Matching slope coefficients in (35) and (36), we have

$$\begin{aligned}\boldsymbol{\alpha}_{A1} &= (1-\theta_A)\left[\mathbf{I}-\theta_A\boldsymbol{\Gamma}'\right]^{-1}\mathbf{e}_r \\ &\quad +(1-\theta_A)\frac{V_A^{(1)}}{\tau q_A}\left[\mathbf{I}-\theta_A\boldsymbol{\Gamma}'\right]^{-1}\left[\mathbf{e}_{s_A}-k^{-1}(1-q_A-q_B)\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)\right] \\ &= \frac{1-\theta_A}{1-\theta_A\rho_r}\mathbf{e}_r+\frac{V_A^{(1)}}{\tau q_A}\left[\frac{1-\theta_A}{1-\theta_A\rho_{s_A}}\mathbf{e}_{s_A}-k^{-1}(1-q_A-q_B)(1-\theta_A)\left[\mathbf{I}-\theta_A\boldsymbol{\Gamma}'\right]^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)\right].\end{aligned} \tag{38}$$

Thus, equilibrium yields take the form

$$\begin{aligned}y_{A,t} &= \alpha_{A0}+\boldsymbol{\alpha}_{A1}'\mathbf{x}_t \\ &= \overbrace{\bar{r}+\frac{1-\theta_A}{1-\theta_A\rho_r}(r_t-\bar{r})}^{\text{Expected future short rates}}+\overbrace{\frac{V_A^{(1)}}{\tau q_A}(\bar{s}_A-(1-q_A-q_B)\delta_{A0})}^{\text{Unconditional term premia}} \\ &\quad +\overbrace{\left[\frac{V_A^{(1)}}{\tau q_A}\left(\frac{1-\theta_A}{1-\theta_A\rho_{s_A}}(s_{A,t}-\bar{s}_A)-(1-\theta_A)(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)'\left[\mathbf{I}-\theta_A\boldsymbol{\Gamma}\right]^{-1}\mathbf{x}_t\right)\right]}^{\text{Conditional term premia}}.\end{aligned} \tag{39}$$

Equilibrium excess returns are given by

$$\begin{aligned}E_t\left[rx_{A,t+1}\right] &= \frac{V_A^{(1)}}{\tau q_A}(\bar{s}_A-(1-q_A-q_B)\delta_{A0}) \\ &\quad +\frac{V_A^{(1)}}{\tau q_A}\left[(s_{A,t}-\bar{s}_A)-(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)'\mathbf{x}_t\right]\end{aligned} \tag{40}$$

B.4.2 Clearing market B

The market-clearing condition for market B is

$$\overbrace{(1-q_A-q_B)k^{-1}d_{B,t}+q_Bb_{B,t}}^{\text{Active demand}}=\overbrace{s_{B,t}-(1-q_A-q_B)(k^{-1}\sum_{i=1}^{k-1}d_{B,t-i})}^{\text{Active supply}}. \tag{41}$$

Letting $V_B^{(1)}=Var_t\left[rx_{B,t+1}\right]=\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\boldsymbol{\Sigma}\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)$, denote the variance of 1-period B-market excess returns, we can rewrite the market-clearing condition for B as

$$\begin{aligned}&\frac{\tau q_B}{V_B^{(1)}}(\alpha_{B0}-\bar{r}-\bar{z})+q_B\left[\tau\frac{\left(\frac{1}{1-\theta_B}\boldsymbol{\alpha}_{B1}-\mathbf{e}_r\right)'-\left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1}+\mathbf{e}_z\right)'\boldsymbol{\Gamma}}{V_B^{(1)}}\right]\mathbf{x}_t \\ &= (\bar{s}_B-(1-q_A-q_B)\delta_{B0})+\left(\mathbf{e}_{s_B}-(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)\right)'\mathbf{x}_t\end{aligned} \tag{42}$$

where $\mathbf{1}_{(B)}=\mathbf{e}_8+\mathbf{e}_9+\mathbf{e}_{10}$. Matching constants in (42), we have

$$\alpha_{B0}=\bar{r}+\bar{z}+\frac{V_B^{(1)}}{q_B\tau}(\bar{s}_B-(1-q_A-q_B)\delta_{B0}) \tag{43}$$

Matching slope coefficients in (42), we have

$$\begin{aligned}\boldsymbol{\alpha}_{B1} &= (1-\theta_B)\left[\mathbf{I}-\theta_B\boldsymbol{\Gamma}'\right]^{-1}\left(\mathbf{e}_r+\boldsymbol{\Gamma}'\mathbf{e}_z+\frac{V_B^{(1)}}{q_B\tau}\left(\mathbf{e}_{s_B}-(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)\right)\right) \\ &= \frac{1-\theta_B}{1-\rho_r\theta_B}\mathbf{e}_r+\frac{1-\theta_B}{1-\rho_z\theta_B}\rho_z\mathbf{e}_z \\ &\quad+\frac{V_B^{(1)}}{\tau q_B}\left(\frac{1-\theta_B}{1-\theta_B\rho_{s_B}}\mathbf{e}_{s_B}-(1-\theta_B)(1-q_A-q_B)k^{-1}\left[\mathbf{I}-\theta_B\boldsymbol{\Gamma}'\right]^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)\right)\end{aligned} \tag{44}$$

Thus, yields are

$$\begin{aligned}y_{B,t} &= \alpha_{B0}+\boldsymbol{\alpha}_{B1}'\mathbf{x}_t \\ &= \overbrace{\left[\overline{r}+\frac{1-\theta_B}{1-\rho_r\theta_B}(r_t-\overline{r})\right]}^{\text{Expected future short rates}}+\overbrace{\left[\overline{z}+\frac{1-\theta_B}{1-\rho_z\theta_B}\rho_z(z_t-\overline{z})\right]}^{\text{Expected future default losses}}+\overbrace{\frac{V_B^{(1)}}{\tau q_B}(\overline{s}_B-(1-q_A-q_B)\delta_{B0})}^{\text{Unconditional term/credit premia}} \\ &\quad+\overbrace{\left[\frac{V_B^{(1)}}{\tau q_B}\left(\frac{1-\theta_B}{1-\theta_B\rho_{s_B}}(s_{B,t}-\overline{s}_B)-(1-\theta_B)(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)'\left[\mathbf{I}-\theta_B\boldsymbol{\Gamma}\right]^{-1}\mathbf{x}_t\right)\right]}^{\text{Conditional term/credit premia}}\end{aligned} \tag{45}$$

Similarly, the expected excess return is

$$\begin{aligned}E_t\left[rx_{B,t+1}\right] &= \frac{V_B^{(1)}}{\tau q_B}(\overline{s}_B-(1-q_A-q_B)\delta_{B0}) \\ &\quad+\frac{V_B^{(1)}}{\tau q_B}\left[(s_{B,t}-\overline{s}_B)-(1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)'\mathbf{x}_t\right].\end{aligned} \tag{46}$$

Note that, because supply risk is priced, there is no clear theoretical separation of bond risk premium into an interest rate risk component and a credit risk component.

B.4.3 Solving for the equilibrium

Solving the model involves finding a solution to a system of $8k$ nonlinear equations in $8k$ unknowns. We need to determine the way that equilibrium yields and active generalist demand in markets A and B respond to shifts in asset supply in A and B: this generates 8 unknowns and 8 corresponding equations. We also need to determine how equilibrium yields and active generalist demand in A and B respond to the holdings of inactive generalists: this generates $8(k-1)$ unknowns and and 8 corresponding equations corresponding equations.

Specifically, an equilibrium solves the following system of equations

$$\begin{aligned}\boldsymbol{\alpha}_{A1} &= \frac{1-\theta_A}{1-\theta_A\rho_r}\mathbf{e}_r \\ &\quad+\frac{V_A^{(1)}(\alpha)}{\tau q_A}\left[\frac{1-\theta_A}{1-\theta_A\rho_{s_A}}\mathbf{e}_{s_A}-k^{-1}(1-q_A-q_B)(1-\theta_A)\left[\mathbf{I}-\theta_A\boldsymbol{\Gamma}(\delta)'\right]^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)\right],\end{aligned} \tag{47}$$

$$\begin{aligned}\boldsymbol{\alpha}_{B1} &= \frac{1-\theta_B}{1-\rho_r\theta_B}\mathbf{e}_r+\frac{1-\theta_B}{1-\rho_z\theta_B}\rho_z\mathbf{e}_z \\ &\quad+\frac{V_B^{(1)}(\alpha)}{\tau q_B}\left(\frac{1-\theta_B}{1-\theta_B\rho_{s_B}}\mathbf{e}_{s_B}-(1-\theta_B)(1-q_A-q_B)k^{-1}\left[\mathbf{I}-\theta_B\boldsymbol{\Gamma}(\delta)'\right]^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)\right),\end{aligned} \tag{48}$$

$$\boldsymbol{\delta}'_{A1} = \tau \frac{\begin{pmatrix} V_B^{(k)}(\alpha,\delta)\left((\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}+\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}'_{A1}\right) \\ -C_{AB}^{(k)}(\alpha,\delta)\left((\boldsymbol{\alpha}_{B1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}+\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}'_{B1}-\mathbf{e}'_z(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}\boldsymbol{\Gamma}(\delta)\right)\end{pmatrix}}{V_A^{(k)}(\alpha,\delta)V_B^{(k)}(\alpha,\delta)-(C_{AB}^{(k)}(\alpha,\delta))^2}\left(\mathbf{I}-\boldsymbol{\Gamma}(\delta)^k\right) \tag{49}$$

and

$$\boldsymbol{\delta}'_{B1} = \tau \frac{\begin{pmatrix} V_A^{(k)}(\alpha,\delta)\left((\boldsymbol{\alpha}_{B1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}+\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}'_{B1}-\mathbf{e}'_z(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}\boldsymbol{\Gamma}(\delta)\right) \\ -C_{AB}^{(k)}(\alpha,\delta)\left((\boldsymbol{\alpha}_{A1}-\mathbf{e}_r)'(\mathbf{I}-\boldsymbol{\Gamma}(\delta))^{-1}+\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}'_{A1}\right)\end{pmatrix}}{V_A^{(k)}(\alpha,\delta)V_B^{(k)}(\alpha,\delta)-(C_{AB}^{(k)}(\alpha,\delta))^2}\left(\mathbf{I}-\boldsymbol{\Gamma}(\delta)^k\right), \tag{50}$$

where we write $V_A^{(1)}(\alpha)$ and $V_B^{(1)}(\alpha)$ to emphasize that the 1-period return variances depend on $\boldsymbol{\alpha}_{A1}$ and $\boldsymbol{\alpha}_{B1}$; $\Gamma(\delta)$ to emphasize that the transition matrix depends on $\boldsymbol{\delta}_{A1}$ and $\boldsymbol{\delta}_{B1}$; and $V_A^{(k)}(\alpha,\delta)$, $V_B^{(k)}(\alpha,\delta)$, and $C_{AB}^{(k)}(\alpha,\delta)$ to emphasize that the k-period return variances depend on $\boldsymbol{\alpha}_{A1}$, $\boldsymbol{\alpha}_{B1}$, $\boldsymbol{\delta}_{A1}$ and $\boldsymbol{\delta}_{B1}$.

We can write this system of non-linear equations more compactly as

$$\begin{aligned} \boldsymbol{\alpha}_{A1} &= \mathbf{f}_{\boldsymbol{\alpha}_{A1}}(\boldsymbol{\alpha}_{A1},\boldsymbol{\alpha}_{B1},\boldsymbol{\delta}_{A1},\boldsymbol{\delta}_{B1}) \\ \boldsymbol{\alpha}_{B1} &= \mathbf{f}_{\boldsymbol{\alpha}_{B1}}(\boldsymbol{\alpha}_{A1},\boldsymbol{\alpha}_{B1},\boldsymbol{\delta}_{A1},\boldsymbol{\delta}_{B1}) \\ \boldsymbol{\delta}_{A1} &= \mathbf{f}_{\boldsymbol{\delta}_{A1}}(\boldsymbol{\alpha}_{A1},\boldsymbol{\alpha}_{B1},\boldsymbol{\delta}_{A1},\boldsymbol{\delta}_{B1}) \\ \boldsymbol{\delta}_{A2} &= \mathbf{f}_{\boldsymbol{\delta}_{A2}}(\boldsymbol{\alpha}_{A1},\boldsymbol{\alpha}_{B1},\boldsymbol{\delta}_{A1},\boldsymbol{\delta}_{B1}) \end{aligned}$$

or simply as

$$\boldsymbol{\omega} = \mathbf{f}(\boldsymbol{\omega}),$$

where $\boldsymbol{\omega} = \left(\boldsymbol{\alpha}'_{A1},\boldsymbol{\alpha}'_{B1},\boldsymbol{\delta}'_{A1},\boldsymbol{\delta}'_{B1}\right)'$.

It is easy to see that, in equilibrium, active generalist demand for A does not depend on r_t or z_t, so $\delta_{A1} = \delta_{A2} = 0$, $\alpha_{A1} = (1-\theta_A)/(1-\rho_r\theta_A)$, and $\alpha_{A2} = 0$. Similarly, active generalist demand for B does not depend on values of r_t or z_t, so $\delta_{B1} = \delta_{B2} = 0$, $\alpha_{B1} = (1-\theta_B)/(1-\rho_r\theta_B)$, and $\alpha_{B2} = \rho_z(1-\theta_B)/(1-\rho_z\theta_B)$. Thus, since we only need to determine the coefficients for the last $2k$ variables in the state vector, equations (47), (48), (49), and (50) represents a system of $8k$ nonlinear equations in $8k$ unknowns. Once we have solved this system of nonlinear equations, the 4 constant terms in (30), (31), (37), and (43) can be obtained by solving a system of linear equations as we show below.

As we discuss below in Appendix C, when there are shocks to asset supply, a solution only exists for sufficiently large τ. And, when a solution exists, there are multiple solutions—i.e., when an equilibrium exists, there are multiple equilibria. However, we always find a unique stable equilibrium: the other equilibria are unstable.

We solve this system of nonlinear equations numerically in Python using the Powell hybrid algorithm. This algorithm performs a quasi-Newton search to find roots of a system of nonlinear equations starting from an initial guess. To find all of the roots, we apply this algorithm by sampling over $10,000$ different initial guesses.

B.5 Bond risk premia

B.5.1 Unconditional, steady-state risk premia

Unconditional, steady-state risk premia are given by

$$\begin{bmatrix} E[rx_A] \\ E[rx_B] \end{bmatrix} = \begin{bmatrix} (\alpha_{A0}-\overline{r}) \\ (\alpha_{B0}-\overline{r}-\overline{z}) \end{bmatrix}. \tag{51}$$

If we substitute in the expressions for δ_{A0} and δ_{B0} from (30) and (31) into the constant terms in the market clearing conditions in (37) and (43) we obtain

$$\tau^{-1}\begin{bmatrix}\bar{s}_A\\ \bar{s}_B\end{bmatrix} = \begin{bmatrix}(1-q_A-q_B)\frac{V_B^{(k)}}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}+\frac{q_A}{kV_A^{(1)}} & -(1-q_A-q_B)\frac{C_{AB}^{(k)}}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}\\ -(1-q_A-q_B)\frac{C_{AB}^{(k)}}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2} & (1-q_A-q_B)\frac{V_A^{(k)}}{V_A^{(k)}V_B^{(k)}-(C_{AB}^{(k)})^2}+\frac{q_B}{kV_B^{(1)}}\end{bmatrix}\begin{bmatrix}k\left(\alpha_{A0}-\bar{r}\right)\\ k\left(\alpha_{B0}-\bar{r}-\bar{z}\right)\end{bmatrix} \tag{52}$$

The key is to figure out the form of this inverse matrix. Using the general fact that

$$\left(c_A\mathbf{V}_A^{-1}+c_B\mathbf{V}_B^{-1}\right)^{-1} = \mathbf{V}_B\left(c_B\mathbf{V}_A+c_A\mathbf{V}_B\right)^{-1}\mathbf{V}_A = \mathbf{V}_A\left(c_B\mathbf{V}_A+c_A\mathbf{V}_B\right)^{-1}\mathbf{V}_B, \tag{53}$$

we have

$$\begin{bmatrix}\alpha_{A0}-\bar{r}\\ \alpha_{B0}-\bar{r}-\bar{z}\end{bmatrix} = \begin{bmatrix}k^{-1}V_A^{(k)} & k^{-1}C_{AB}^{(k)}\\ k^{-1}C_{AB}^{(k)} & k^{-1}V_B^{(k)}\end{bmatrix}\left(\begin{matrix}\begin{bmatrix}k^{-1}V_A^{(k)} & k^{-1}C_{AB}^{(k)}\\ k^{-1}C_{AB}^{(k)} & k^{-1}V_B^{(k)}\end{bmatrix}\\ +(1-q_A-q_B)\begin{bmatrix}\frac{V_A^{(1)}}{q_A} & 0\\ 0 & \frac{V_B^{(1)}}{q_B}\end{bmatrix}\end{matrix}\right)^{-1}\begin{bmatrix}\frac{V_A^{(1)}}{\tau q_A}\bar{s}_A\\ \frac{V_B^{(1)}}{\tau q_B}\bar{s}_B\end{bmatrix}. \tag{54}$$

Alternately, we have

$$\begin{bmatrix}\alpha_{A0}-\bar{r}\\ \alpha_{B0}-\bar{r}-\bar{z}\end{bmatrix} = \begin{bmatrix}\frac{V_A^{(1)}}{q_A} & 0\\ 0 & \frac{V_B^{(1)}}{q_B}\end{bmatrix}\left(\begin{matrix}\begin{bmatrix}k^{-1}V_A^{(k)} & k^{-1}C_{AB}^{(k)}\\ k^{-1}C_{AB}^{(k)} & k^{-1}V_B^{(k)}\end{bmatrix}\\ +(1-q_A-q_B)\begin{bmatrix}\frac{V_A^{(1)}}{q_A} & 0\\ 0 & \frac{V_B^{(1)}}{q_B}\end{bmatrix}\end{matrix}\right)^{-1} k^{-1}\tau^{-1}\begin{bmatrix}V_A^{(k)}\bar{s}_A+C_{AB}^{(k)}\bar{s}_B\\ V_B^{(k)}\bar{s}_B+C_{AB}^{(k)}\bar{s}_A\end{bmatrix}. \tag{55}$$

Takeaways:

1. Symmetric cross-market effects in return space. We have

$$\frac{\partial}{\partial\bar{s}_A}\left(\alpha_{B0}-\bar{r}-\bar{z}\right) = \frac{\partial}{\partial\bar{s}_B}\left(\alpha_{A0}-\bar{r}\right). \tag{56}$$

2. If there are no generalists—i.e., $(1-q_A-q_B)=0$, markets are completed segmented at all horizons and unconditional risk premia satisfy

$$\begin{bmatrix}\alpha_{A0}-\bar{r}\\ \alpha_{B0}-\bar{r}-\bar{z}\end{bmatrix} = \begin{bmatrix}\frac{V_A^{(1)}}{\tau q_A}\bar{s}_A\\ \frac{V_B^{(1)}}{\tau q_B}\bar{s}_B\end{bmatrix}. \tag{57}$$

3. If there are no specialists—i.e., $(1-q_A-q_B)=1$, markets are integrated in the long run and unconditional risk premia satisfy a k-period CAPM. That is

$$\begin{bmatrix}\alpha_{A0}-\bar{r}\\ \alpha_{B0}-\bar{r}-\bar{z}\end{bmatrix} = k^{-1}\tau^{-1}\begin{bmatrix}V_A^{(k)}\bar{s}_A+C_{AB}^{(k)}\bar{s}_B\\ V_B^{(k)}\bar{s}_B+C_{AB}^{(k)}\bar{s}_A\end{bmatrix} \tag{58}$$

4. In the general case where $0<(1-q_A-q_B)<1$, unconditional risk premia do not satisfy a CAPM of any horizon. Thus, different agents will price exposures to common risk factors in different ways. In this case, unconditional risk premia can either be written (i) as a linear combination of risk premia that would prevail under perfectly segmented markets with only short-horizon specialists as shown in

equation (54) or (ii) as a linear combination of risk premia that would prevail under perfect long-run integration with only generalists investors as shown in equation (55). In particular:

(a) Own-market long-run demand curves are steeper than in the fully integrated case, but are not as steep as in the fully segmented case. For instance, we have

$$\frac{\partial E\left[rx_A\right]}{\partial \overline{s}_A}|_{(1-q_A-q_B)=1} < \frac{\partial E\left[rx_A\right]}{\partial \overline{s}_A}|_{0<(1-q_A-q_B)<1} < \frac{\partial E\left[rx_A\right]}{\partial \overline{s}_A}|_{(1-q_A-q_B)=0} \tag{59}$$

(b) Long-run cross-market price impact is more limited than in the fully integrated case, but exceeds that in the fully segmented case. For instance, we have

$$\frac{\partial E\left[rx_A\right]}{\partial \overline{s}_B}|_{(1-q_A-q_B)=0} < \frac{\partial E\left[rx_A\right]}{\partial \overline{s}_B}|_{0<(1-q_A-q_B)<1} < \frac{\partial E\left[rx_A\right]}{\partial \overline{s}_B}|_{(1-q_A-q_B)=1}. \tag{60}$$

We can also examine the portfolio's held by generalists in the long-run stead state. We have

$$\begin{aligned}
\begin{bmatrix}\delta_{A0}\\ \delta_{B0}\end{bmatrix} &= \tau\begin{bmatrix}V_A^{(k)} & C_{AB}^{(k)}\\ C_{AB}^{(k)} & V_B^{(k)}\end{bmatrix}^{-1}\begin{bmatrix}\alpha_{A0}-\overline{r}\\ \alpha_{B0}-\overline{r}-\overline{z}\end{bmatrix} \qquad (61)\\
&= \begin{bmatrix}\overline{s}_A\\ \overline{s}_B\end{bmatrix} + \begin{bmatrix} q_B\overline{s}_A\frac{1-q_A\left(1-\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}\right)}{(1-q_A)(1-q_B)-q_Aq_B\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\\ q_A\overline{s}_B\frac{1-q_B\left(1-\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}\right)}{(1-q_A)(1-q_B)-q_Aq_B\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\end{bmatrix} - \begin{bmatrix} q_A\overline{s}_B\frac{\frac{C_{AB}^{(k)}}{V_A^{(k)}}}{(1-q_A)(1-q_B)-q_Aq_B\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\\ q_B\overline{s}_A\frac{\frac{C_{AB}^{(k)}}{V_B^{(k)}}}{(1-q_A)(1-q_B)-q_Aq_B\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\end{bmatrix}
\end{aligned}$$

When $\overline{s}_A = \overline{s}_B = \overline{s}$ and $q_A = q_B = q$, this becomes

$$\begin{bmatrix}\delta_{A0}\\ \delta_{B0}\end{bmatrix} = \begin{bmatrix}\overline{s}\\ \overline{s}\end{bmatrix} + \begin{bmatrix} q\overline{s}\frac{1-q\left(1-\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}\right)}{(1-q)^2-q^2\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\\ q\overline{s}\frac{1-q\left(1-\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}\right)}{(1-q)^2-q^2\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\end{bmatrix} - \begin{bmatrix} q\overline{s}\frac{\frac{C_{AB}^{(k)}}{V_A^{(k)}}}{(1-q)^2-q^2\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\\ q\overline{s}\frac{\frac{C_{AB}^{(k)}}{V_B^{(k)}}}{(1-q)^2-q^2\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\end{bmatrix}. \tag{62}$$

Thus, if $V_B^{(k)} > V_A^{(k)}$ we have

$$\delta_{B0} - \delta_{A0} = \frac{C_{AB}^{(k)}q\overline{s}}{(1-q)^2 - q^2\frac{C_{AB}^{(k)2}}{V_A^{(k)}V_B^{(k)}}}\left(\frac{1}{V_A^{(k)}} - \frac{1}{V_B^{(k)}}\right) > 0. \tag{63}$$

B.5.2 Conditional risk-premia

In the special case where $k = 1$, we can use a similar approach to characterize the conditional reaction of A and B yields to supply shocks. However, this approach cannot be use to characterize the reaction when $k > 1$. Specifically, when $k = 1$, we have

$$\begin{bmatrix}E_t\left[rx_{A,t+1}\right] - E\left[rx_{A,t+1}\right]\\ E_t\left[rx_{B,t+1}\right] - E\left[rx_{B,t+1}\right]\end{bmatrix} = \begin{bmatrix}\alpha_{A_{s_A}}\frac{1-\theta_A\rho_{s_A}}{1-\theta_A} & \alpha_{A_{s_B}}\frac{1-\theta_A\rho_{s_B}}{1-\theta_A}\\ \alpha_{B_{s_A}}\frac{1-\theta_B\rho_{s_A}}{1-\theta_B} & \alpha_{B_{s_B}}\frac{1-\theta_B\rho_{s_B}}{1-\theta_B}\end{bmatrix}\begin{bmatrix}s_{A,t}-\overline{s}_A\\ s_{A,t}-\overline{s}_B\end{bmatrix} \tag{64}$$

where

$$
\begin{bmatrix} \alpha_{A_{s_A}} \frac{1-\theta_A \rho_{s_A}}{1-\theta_A} & \alpha_{A_{s_B}} \frac{1-\theta_A \rho_{s_B}}{1-\theta_A} \\ \alpha_{B_{s_A}} \frac{1-\theta_B \rho_{s_A}}{1-\theta_B} & \alpha_{B_{s_B}} \frac{1-\theta_B \rho_{s_B}}{1-\theta_B} \end{bmatrix} \tag{65}
$$

$$
= \frac{1}{\tau} \frac{\begin{bmatrix} V_A^{(1)} \times ((1-q_A) V_A^{(1)} V_B^{(1)} - q_B (C_{AB}^{(1)})^2) & C_{AB}^{(1)} \times (1-q_A-q_B) V_A^{(1)} V_B^{(1)} \\ C_{AB}^{(1)} \times (1-q_A-q_B) V_A^{(1)} V_B^{(1)} & V_B^{(1)} \times ((1-q_B) V_A^{(1)} V_B^{(1)} - q_A (C_{AB}^{(1)})^2) \end{bmatrix}}{(1-q_A)(1-q_B) V_A^{(1)} V_B^{(1)} - q_A q_B (C_{AB}^{(1)})^2}
$$

and

$$
\begin{aligned}
V_A^{(1)} &= \left(\frac{\theta_A}{1-\rho_r \theta_A}\right)^2 \sigma_r^2 + \left(\frac{\theta_A}{1-\theta_A} \alpha_{As_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_A}{1-\theta_A} \alpha_{As_B}\right)^2 \sigma_{s_B}^2 \\
V_B^{(1)} &= \left(\frac{\theta_B}{1-\rho_r \theta_B}\right)^2 \sigma_r^2 + \left(\frac{1}{1-\rho_z \theta_B}\right)^2 \sigma_z^2 + \left(\frac{\theta_B}{1-\theta_B} \alpha_{Bs_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_B}{1-\theta_B} \alpha_{Bs_B}\right)^2 \sigma_{s_B}^2 \\
C_{AB}^{(1)} &= \left(\frac{\theta_A}{1-\rho_r \theta_A}\right)\left(\frac{\theta_B}{1-\rho_r \theta_B}\right) \sigma_r^2 + \left(\frac{\theta_A}{1-\theta_A} \alpha_{As_A}\right)\left(\frac{\theta_B}{1-\theta_B} \alpha_{Bs_A}\right) \sigma_{s_A}^2 \\
&\quad + \left(\frac{\theta_A}{1-\theta_A} \alpha_{As_B}\right)\left(\frac{\theta_B}{1-\theta_B} \alpha_{Bs_B}\right) \sigma_{s_B}^2 .
\end{aligned} \tag{66}
$$

Takeaways:

1. Cross-market, short-run price impact is symmetric in excess return space:

$$
\alpha_{B_{s_A}} \frac{1-\theta_B \rho_{s_A}}{1-\theta_B} = \frac{\partial E_t [rx_{B,t+1}]}{\partial s_{A,t}} = \frac{\partial E_t [rx_{A,t+1}]}{\partial s_{B,t}} = \alpha_{A_{s_B}} \frac{1-\theta_A \rho_{s_B}}{1-\theta_A}.
$$

2. Where there are no specialists—i.e., $(1-q_A-q_B) = 1$, markets are integrated in the short run and conditional risk premia satisfy a 1-period conditional CAPM. Specifically, we have

$$
\begin{bmatrix} \alpha_{A_{s_A}} \frac{1-\theta_A \rho_{s_A}}{1-\theta_A} & \alpha_{A_{s_B}} \frac{1-\theta_A \rho_{s_B}}{1-\theta_A} \\ \alpha_{B_{s_A}} \frac{1-\theta_B \rho_{s_A}}{1-\theta_B} & \alpha_{B_{s_B}} \frac{1-\theta_B \rho_{s_B}}{1-\theta_B} \end{bmatrix} = \tau^{-1} \begin{bmatrix} V_A^{(1)} & C_{AB}^{(1)} \\ C_{AB}^{(1)} & V_B^{(1)} \end{bmatrix}.
$$

3. In the general case where $0 < (1-q_A-q_B) < 1$, conditional risk premia do not satisfy conditional CAPM. Thus, different agents will price exposures to common risk factors in different ways in the short-run. As above, conditional risk premia can either be written (i) as a linear combination of risk premia that would prevail under perfectly segmented markets with only short-horizon specialists or (ii) as a linear combination of risk premia that would prevail under perfect short-run integration with only short-run generalists:

   (a) Since $[(1-q_A) V_A^{(1)} V_B^{(1)} - q_B (C_{AB}^{(1)})^2]/[(1-q_A)(1-q_B) V_A^{(1)} V_B^{(1)} - q_A q_B (C_{AB}^{(1)})^2] > 1$ and $[(1-q_B) V_A^{(1)} V_B^{(1)} - q_A (C_{AB}^{(1)})^2]/[(1-q_A)(1-q_B) V_A^{(1)} V_B^{(1)} - q_A q_B (C_{AB}^{(1)})^2] > 1$, the equilibrium with partially segmented markets features greater own-market price impact than the fully integrated markets limit ($q_A = q_B = 0$). For example, increases in $s_{A,t}$ have larger effects on $E_t [rx_{A,t+1}]$ and $y_{A,t}$ than they would in the integrated markets case (i.e., $\alpha_{A_{s_A}}$ is larger). However, own-market price-impact is not as great in the completely segmented market limit $(1-q_A-q_B = 0)$.

   (b) Since $[(1-q_A-q_B) V_B^{(1)} V_A^{(1)}]/[(1-q_A)(1-q_B) V_A^{(1)} V_B^{(1)} - q_A q_B (C_{AB}^{(1)})^2] < 1$, the equilibrium with partially segmented markets features smaller cross-market price impact than the fully integrated markets limit ($q_A = q_B = 0$). For example, increases in $s_{A,t}$ have a smaller impact on $E_t[rx_{B,t+1}]$ and $y_{B,t}$ than they would in the integrated markets case (i.e., $\alpha_{B_{s_A}}$ is smaller). However, cross-market price-impact is clearly greater than in the completely segmented market limit $(1 - q_A - q_B = 0)$ where there is zero cross-market impact.

C Equilibrium existence and multiplicity

C.1 Overview

Equilibrium non-existence and multiplicity are common issues in overlapping-generations, rational-expectations models such as ours where mean-variance (CARA-normal) investors with finite investment horizons trade an infinitely-lived asset that is subject to supply shocks. Specifically, if arbitrageurs are not sufficiently risk-tolerant, then linear equilibria fail to exist in models of this sort. And, if arbitrageurs are sufficiently risk-tolerant, then there will be multiple equilibria.[1] As shown in prior studies, equilibrium non-existence and multiplicity arise in these models because finitely-lived agents trade an infinitely-lived asset.[2] In this Appendix, we clarify why multiple equilibria arise in our model, our approach to equilibrium selection in our numerical exercise (we focus on the model's unique "stable" equilibrium), and the properties of the model's other "unstable" equilibria.

Clarifying the key issues in a simple case The intuition for equilibrium multiplicity can be seen most clearly in the simple case when there is a single risky asset and only fast-moving investors. We now treat this special case in some detail below.

If an equilibrium exists in this simple case, there are two equilibria: a low price-impact (or low return volatility) equilibrium and a high price-impact (or high return volatility) equilibrium. If investors belief that supply shocks will have a large impact on prices, then the asset becomes highly risky for investors with finite lives. As a result, investors will only absorb supply shocks if they are compensated by large price changes, making the belief that supply shocks have a large impact on price self-fulfilling. Conversely, if investors believe that prices are will be less sensitive to supply shocks, then the asset becomes less risky and investors will absorb supply shocks even if they are only compensated by modest changes in prices.

Formally, a rational expectations equilibrium in this simple case a fixed point of a specific operator involving the price-impact coefficient, α_s. Let $f(\alpha_s)$ denote the price-impact coefficient that will clear the market for the risky asset today if agents conjecture that the price-impact coefficient will be α_s in all future periods. In this simple case, $f(\alpha_s) = A \times \alpha_s^2 + C$ where A and C are positive constants. Thus, an rational expectations equilibrium α_s^* satisfies $\alpha_s^* = f(\alpha_s^*)$—i.e., an equilibrium is the solution to a specific quadratic equation. When aggregate risk tolerance τ is low, there are no real solutions to this quadratic. When τ is higher, there are two solutions $\alpha_s^{*(-)}$ and $\alpha_s^{*(+)}$ which satisfy $0 < \alpha_s^{*(-)} < \alpha_s^{*(+)}$—i.e., there are low and high price-impact equilibria.

The multiplicity of equilibrium capture an important set of economic intuitions. And, in particular, the high price-impact equilibrium may be useful for thinking about the behavior of markets during times of distress. This implication has previously been emphasized by Spiegel (1998) and Watanabe (2008) among others. However, the low price-impact equilibrium has several important advantages and is the only equilibrium that is "well-behaved" in the following three ways:

[1] For previous treatments of these issues, see Spiegel (1998), Bacchetta and van Wincoop (2003), Watanabe (2008), Banerjee (2011), Greenwood and Vayanos (2014), and Albagli (2015).

[2] First, when a sequence of finitely-lived generations of mean-variance agents trade a finitely-lived asset, standard backwards induction arguments show that a unique linear equilibrium always exists. And, as Banerjee (2011) shows, the low price-impact equilibrium of the infinite horizon economy corresponds to the limit of this finite-horizon economy as the economy's horizon approaches infinity.

Second, when the asset is infinitely-lived, Albagli (2015) shows that a unique linear equilibrium always exists when agents are infinitely-lived and consume on an interim basis. Indeed, as the length of agents' lives increases, Albagli (2015) shows that the equilibrium with low price-impact with finitely-lived agents converges to the unique equilibrium with infinitely-lived agents. In other words, the high price-impact equilibria (if it exists with finitely-lived agents) disappears in the limit where agents live forever.

1. **Limiting behavior:** The low price-impact equilibrium has well-defined limiting behavior whereas the high price-impact equilibrium has divergent limiting behavior (Watanabe (2008), Greenwood and Vayanos (2014)). As the volatility of supply shocks approaches zero, the low-price impact equilibrium converges to the equilibrium with deterministic supply; by contrast, the high-price impact equilibrium explodes with both the price-impact coefficient and the variance of returns diverging to infinity. Similarly, in the risk-neutral limit where investor risk tolerance approaches infinity, the low-price impact equilibrium converges to an equilibrium with zero price-impact; by contrast, the high-price impact equilibrium explodes with both the price-impact coefficient and the variance of returns diverging to infinity. Intuitively, in order to sustain the high-price impact equilibrium in these limiting cases, investors must fear that tiny supply shocks will have a massive impact on prices.

2. **Equilibrium stability:** The low-price impact equilibrium is "stable" in the sense that it is robust to a small perturbation in investors' beliefs regarding tomorrow's price-impact parameter (Bacchetta and van Wincoop (2003), Albagli (2015), Bogousslavsky (2016)). By contrast, the high high-impact equilibrium is unstable. An initial equilibrium α_s^* is stable (unstable) if, following a small initial change from α_s^* to $\alpha_s^* + \xi$ for ξ small, iterating on the price-impact operator drives the system back towards (away from) the initial equilibrium. Formally, letting $\alpha_s^{(1)} = \alpha_s^* + \xi$ and defining $\alpha_s^{(n)} = f(\alpha_s^{(n-1)})$ an equilibrium is stable if $\lim_{n\to\infty} \alpha_s^{(n)} = \alpha_s^*$ and is unstable if $\lim_{n\to\infty} \alpha_s^{(n)} \neq \alpha_s^*$. In this simple case with a single endogenous parameter, an equilibrium α_s^* is stable if $|f'(\alpha_s^*)| < 1$ and is unstable if $|f'(\alpha_s^*)| > 1$. Thus, there is a clear formal sense in which the high price-impact equilibrium is a fragile, knife-edge outcome.

3. **Comparative statics:** The comparative statics at the low price-impact equilibrium accord with standard economic intuition, while the comparative statics at the high price-impact equilibrium run contrary to standard economic intuition (see for example Albagli (2015)). For instance, at the low-price impact equilibrium, an increase in fundamental volatility or the volatility of supply shocks is associated with an increase the price-impact coefficient and an increase in the volatility of asset returns. By contrast, at the high-price impact equilibrium, an increase in fundamental volatility or the volatility of supply shocks is associated with a decline the price-impact coefficient and a decline return volatility. This is an instance of Samuelson's (1947) "correspondence principle."

   Consider the impact of some exogenous parameter γ on the equilibrium price-impact parameter α_s^*. An equilibrium solves $\alpha_s^* = f(\alpha_s^*, \gamma)$ where $\partial f(\alpha_s, \gamma)/\partial \alpha_s > 0$. Thus, we have $\partial \alpha_s^*/\partial \gamma = [\partial f(\alpha_s^*, \gamma)/\partial \gamma] / [1 - \partial f(\alpha_s^*, \gamma)/\partial \alpha_s]$. The low price-impact equilibrium ($\alpha_s^{*(-)}$) is stable ($0 < \partial f(\alpha_s^{*(-)}, \gamma)/\partial \alpha_s <$ 1), so $\partial \alpha_s^{*(-)}/\partial \gamma \propto \partial f(\alpha_s^*, \gamma)/\partial \gamma$ which will have the same sign as that suggested by standard economic intuition.

4. By contrast, the high price-impact equilibrium ($\alpha_s^{*(+)}$) is unstable ($\partial f(\alpha_s^{*(-)}, \gamma)/\partial \alpha_s > 1$), so $\partial \alpha_s^{*(-)}/\partial \gamma \propto -\partial f(\alpha_s^*, \gamma)/\partial \gamma$ which will have the opposite sign as that suggested standard economic intuition.

For these reasons, papers in this class that emphasize comparative statics results focus on the low price-impact equilibrium (see, e.g., Banerjee (2011) and Greenwood and Vayanos (2014)).

Equilibrium multiplicity in our general model Things are somewhat more complicated in our general model, but the basic insights from the single asset case largely carry through. Specifically, the addition of multiple risky assets, the partial segmentation of markets, and the introduction of slow-moving capital, give rise to additional unstable equilibria. However, when an equilibrium exists—which is always the case for a sufficiently large choice of τ, we always find a single equilibrium that is stable in the sense that it is robust to a small perturbation in investors' beliefs regarding future asset price volatility (and the reaction of slow-moving generalists). Thus, we use equilibrium stability as our sole criteria for selecting amongst equilibria.

To build intuition, we study two special cases below.

Special Case #1 (Two symmetric assets with only fast-moving generalists): If there are only fast-moving generalists investors ($k = 1$, $q_A = q_B = 0$)—i.e., if we switch off both of the key asset pricing frictions that are the focus of our paper—and if the two assets are symmetric ($\theta_A = \theta_B = \theta$, and $\rho_{s_A} = \rho_{s_B} = \rho_s$), then our model reduces to a variant of Spiegel (1998) and we have a closed-form solution for the four price-impact coefficients that determine an equilibrium.[3] In this case, the logic in Spiegel (1998) and Wantanabe (1998) shows that generically there are either no equilibrium or four distinct equilibrium.[4] This is the special case of our model shown in Panel A of Figure 1.

As explained in Spiegel (1998) and Wantanabe (1998), the different equilibria correspond to different self-fulfilling beliefs that investors hold about the equilibrium volatility of two portfolios of the two risky assets. Specifically, the first portfolio is long both assets and the second is a long-short portfolio (the returns on the two portfolios are orthogonal). The unique stable equilibrium obtains when investors believe both of these portfolios will have low volatility. This equilibrium features modest own-market price-impact and modest, positive cross-market price-impact. This stable equilibrium has well-defined limiting behavior and generates comparative statics that accord with standard intuition. The three unstable equilibria obtain when investors believe that one or more of these portfolios will have high volatility. These unstable equilibria have divergent limiting behavior as $\tau \to \infty$ or as $\sigma^2_{s_A}, \sigma^2_{s_A} \to 0$ and feature local comparative statics that run contrary to standard intuition.

Special Case #2 (Fast-moving specialists and fast-moving generalists): In this case, markets are partially segmented, but there is not slow-moving capital—i.e., when $k = 1$, $q_A, q_B > 0$, and $(1 - q_A - q_B) > 0$. This is the special case of our model shown in Panel B of Figure 2. There are no closed-form solutions in this case. However, when an equilibrium exists, we always find six distinct equilibria. Four of these equilibria correspond to the equilibria that obtain in Special Case #1 where $q_A = q_B = 0$—although the price-impact coefficients will differ due to the partially-segmented nature of the two markets.

However, when $q_A, q_B > 0$, there are two additional equilibria where there is only high price-impact in one of the two markets. The existence of these two asymmetric equilibria means that the degree of equilibrium segmentation between the A and B can be self-fulfilling. If generalists believe that markets A and B will be highly segmented, then cross-market arbitrage becomes very risky, so generalists do not aggressively integrate markets in response to supply shocks. Specifically, yields in A are not highly responsive to B supply shocks, and vice versa.

Again, when an equilibrium exists in this special case, we always find a single stable equilibrium which corresponds to the stable equilibrium when there are only fast-moving generalists. Naturally, relative to the integrated market equilibrium in Special case #1, the introduction of specialists increases the own-market price-impact coefficients and reduces the cross-market price impact coefficients. As above, this stable equilibrium has well-defined limiting behavior and local comparative statics that square with standard intuition. This is not the case for the five unstable equilibria that we find.

General Case (Both partial segmentation and slow-moving capital): In the general case, where $q_A, q_B > 0$ and $k > 1$, there is no closed-form solution and again we need to find equilibria numerically. We find (i)

[3] There is no closed-form solution when $k = 1$, $q_B = q_B = 0$, and the assets are not symmetric is the sense that either $\theta_A \neq \theta_B$ or $\rho_{s_A} \neq \rho_{s_B}$. However, we also always find four equilibria in this case.

[4] As in Spiegel (1998) and Watanabe (2008), if there are N assets and an equilibrium exists, then there will generically be 2^N equilibria. This is because the $N \times N$ matrix of price-impact coefficients satisfies a quadratic matrix equation. The solution involves taking the matrix square root of a specific symmetric matrix. If this matrix is positive-definite (i.e., if all of its eigenvalues λ_n are positive), then there generically are 2^N real matrix square roots corresponding to choosing $\sqrt{\lambda_n}$ or $-\sqrt{\lambda_n}$ for each eigenvalue $n = 1, ..., N$. If this matrix is not positive-semidefinite (i.e., if some of its eigenvalues are negative), then there are no real matrix square roots and there are no equilibrium. Furthermore, the relevant matrix will only be positive-definite if aggregate risk tolerance is sufficiently high.

As shown in Watanabe (2008), there are infinitely many equilibria in the knife-edge case where (i) the fundamental shocks affecting the N assets are uncorrelated and have the same volatility and (ii) the supply shocks affecting the N assets are uncorrelated and have the same volatility (assuming that risk tolerance is sufficiently large so equilibria exist). Mathematically, the infinitude of equilibria is a consequence of the fact that the identity matrix has an infinite number of square roots. In this case, there is a single stable equilibrium that features no cross-asset price impact and an infinite number of unstable equilibria, some of which feature cross-market price impact.

no equilibrium exists when τ is sufficiently low; (ii) when an equilibrium exists—i.e., when τ is sufficiently large, there are generically an even number of equilibrium; (iii) we always find a single stable equilibrium.

A rational expectations equilibrium of our general model is a fixed point of a specific operator involving the price-impact coefficients, $(\boldsymbol{\alpha}'_{A1}, \boldsymbol{\alpha}'_{B1})$, which show how bond supply and inactive generalist demand impact current yields, and the demand-impact coefficients, $(\boldsymbol{\delta}'_{A1}, \boldsymbol{\delta}'_{B1})$, which show how bond supply and inactive generalist demand impact current generalist demands. Specifically, let $\boldsymbol{\omega} = (\boldsymbol{\alpha}'_{A1}, \boldsymbol{\alpha}'_{B1}, \boldsymbol{\delta}'_{A1}, \boldsymbol{\delta}'_{B1})'$ and consider the operator $\mathbf{f}(\boldsymbol{\omega}_0)$ which gives (i) the price-impact coefficients that will clear the two markets and (ii) the demand-impact coefficients consistent with optimization on the part of generalists if agents conjecture that $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ at all future dates. Thus, a rational expectations equilibrium of our model is a fixed point $\boldsymbol{\omega}^* = \mathbf{f}(\boldsymbol{\omega}^*)$.

A stable (unstable) equilibrium is robust (is not robust) to a small perturbation in investors' beliefs regarding that equilibrium that will prevail in the future. Formally, an initial equilibrium $\boldsymbol{\omega}^*$ is stable (unstable) if, following a small initial change from $\boldsymbol{\omega}^*$ to $\boldsymbol{\omega}^* + \boldsymbol{\xi}$ for $\boldsymbol{\xi}$ small, iterating on this operator drives the system back towards (away from) the initial equilibrium. Formally, letting $\boldsymbol{\omega}^{(1)} = \boldsymbol{\omega}^* + \boldsymbol{\xi}$ and defining $\boldsymbol{\omega}^{(n)} = \mathbf{f}(\boldsymbol{\omega}^{(n-1)})$ an equilibrium is stable if $\lim_{n\to\infty} \boldsymbol{\omega}^{(n)} = \boldsymbol{\omega}^*$ and is unstable if $\lim_{n\to\infty} \boldsymbol{\omega}^{(n)} \neq \boldsymbol{\omega}^*$. Let $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*)$ denote the Jacobian of $\mathbf{f}(\boldsymbol{\omega})$ evaluated at $\boldsymbol{\omega}^*$ and $\lambda_i^{\mathbf{D}}$ denote the eigenvalues of $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*)$. If $\max_i |\lambda_i^{\mathbf{D}}| < 1$, the equilibrium is stable; if $\max_i |\lambda_i^{\mathbf{D}}| > 1$, the equilibrium is unstable. When equilibria exist, we always find a unique stable equilibrium—-all other equilibria are unstable—and we focus on the unique stable equilibrium in our numerical illustrations.

Some of the unstable equilibria in our general model with both partial segmentation, correspond to those discussed in Special Case #2 when there is no slow-moving capital. However, the introduction of slow-moving capital opens the door to a new class of unstable equilibria: equilibria that feature unstable time-series dynamics. Specifically, it is natural to require that the VAR for the vector of state variables $\mathbf{x}_{t+1} = \Gamma(\boldsymbol{\delta}^*)\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$ has stable time-series dynamics in equilibrium—i.e., implies non-explosive dynamics and well-defined unconditional second moments (Hamilton [1994]). Because these dynamics depend on generalists' demand functions, they must be pinned down in equilibrium. Letting λ_i^{Γ} denote the eigenvalues of $\Gamma(\boldsymbol{\delta}^*)$, this is equivalent to the requirement that $\max_i |\lambda_i^{\boldsymbol{\Gamma}}| < 1$ (Hamilton [1994]). When $k > 1$, the system admits solutions with unstable VAR dynamics where $\Gamma(\boldsymbol{\delta}^*)$ has an eigenvalue that is less than -1, implying unstable dynamics for the state vector.[5] When $k = 2$, this corresponds to solutions where $\delta_{A1}[d_{A,t-1}] < -1$ or $\delta_{B1}[d_{B,t-1}] < -1$—i.e., where active generalists reduce their holdings more than one-for-one in response to the holdings of inactive generalists. These solutions imply divergent and explosive oscillations in the holdings of generalists and, thus, in equilibrium yields. Agents in the model expect these divergent, oscillatory dynamics in the future, but the impact on current yields is finite because (i) future risk premia are expected to oscillate and (ii) because expected risk-premia in the distant future have a smaller effect on current yields.

Why do we focus on the unique stable equilibrium? First, consistent with Samuelson's (1947) correspondence principle, the single stable equilibrium has local comparative statics that comport with common sense economic intuition. By contrast, the unstable equilibria feature comparative statics that conflict with standard intuition. To understand the intuition for this result, consider the impact of some parameter γ on the equilibrium. An equilibrium satisfies $\boldsymbol{\omega}^* = \mathbf{f}(\boldsymbol{\omega}^*, \gamma)$. By the implicit function theorem, we have

$$\mathbf{D}_\gamma \boldsymbol{\omega}^* = [\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma)]^{-1} \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma).$$

If an equilibrium is stable (as well as isolated and non-degenerate) then all of the eigenvalues of $\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma)$ have a modulus less than 1. Thus, we have $[\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma)]^{-1} = [\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma))^i]$ so we can write

$$\mathbf{D}_\gamma \boldsymbol{\omega}^* = [\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma)]^{-1} \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma) = [\textstyle\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\omega}}\mathbf{f}(\boldsymbol{\omega}^*, \gamma))^i] \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma).$$

[5] Equilibria that feature unstable VAR dynamics are also not robust to small perturbations to investors' beliefs regarding that equilibrium that will prevail in the future—i.e., $\max_i |\lambda_i^{\boldsymbol{\Gamma}}| > 1$ implies $\max_i |\lambda_i^{\mathbf{D}}| > 1$. Thus, our sole criterion for equilibrium selection involves the eigenvalues of $\mathbf{D}_{\boldsymbol{\theta}}\mathbf{f}(\boldsymbol{\theta}^*)$.

This says that comparative statics on $\boldsymbol{\omega}^*$ have a straightforward interpretation in terms of a dynamic adjustment process. The first-round direct effect is $\mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma)$. The second-round indirect effect is then $\mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma) \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma)$. The third-round indirect effect is $(\mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma))^2 \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\theta}^*, \gamma)$. The total effect is the sum across all rounds.[6] Samuelson's correspondence principle refers to this correspondence between equilibrium comparative statics and the result of this dynamic adjustment process.

As discussed above, when equilibrium only involves a single variable, knowledge that an equilibrium is stable (unstable) allows one to unambiguously determine the sign of equilibrium comparative statics (Samuelson (1947)). Things are more complicated when an equilibrium involves multiple unknowns as it does in our general (Arrow and Hahn (1971) and Echenique (2002, 2008)). In multivariate settings, knowledge that an equilibrium is stable (or unstable) only allows one to unambiguously sign equilibrium comparative statics in very special cases. However, the fact that an equilibrium is stable, still has qualitative implications for comparative statics.[7]

Second, as in the single asset case, the unique stable equilibrium of our general model has a well-behaved limit as investors' risk tolerance grows large ($\tau \to \infty$) in the sense that it converges to the equilibrium with risk-neutral investors. By contrast, the unstable equilibria explode in this limit with one or more price-impact (or demand-impact) coefficients going to infinity. Similarly, as the volatility of supply shocks vanishes ($\sigma^2_{s_A}, \sigma^2_{s_A} \to 0$), the stable equilibrium converges to the equilibrium with deterministic supply. Again, the unstable equilibria explode in this limit.

C.2 A simple case

In this section, we solve the model in the special case where there is only a default-free long-term bond, denoted L, and where there are no slow-moving investors. The result is a simplified, discrete-time version of the default-free term structure models developed in Vayanos and Vila (2009) and Greenwood and Vayanos (2014) in which long-term bonds are priced by a set of specialized, risk-averse bond arbitrageurs who have short-horizons. Since this special case can be solved with pencil and paper, it is useful for understanding (i) why linear equilibria may fail to exist in our more general model; (ii) why there are multiple equilibria if one exists; and (iii) why, for the purposes of examining comparative statics, it makes sense to focus on the unique stable equilibrium.

Model setting The excess returns on the long-term default-free bond, denoted L, are given by

$$rx_{L,t+1} = \frac{1}{1-\theta} y_{L,t} - \frac{\theta}{1-\theta} y_{L,t+1} - r_t. \tag{67}$$

In this special case, there are just two state variables:

- **Short-term interest rates:** The log short-term riskless rate available to investors between time t and $t+1$, denoted r_t, is known at time t. We assume that r_t evolves according to

$$r_{t+1} = \overline{r} + \rho_r (r_t - \overline{r}) + \varepsilon_{r,t+1}, \tag{68}$$

  where $Var_t[\varepsilon_{r,t+1}] = \sigma^2_r$.

[6] By contrast, if an equilibrium is unstable then some of the eigenvalues of $\mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma)$ have a modulus greater than 1. Thus, we have $[\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma)]^{-1} \neq \left[\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma))^i\right]$ so $\mathbf{D}_\gamma \boldsymbol{\omega}^* \neq \left[\sum_{i=0}^{\infty} (\mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma))^i\right] \mathbf{D}_\gamma \mathbf{f}(\boldsymbol{\omega}^*, \gamma)$ and comparative statics don't have this intuitive interptation.

[7] Specifcally, if $\boldsymbol{\omega}^*$ is an isolated, stable, non-degenerate equilibrium equilibrium, then this implies that $\Delta \equiv \det(\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma)) > 0$. And the fact that $\Delta > 0$ has qualitative implications for comparative statics. To see the implications, note that

$$\frac{\partial \boldsymbol{\omega}^*_n}{\partial \gamma} = \sum\nolimits_j \frac{\Delta_{jn}}{\Delta} \frac{\partial f_n(\boldsymbol{\omega}^*, \gamma)}{\partial \gamma}$$

where Δ_{ij} is the (i,j) co-factor of $[\mathbf{I} - \mathbf{D}_{\boldsymbol{\omega}} \mathbf{f}(\boldsymbol{\omega}^*, \gamma)]$. Thus, the fact that $\Delta > 0$ has qualitative implications for comparitive statics.

- **Supply:** The net supply of long-term bonds that arbitrageurs must hold evolves according to

$$s_{t+1} = \overline{s} + \rho_s (s_t - \overline{s}) + \varepsilon_{s,t+1}, \tag{69}$$

where $Var_t[\varepsilon_{s,t+1}] = \sigma_s^2$. For simplicity, we assume that $Cov_t[\varepsilon_{r,t+1}, \varepsilon_{s,t+1}] = 0$.

There is a unit mass of specialized bond arbitrageurs, each with risk tolerance τ. Specialist arbitrageurs can earn an uncertain future return of $r_{L,t+1}$ from to t to $t+1$ by investing in the default-free long-term bond. Alternatively, they can earn a certain return of r_t by investing at the short-term interest rate. Specialist arbitrageurs are concerned with their interim wealth. Formally, we assume that at date t specialist arbitrageurs have mean-variance preferences over their wealth at $t+1$. This means that arbitrageurs choose their holdings of the perpetuity to solve

$$\max_{b_t} \left\{ b_t E_t [rx_{L,t+1}] - (2\tau)^{-1} (b_t)^2 Var_t [rx_{L,t+1}] \right\}, \tag{70}$$

where $rx_{L,t+1} \equiv r_{L,t+1} - r_t$ is the log excess return on the default-free long–term bond over the short-term interest rate between t and $t+1$. Thus, arbitrageur demand for the risky bond is

$$b_t = \tau \frac{E_t [rx_{L,t+1}]}{Var_t [rx_{L,t+1}]}. \tag{71}$$

Market clearing ($b_t = s_t$) implies that conditional bond risk premia, $E_t[rx_{L,t+1}]$ are given by

$$E_t [rx_{L,t+1}] = \tau^{-1} V_L s_t. \tag{72}$$

where $V_L \equiv Var_t [rx_{L,t+1}]$ is the equilibrium conditional variance of 1-period excess returns to be determined below. Rewriting equation (67) as $y_{L,t} = E_t [(1-\theta)(r_t + rx_{L,t+1}) + \theta y_{L,t+1}]$, making use of (72), and iterating forward, we see that the equilibrium yield on the defaultable perpetuity is a weighted average of expected future short rates and future risk premia which in turn are governed by future bond supply:ℓ

$$y_{L,t} = (1-\theta) \sum_{i=0}^{\infty} \theta^i E_t [\overbrace{r_{t+i}}^{\text{Short rate}} + \overbrace{\tau^{-1} V_L s_{t+i}}^{\text{Risk premium}}]. \tag{73}$$

Making use of the assumed AR(1) dynamics for r_t and s_t, we can express the equilibrium yield as

$$y_{L,t} = \overbrace{\left[\overline{r} + \frac{1-\theta}{1-\rho_r \theta} (r_t - \overline{r}) \right]}^{\text{Expected future short rates}} + \overbrace{\left[\tau^{-1} V_L \overline{s} + \tau^{-1} V_L \frac{1-\theta}{1-\rho_s \theta} (s_t - \overline{s}) \right]}^{\text{Risk premium}}. \tag{74}$$

To formally solve the single asset model, we only need to determine V_L, the equilibrium conditional variance of 1-period excess returns.[8] To do, we conjecture that equilibrium bond yields take the linear form

$$y_{L,t} = \alpha_0 + \alpha_r (r_t - \overline{r}) + \alpha_s (s_t - \overline{s}). \tag{75}$$

[8] In the single asset model, one can solve directly for V_L. Specifically, assuming that yields take the form given in (74), V_L must satisfy the following quadratic equation

$$V_L = \left(\frac{\theta}{1-\rho_r \theta} \sigma_r \right)^2 + \left(\tau^{-1} \frac{\theta}{1-\rho_s \theta} \sigma_s \right)^2 (V_L)^2.$$

Below we will use the equivalent approach of solving for a fixed point involving the price-impact parameter α_s since this is how we must solve our general model.

Given this conjecture, we have

$$E_t\left[rx_{L,t+1}\right]=\left[\alpha_0-\overline{r}\right]+\left[\frac{1-\theta\rho_r}{1-\theta}\alpha_r-1\right]\left(r_t-\overline{r}\right)+\left[\frac{1-\theta\rho_s}{1-\theta}\alpha_s\right]\left(s_t-\overline{s}\right)$$

Matching terms on 1 and $(r_t-\overline{r})$ in the market-clearing equation (72), we obtain

$$\alpha_0-\overline{r}=\tau^{-1}V_L\overline{s}\text{ and }\alpha_r=\frac{1-\theta}{1-\theta\rho_r}$$

as suggested by equation (74). Next note that our conjecture implies that

$$V_L\equiv Var_t\left[rx_{L,t+1}\right]=\left(\frac{\theta}{1-\theta\rho_r}\right)^2\sigma_r^2+\left(\frac{\theta}{1-\theta}\alpha_s\right)^2\sigma_s^2.$$

Matching terms on $(s_t-\overline{s})$ in the market-clearing equation (72), we obtain

$$\alpha_s=\tau^{-1}\frac{1-\theta}{1-\rho_s\theta}V_L=\tau^{-1}\frac{1-\theta}{1-\rho_s\theta}\left(\left(\frac{\theta}{1-\rho_r\theta}\sigma_r\right)^2+\left(\frac{\theta}{1-\theta}\sigma_s\right)^2\alpha_s^2\right).$$

Thus, α_s satisfies the following quadratic equation

$$0=\overbrace{\tau^{-1}\frac{1-\theta}{1-\rho_s\theta}\left(\frac{\theta}{1-\theta}\sigma_s\right)^2}^{A_{\alpha_s}}\alpha_s^2-\alpha_s+\overbrace{\tau^{-1}\frac{1-\theta}{1-\rho_s\theta}\left(\frac{\theta}{1-\rho_r\theta}\sigma_r\right)^2}^{C_{\alpha_s}}. \tag{76}$$

Equilibrium existence and multiplicity Equation (76) has a real solution if and only if the discriminant $(1-4A_{\alpha_s}C_{\alpha_s}>0)$ is positive which is equivalent to

$$\frac{\tau}{2}>\frac{\theta}{1-\rho_s\theta}\sigma_s\times\frac{\theta}{1-\rho_r\theta}\sigma_r. \tag{77}$$

This equation shows that when $\sigma_s>0$, a linear equilibrium only exists if arbitrageur tolerance is large enough—i.e. for sufficiently large values of τ. If arbitrageurs' risk-bearing capacity is low (i.e., if τ is low), the existence of supply shocks makes it extremely risky for risk-averse arbitrageurs with short horizons to hold long-term bonds, so it becomes impossible to clear the market. This effect obtains in overlapping generations models in which agents with finite lives trade a long-lived asset that is subject to both fundamental risk ($\sigma_r>0$) and supply shocks ($\sigma_s>0$)—see, for instance, Spiegel (1998), Watanabe (2008), Vayanos and Vila (2009), Greenwood and Vayanos (2014), and Albagli (2015).

If τ is large enough, there are two possible solutions: one in which yields are highly sensitive to supply shocks and one in which yields are less sensitive. Specifically, the smaller root gives rise to a stable equilibrium:

$$\alpha_s^{*(-)}=\frac{1-\sqrt{1-4A_{\alpha_s}C_{\alpha_s}}}{2A_{\alpha_s}}=\frac{1}{2}\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}-\sqrt{\frac{1}{4}\left(\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}\right)^2-\frac{(1-\theta)^2}{\theta^2}\frac{V_F}{\sigma_s^2}} \tag{78}$$

where $V_F=\left(\theta/\left(1-\rho_r\theta\right)\right)^2\sigma_r^2$ is the "fundamental" variance of bond returns in the absence of supply shocks. The larger root gives rise to a unstable equilibrium:

$$\alpha_s^{*(+)}=\frac{1+\sqrt{1-4A_{\alpha_s}C_{\alpha_s}}}{2A_{\alpha_s}}=\frac{1}{2}\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}+\sqrt{\frac{1}{4}\left(\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}\right)^2-\frac{(1-\theta)^2}{\theta^2}\frac{V_F}{\sigma_s^2}}. \tag{79}$$

Equilibrium stability A rational expectations equilibrium is a fixed point of a specific operator involving the price-impact coefficient, α_s. Specifically, consider the price operator $f(\alpha_s) = C_{\alpha_s} + A_{\alpha_s}\alpha_s^2$. In words, $f(\alpha_s)$ is the price-impact coefficient that will clear the market for long-term bonds today if agents conjecture that the price-impact coefficient will be α_s in all future periods. Thus, a rational expectations equilibrium α_s^* satisfies $\alpha_s^* = f(\alpha_s^*)$.

An initial equilibrium α_s^* is stable (unstable) if, following a small initial change from α_s^* to $\alpha_s^* + \xi$ for ξ small, iterating on the price operator drives the system back towards (away from) the initial equilibrium. Formally, letting $\alpha_s^{(1)} = \alpha_s^* + \xi$ and defining $\alpha_s^{(n)} = f(\alpha_s^{(n-1)})$ an equilibrium is stable if $\lim_{n\to\infty} \alpha_s^{(n)} = \alpha_s^*$ and is unstable if $\lim_{n\to\infty} \alpha_s^{(n)} \neq \alpha_s^*$. In this simple case with a single endogenous parameter, an equilibrium α_s^* is stable if $|f'(\alpha_s^*)| < 1$ and is unstable if $|f'(\alpha_s^*)| > 1$.

Thus, since $f'(\alpha_s^{*(-)}) = 2A_{\alpha_s}\alpha_s^{*(-)} = 1 - \sqrt{1 - 4A_{\alpha_s}C_{\alpha_s}} < 1$ and $f'(\alpha_s^{*(+)}) = 2A_{\alpha_s}\alpha_s^{*(+)} = 1 + \sqrt{1 - 4A_{\alpha_s}C_{\alpha_s}} > 1$, we see that the low-volatility equilibrium is stable and the high-volatility equilibrium is unstable. See Albagli (2015) for a related analysis of equilibrium stability in a similar context.

Price-impact coefficients: Comparative statics and limiting behavior We can compute comparative statics and the limiting behavior of the price-impact coefficient α_s^*. Specifically, comparative statics for a change in any exogenous parameter γ follow from

$$\frac{\partial \alpha_s^*}{\partial \gamma} = \frac{\frac{\partial A_{\alpha_s}}{\partial \gamma}(\alpha_s^*)^2 + \frac{\partial C_{\alpha_s}}{\partial \gamma}}{1 - 2A_{\alpha_s}\alpha_s^*}. \tag{80}$$

Note that $2A_{\alpha_s}\alpha_s^{*(-)} < 1$ at the smaller root with lower price-impact and $2A_{\alpha_s}\alpha_s^{*(-)} > 1$ at the larger root with higher price-impact. As noted in Spiegel (1998), Watanabe (2008), and Albagli (2015), all of the relevant comparative statics for the price impact parameter α_s^* have the intuitive signs at $\alpha_s^{*(-)}$ and take the opposite signs at $\alpha_s^{*(+)}$. Specifically, we have $\partial\alpha_s^{*(-)}/\partial\tau < 0$; $\partial\alpha_s^{*(-)}/\partial\sigma_r^2 > 0$; $\partial\alpha_s^{*(-)}/\partial\sigma_s^2 > 0$; $\partial\alpha_s^{*(-)}/\partial\rho_r > 0$; and $\partial\alpha_s^{*(-)}/\partial\rho_s > 0$.[9] By contrast, we have $\partial\alpha_s^{*(+)}/\partial\tau > 0$; $\partial\alpha_s^{*(+)}/\partial\sigma_r^2 < 0$; $\partial\alpha_s^{*(+)}/\partial\sigma_s^2 < 0$; $\partial\alpha_s^{*(+)}/\partial\rho_r < 0$; and $\partial\alpha_s^{*(+)}/\partial\rho_s < 0$. Thus, as in Samuelson's (1947) correspondence principle, the stability of the equilibrium has qualitative implications for the signs of various comparative statics. Specifically, comparative statics take on expected signs at the stable equilibrium.

In terms of limiting behavior, it is easy to see that

$$\lim_{\sigma_s^2\to 0} \alpha_s^{*(-)} = \tau^{-1}\frac{1-\theta}{1-\rho_s\theta}\left(\frac{\theta}{1-\rho_r\theta}\sigma_r\right)^2 \text{ and } \lim_{\sigma_s^2\to 0} \alpha_s^{*(+)} = \infty.$$

Thus, $\alpha_s^{*(-)}$ convergences to the level we would expect with a deterministic asset supply. By contrast, $\alpha_s^{*(+)}$ diverges to infinity as supply shocks become smaller and smaller; intuitively, agents need to fear that these tiny shocks have a massive impact on price in order for the high price impact equilibrium to be sustained. Relatedly, we have $\lim_{\tau\to\infty} \alpha_s^{*(-)} = 0$ and $\lim_{\tau\to\infty} \alpha_s^{*(+)} = \infty$.

Return volatility: Comparative statics and limiting behavior The stable solution for α_s^* in equation (78) corresponds to an low-volatility equilibrium where the volatility of excess returns is $V_L^{*(-)} = \tau\left[(1-\rho_s\theta)/(1-\theta)\right]\alpha_s^{*(-)}$; and the unstable solution for α_s^* in equation (79) corresponds to the high-volatility equilibrium where the volatility of excess returns is $V_L^{*(+)} = \tau\left[(1-\rho_s\theta)/(1-\theta)\right]\alpha_s^{*(+)}$.

[9] The sign of $\partial\alpha_s^{*(-)}/\partial\theta = \tau^{-1}\frac{1-\theta}{1-\rho_s\theta}V_L^{*(-)}$ is ambiguous since $\partial V_L^{*(-)}/\partial\theta > 0$, but $\partial\left[(1-\theta)/(1-\rho_s\theta)\right]/\partial\theta < 0$. This corresponds to the finding in Vayanos and Greenwood (2014) that, depending on the persistence of supply shocks, a current increase in bond supply can have a greater impact on the yields of intermediate or long-dated bonds. Specifically, highly persistent supply shocks have the greatest impact on long-dated yields, while transitory supply shocks have the greatest impact on intermediate-dated yields.

Turning to comparative statics, we can show that $\partial V_L^{*(-)}/\partial \sigma_r^2 > 0$; $\partial V_L^{*(-)}/\partial \sigma_s^2 > 0$; $\partial V_L^{*(-)}/\partial \rho_r > 0$; $\partial V_L^{*(-)}/\partial \rho_s > 0$; $\partial V_L^{*(-)}/\partial \tau < 0$ when $\sigma_s^2 > 0$; and $\partial V_L^{*(-)}/\partial \theta > 0$, implying that risk premia are larger when the perpetuity has a longer duration. By contrast, we have $\partial V_L^{*(+)}/\partial \sigma_r^2 < 0$; $\partial V_L^{*(+)}/\partial \sigma_s^2 < 0$; $\partial V_L^{*(+)}/\partial \rho_r < 0$; $\partial V_L^{*(+)}/\partial \rho_s < 0$; $\partial V_L^{*(+)}/\partial \tau > 0$ when $\sigma_s^2 > 0$; and $\partial V_L^{*(+)}/\partial \theta < 0$. Since bond risk premia are given by $E_t\left[rx_{L,t+1}\right] = \tau^{-1} V_L^* s_t$ this means that the comparative statics for risk premia have the intuitive signs that we expect to hold in practice in the low-volatility equilibrium and have the opposite signs in the high-volatility equilibrium.

In terms of limiting behavior, we have $\lim_{\sigma_s^2 \to 0} V_L^{*(-)} = V_F$ where $V_F \equiv \left(\theta/\left(1-\rho_r\theta\right)\right)^2 \sigma_r^2$ is the "fundamental" variance of bond returns in the absence of supply shocks. By contrast, we have $\lim_{\sigma_s^2 \to 0} V_L^{*(+)} = \infty$. Relatedly, we have $\lim_{\tau \to \infty} V_L^{*(-)} = V_F$. By contrast, we have $\lim_{\tau \to \infty} V_L^{*(+)} = \infty$.

C.3 General model solution when $k = 1$

When there is no slow-moving capital in our general model—i.e., when $k = 1$, we only need to determine four unknowns, namely, $\alpha_{A_{s_A}}$, $\alpha_{A_{s_B}}$, $\alpha_{B_{s_A}}$, and $\alpha_{B_{s_B}}$. As shown above in equation (65), these four unknowns must satisfy

$$\begin{bmatrix} \alpha_{A_{s_A}} & \alpha_{A_{s_B}} \\ \alpha_{B_{s_A}} & \alpha_{B_{s_B}} \end{bmatrix} = \tau^{-1} \begin{bmatrix} \frac{1-\theta_A}{1-\theta_A \rho_{s_A}} & \frac{1-\theta_A}{1-\theta_A \rho_{s_B}} \\ \frac{1-\theta_B}{1-\theta_B \rho_{s_A}} & \frac{1-\theta_B}{1-\theta_B \rho_{s_B}} \end{bmatrix} \tag{81}$$

$$\cdot\times \frac{\begin{bmatrix} \left(1-q_A\right) V_A^{(1)} V_B^{(1)} - q_B (C_{AB}^{(1)})^2 & \left(1-q_A-q_B\right) V_A^{(1)} V_B^{(1)} \\ \left(1-q_A-q_B\right) V_A^{(1)} V_B^{(1)} & \left(1-q_B\right) V_A^{(1)} V_B^{(1)} - q_A (C_{AB}^{(1)})^2 \end{bmatrix}}{\left(1-q_A\right)\left(1-q_B\right) V_A^{(1)} V_B^{(1)} - q_A q_B (C_{AB}^{(1)})^2}$$

$$\cdot\times \begin{bmatrix} V_A^{(1)} & C_{AB}^{(1)} \\ C_{AB}^{(1)} & V_B^{(1)} \end{bmatrix}$$

where we use $\cdot\times$ to denote element-wise array multiplication (as opposed to matrix multiplication) and where

$$\begin{aligned} V_A^{(1)} &= \left(\frac{\theta_A}{1-\rho_r\theta_A}\right)^2 \sigma_r^2 + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_B}\right)^2 \sigma_{s_B}^2 \\ V_B^{(1)} &= \left(\frac{\theta_B}{1-\rho_r\theta_B}\right)^2 \sigma_r^2 + \left(\frac{1}{1-\rho_z\theta_B}\right)^2 \sigma_z^2 + \left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_A}\right)^2 \sigma_{s_A}^2 + \left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_B}\right)^2 \sigma_{s_B}^2 \\ C_{AB}^{(1)} &= \left(\frac{\theta_A}{1-\rho_r\theta_A}\right)\left(\frac{\theta_B}{1-\rho_r\theta_B}\right)\sigma_r^2 + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_A}\right)\left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_A}\right)\sigma_{s_A}^2 \\ &\quad + \left(\frac{\theta_A}{1-\theta_A}\alpha_{As_B}\right)\left(\frac{\theta_B}{1-\theta_B}\alpha_{Bs_B}\right)\sigma_{s_B}^2. \end{aligned} \tag{82}$$

This means that, when there is no slow-moving capital, an equilibrium is a solution to four non-linear equations in four unknowns. And note that in the integrated markets case where $q_A = q_B = 1$, this middle term in equation (81) is simply a matrix of 1s.

Closed form in special case: Symmetric assets and no partial segmentation In addition to assuming $k = 1$, suppose that markets are fully integrated $q_B = q_B = 0$ and that the two assets are symmetric in the sense that $\theta_A = \theta_B = \theta$, and $\rho_{s_A} = \rho_{s_B} = \rho_s$. In this case, we can obtain a closed form expression for the price-impact coefficients by solving a quadratic matrix equation as in Spiegel (1998).

Specifically, let

$$\mathbf{A}=\begin{bmatrix}\alpha_{A_{s_A}} & \alpha_{A_{s_B}}\\ \alpha_{B_{s_A}} & \alpha_{B_{s_B}}\end{bmatrix},\ \mathbf{\Sigma}_s=\begin{bmatrix}\sigma_{s_A}^2 & 0\\ 0 & \sigma_{s_B}^2\end{bmatrix},\text{ and }\mathbf{V}^{(f)}=\begin{bmatrix}\left(\frac{\theta}{1-\rho_r\theta}\right)^2\sigma_r^2 & \left(\frac{\theta}{1-\rho_r\theta}\right)^2\sigma_r^2\\ \left(\frac{\theta}{1-\rho_r\theta}\right)^2\sigma_r^2 & \left(\frac{\theta}{1-\rho_r\theta}\right)^2\sigma_r^2+\left(\frac{1}{1-\rho_z\theta}\right)^2\sigma_z^2\end{bmatrix},$$

where $\mathbf{V}^{(f)}$ is the covariance matrix of fundamental news. In this case, equations **(81)** and (82) reduce to

$$\mathbf{A}=\frac{1-\theta}{1-\theta\rho_s}\tau^{-1}\mathbf{V}^{(f)}+\frac{1-\theta}{1-\theta\rho_s}\left(\frac{\theta}{1-\theta}\right)^2\tau^{-1}\mathbf{A}\mathbf{\Sigma}_s\mathbf{A}'.$$

Since the right-hand side of this equation is symmetric, $\mathbf{A}$ must be symmetric (i.e., $\mathbf{A}=\mathbf{A}'$). Pre- and post-multiplying by the unique positive matrix square root of $\mathbf{\Sigma}_s$ and re-arranging, we obtain

$$\begin{aligned}\frac{(1-\theta)^2}{\theta^2}\mathbf{\Sigma}_s^{1/2}\mathbf{V}^{(f)}\mathbf{\Sigma}_s^{1/2} &= \frac{\tau}{\theta^2}(1-\theta\rho_s)(1-\theta)\,\mathbf{\Sigma}_s^{1/2}\mathbf{A}\mathbf{\Sigma}_s^{1/2}-\mathbf{\Sigma}_s^{1/2}\mathbf{A}\mathbf{\Sigma}_s\mathbf{A}\mathbf{\Sigma}_s^{1/2}\\ &= \frac{\tau}{\theta^2}(1-\theta\rho_s)(1-\theta)\,\mathbf{Y}-\mathbf{Y}^2\\ &= -\left(\mathbf{Y}-\frac{1}{2}\frac{\tau}{\theta^2}(1-\theta\rho_s)(1-\theta)\,\mathbf{I}\right)^2+\frac{1}{4}\frac{\tau^2}{\theta^4}(1-\theta\rho_s)^2(1-\theta)^2\,\mathbf{I}\end{aligned}$$

where the first line follows from substituting $\mathbf{Y}=\mathbf{\Sigma_s^{1/2}}\mathbf{A}\mathbf{\Sigma}_s^{1/2}$ and the second follows from completing the square. Thus, we have

$$\begin{aligned}\mathbf{A}^* &= \frac{1}{2}\frac{\tau}{\theta^2}(1-\theta\rho_s)(1-\theta)\,\mathbf{\Sigma}_s^{-1}+\left[\frac{1}{4}\frac{\tau^2}{\theta^4}(1-\theta\rho_s)^2(1-\theta)^2\,\mathbf{\Sigma}_s^{-2}-\frac{(1-\theta)^2}{\theta^2}\mathbf{\Sigma}_s^{-1/2}\mathbf{V}^{(f)}\mathbf{\Sigma}_s^{-1/2}\right]^{1/2} \\ &= \frac{1}{2}\frac{\tau}{\theta^2}(1-\theta\rho_s)(1-\theta)\,\mathbf{\Sigma}_s^{-1}+\mathbf{Q}\mathbf{\Lambda}^{1/2}\mathbf{Q}'\end{aligned}\quad (83)$$

where $\mathbf{Q\Lambda Q'}$ is the unique spectral decomposition of the symmetric matrix in square brackets:

$$\mathbf{Q\Lambda Q'}=\left[\frac{1}{4}\frac{\tau^2}{\theta^4}(1-\theta\rho_s)^2(1-\theta)^2\,\mathbf{\Sigma}_s^{-2}-\frac{(1-\theta)^2}{\theta^2}\mathbf{\Sigma}_s^{-1/2}\mathbf{V}^{(f)}\mathbf{\Sigma}_s^{-1/2}\right].$$

Thus, $\mathbf{Q}$ contains an orthonormal basis of eigenvectors as its columns ($\mathbf{QQ'}=\mathbf{I}$) and $\mathbf{\Lambda}$ is a diagonal matrix that contains the associated eigenvalues (all of which are real). This expression is the precise matrix analog to the solution in the single asset case:

$$\alpha_s^*=\frac{1}{2}\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}\pm\sqrt{\frac{1}{4}\left(\frac{\tau}{\sigma_s^2}\frac{(1-\rho_s\theta)(1-\theta)}{\theta^2}\right)^2-\frac{(1-\theta)^2}{\theta^2}\frac{V_{(f)}}{\sigma_s^2}}.$$

There are no real solutions to this quadratic matrix equation unless the matrix in square brackets in equation (83) is positive definite—i.e., all of its eigenvalues are positive. The relevant matrix will only be positive definite when $\left[\frac{1}{4}\tau^2\frac{(1-\theta\rho_s)^2}{\theta^2}\mathbf{I}-\mathbf{\Sigma}_s^{1/2}\mathbf{V}^{(f)}\mathbf{\Sigma}_s^{1/2}\right]$ is positive definite and the latter is clearly negative definite for sufficiently small τ. Thus, an equilibrium will only exist for sufficiently high τ. And one needs a larger τ when there is greater cash flow or discount rate risk—formally when the eigenvalues of $\mathbf{\Sigma}_s^{1/2}\mathbf{V}^{(f)}\mathbf{\Sigma}_s^{1/2}$ are greater. If the matrix the matrix is positive definite, then there are four possible equilibria corresponding to the four possible matrix square roots, which are obtained by taking the positive or negative square roots of each eigenvalue.

As explained in Spiegel (1998) and Wantanabe (1998), the different equilibria correspond to different self-fulfilling beliefs that investors hold about the equilibrium volatility of a set of 2-asset portfolios. Specifically, the first portfolio is long both assets and the second is a long-short portfolio; the returns on the two portfolios are orthogonal. The properties of the four equilibria as are follows:

1. The *unique stable equilibrium* obtains when investors believe both of these portfolios will have low volatility. This is the solution with two negative eigenvalues. This equilibrium features modest own-market price-impact and modest, positive cross-market spillovers. This stable equilibrium has well-defined limiting behavior and generates local comparative statics that accord with standard economic intuition.

2. A *first unstable equilibrium* obtains when investors believe the long-only portfolio will have high volatility, but the long-short portfolio will have low volatility. This is the solution where the larger eigenvalue is negative and the smaller eigenvalue is positive. This equilibrium features large own-market price-impact and large, positive cross-market price-impact. Like the other unstable equilibria discussed below, this equilibrium has divergent limiting behavior as $\tau \to \infty$ or as $\sigma^2_{s_A}, \sigma^2_{s_A} \to 0$ and features local comparative statics that run contrary to standard economic intuition.

3. A *second unstable equilibrium* obtains when investors believe the long-only portfolio will have low volatility, but the long-short portfolio with have high volatility. This is the solution where the larger eigenvalue is positive and the smaller eigenvalue is negative. This equilibrium features large own-market price-impact and large, *negative* cross-market spillovers. Because these negative cross-market spillovers have roughly the same magnitude as the own-market price-impact, the long-short strategy has high volatility but the long-only strategy has low volatility.

4. A *third unstable equilibrium* obtains when investors believe both of these portfolios will have high volatility. This is the solution with two positive eigenvalues. This equilibrium features large own-market price-impact and large, *negative* cross-market spillovers. However, because the negative cross-market spillovers are smaller in magnitude than the own-market price-impact, both the long-only and the long-short portfolio have high volatility.

While the stable equilibrium and the first unstable equilibrium capture natural intuitions that one might have about real-world markets, we have had a harder time finding a real-world analogs to the second and third unstable equilibria—i.e., we can't think of examples where supply shocks lead to excess, *negative* comovement between assets.

D Model variations and extensions

D.1 Two-stock version of our model

The three key conceptual ingredients of our model are that:

1. The returns for risky asset class A and asset class B are exposed to a common risk factor. It is easiest to think of this risk factor as arising from a common source of fundamental risk—i.e., correlated "cash flow news".

2. The markets for the two asset classes are partially segmented as in Gromb and Vayanos (2002). Formally, one subset of specialists can only trade risky asset A and the riskless asset, another subset of specialists can only trade asset B and the riskless asset, and only the final subset of generalist investors can trade both asset A and asset B.

3. The generalist investors only rebalance their portfolio gradually over time as in Duffie (2010). Thus, capital moves slowly across asset classes.

In the main text, we focus on an example where asset A represents long-term, default-free bonds (e.g., Treasuries) and asset B represents long-term, defaultable bonds (e.g., corporate bonds). However, the exact same insights would emerge if we recast our model to study any pair of asset classes. For instance, asset A could be Treasuries and asset B could be stocks, or asset A could be U.S. stocks and asset B could be European stocks.

While the basic insights of our model can be expressed in terms of equilibrium expected returns, in order to compute the model's equilibrium we need to move from returns to prices (or equivalently yields). Specifically, we need to link returns back to an equilibrium pricing function that solves a particular fixed point problem. And, moving from returns to prices, means that we must make *some* set of stylized assumptions about the assets we are considering.[10] To illustrate this, below we work out a very similar version of the model in which both assets are risky, perpetual stocks.

D.1.1 Securities

There is a riskless asset, available in perfectly elastic supply, that pays a gross interest rate of $R = 1 + r$ per period. Suppose there are two perpetual risky stocks, A and B, with prices $p_{A,t}$ and $p_{B,t}$. Stock A pays a dividend $c_{A,t}$ at time t and the law of motion for these dividends is:

$$c_{A,t+1} = \bar{c}_A + \rho_{c_A}(c_{A,t} - \bar{c}_A) + \varepsilon_{c_A,t+1}.$$

Similarly, stock B pays a dividend $c_{B,t}$ at time t with law of motion:

$$c_{B,t+1} = \bar{c}_B + \rho_{c_B}(c_{B,t} - \bar{c}_B) + \varepsilon_{c_B,t+1}.$$

The net share supply that investors must hold of stock A evolves according to

$$s_{A,t+1} = \bar{s}_A + \rho_{s_A}(s_{A,t} - \bar{s}_A) + \varepsilon_{s_A,t+1},$$

and the net supply that investors must hold of stock B evolves according to

$$s_{B,t+1} = \bar{s}_B + \rho_{s_B}(s_{B,t} - \bar{s}_B) + \varepsilon_{s_B,t+1}.$$

Although the model can be solved for an arbitrary correlation structure between $\varepsilon_{c_A,t+1}$, $\varepsilon_{c_B,t+1}$, $\varepsilon_{s_A,t+1}$, and $\varepsilon_{s_B,t+1}$, it might be natural to assume that A and B are both exposed to a common source of cash flow risk but that the supply shocks to A and B are independent. Thus, one might assume that

$$Var_t \begin{bmatrix} \varepsilon_{c_A,t+1} \\ \varepsilon_{c_B,t+1} \\ \varepsilon_{s_A,t+1} \\ \varepsilon_{s_B,t+1} \end{bmatrix} = \begin{bmatrix} \sigma_{c_A}^2 & \rho_{c_A c_B}\sigma_{c_A}\sigma_{c_B} & 0 & 0 \\ \rho_{c_A c_B}\sigma_{c_A}\sigma_{c_B} & \sigma_{c_B}^2 & 0 & 0 \\ 0 & 0 & \sigma_{s_A}^2 & 0 \\ 0 & 0 & 0 & \sigma_{s_B}^2 \end{bmatrix},$$

where $\rho_{c_A c_B} > 0$.

D.1.2 Market participants

There are three types of investors, all with identical risk tolerance τ. Fast-moving A-specialists are free to adjust their holdings of stock A asset and the riskless asset each period; however, A-specialists cannot hold stock B. A-specialists are present in mass q_A and we denote their demand for A by $b_{A,t}$. Analogously, fast-moving B-specialists can freely adjust their holdings of stock B and the riskless asset each period, but cannot hold stock A asset. B-specialists are present in mass q_B and their demand for stock B is $b_{B,t}$. The

[10] Technically, we also need to assume that both prices (or yields) and expected returns are affine functions of some state vector. Although also stylized, this linearity assumption is commonplace in rational expectations models with noisy asset supply and is made solely to increase the analytical tractability of these models.

third group of investors is a set of slow-moving generalists who can adjust their holdings of stocks A and B, as well as the riskless asset, but can do so only every k periods. Generalists are present in mass $1 - q_A - q_B$. Fraction $1/k$ of these generalists investors are active each period and can reallocate their portfolios between stocks A and B. However, they must then maintain this same portfolio allocation for the next k periods.

The excess return from holding stock A for 1-period is

$$rx_{A,t+1} = (p_{A,t+1} + c_{A,t+1}) - (1+r)\, p_{A,t},$$

and that for stock B is

$$rx_{B,t+1} = (p_{B,t+1} + c_{B,t+1}) - (1+r)\, p_{B,t}.$$

Fast-moving A-specialists and B-specialists have mean-variance preferences over 1-period ahead wealth. Thus, their demands are given by

$$b_{A,t} = \tau \frac{E_t\left[rx_{A,t+1}\right]}{Var_t\left[rx_{A,t+1}\right]} \text{ and } b_{B,t} = \tau \frac{E_t\left[rx_{B,t+1}\right]}{Var_t\left[rx_{B,t+1}\right]}.$$

The k-period excess return from holding stock A is

$$rx_{A,t\to t+k} = p_{A,t+k} + \textstyle\sum_{j=1}^{k} (1+r)^{k-j}\, c_{A,t+j} - (1+r)^k\, p_{A,t} = \textstyle\sum_{j=1}^{k} (1+r)^{k-j}\, rx_{A,t+j}.$$

and that for stock B

$$rx_{B,t\to t+k} = p_{B,t+k} + \textstyle\sum_{j=1}^{k} (1+r)^{k-j}\, c_{B,t+j} - (1+r)^k\, p_{B,t} = \textstyle\sum_{j=1}^{k} (1+r)^{k-j}\, rx_{B,t+j}.$$

Thus, as in Duffie (2010), we assume that dividends are reinvested at the riskfree rate. This modelling device ensures that prices and expected returns are linear functions of the state vector, keeping the model tractable. Since they only rebalance their portfolios every k periods, slow-moving generalist investors have mean-variance preferences over their k-period ahead wealth. Thus, the demands of generalist investors who are active at time t are

$$\begin{bmatrix} d_{A,t} \\ d_{B,t} \end{bmatrix} = \tau \begin{bmatrix} V_A^{(k)} & C_{AB}^{(k)} \\ C_{AB}^{(k)} & V_B^{(k)} \end{bmatrix}^{-1} \begin{bmatrix} E_t\left[rx_{A,t\to t+k}\right] \\ E_t\left[rx_{B,t\to t+k}\right] \end{bmatrix}.$$

D.1.3 Equilibrium

We conjecture that equilibrium yields and generalist demands are linear functions of a state vector, $\mathbf{x}_t$. Formally, we conjecture that prices of stock A and B are

$$p_{A,t} = \alpha_{A0} + \boldsymbol{\alpha}'_{A1}\mathbf{x}_t \text{ and } p_{B,t} = \alpha_{B0} + \boldsymbol{\alpha}'_{B1}\mathbf{x}_t,$$

and that the demands of slow-moving generalists are

$$d_{A,t} = \delta_{A0} + \boldsymbol{\delta}'_{A1}\mathbf{x}_t \text{ and } d_{B,t} = \delta_{B0} + \boldsymbol{\delta}'_{B1}\mathbf{x}_t,$$

where the $2(1+k) \times 1$ dimensional state vector, $\mathbf{x}_t$, is given by

$$\mathbf{x}_t = [c_{A,t} - \bar{c}_A, c_{B,t} - \bar{c}_B, s_{B,t} - \bar{s}_B, s_{A,t} - \bar{s}_A, d_{A,t-1} - \delta_{A0}, \cdots, d_{A,t-(k-1)} - \delta_{A0}, d_{B,t-1} - \delta_{B0}, \cdots, d_{B,t-(k-1)} - \delta_{B0}]'.$$

These assumptions imply that the state vector follows an AR(1) process

$$\mathbf{x}_{t+1} = \mathbf{\Gamma}\mathbf{x}_t + \boldsymbol{\epsilon}_{t+1},$$

where the transition matrix $\mathbf{\Gamma}$ depends on generalist demands.

Assuming that (i) specialists and generalist optimally choose their stock holdings, (ii) the markets for both stock A and B clear each period, and (iii) agents correctly perceive the pricing functions and laws of motion for the state variables, then a rational expectation equilibrium is a fixed point of the form:

$$\begin{aligned}
\boldsymbol{\alpha}_{A1} &= \mathbf{f}_{\boldsymbol{\alpha}_{A1}}\left(\boldsymbol{\alpha}_{A1}, \boldsymbol{\alpha}_{B1}, \boldsymbol{\delta}_{A1}, \boldsymbol{\delta}_{B1}\right) \\
\boldsymbol{\alpha}_{B1} &= \mathbf{f}_{\boldsymbol{\alpha}_{B1}}\left(\boldsymbol{\alpha}_{A1}, \boldsymbol{\alpha}_{B1}, \boldsymbol{\delta}_{A1}, \boldsymbol{\delta}_{B1}\right) \\
\boldsymbol{\delta}_{A1} &= \mathbf{f}_{\boldsymbol{\delta}_{A1}}\left(\boldsymbol{\alpha}_{A1}, \boldsymbol{\alpha}_{B1}, \boldsymbol{\delta}_{A1}, \boldsymbol{\delta}_{B1}\right) \\
\boldsymbol{\delta}_{A2} &= \mathbf{f}_{\boldsymbol{\delta}_{A2}}\left(\boldsymbol{\alpha}_{A1}, \boldsymbol{\alpha}_{B1}, \boldsymbol{\delta}_{A1}, \boldsymbol{\delta}_{B1}\right).
\end{aligned}$$

Each step in the argument perfectly mirrors that in our baseline model.

Proceeding as above, realized 1-period excess returns for stock A are

$$rx_{A,t+1} = (\bar{c}_A - r\alpha_{A0}) + (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{x}_{t+1} - (1+r)\boldsymbol{\alpha}'_{A1}\mathbf{x}_t.$$

This implies that

$$E_t\left[rx_{A,t+1}\right] = (\bar{c}_A - r\alpha_{A0}) + [(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Gamma - (1+r)\, \boldsymbol{\alpha}'_{A1}]\mathbf{x}_t$$

and

$$Var_t\left[rx_{A,t+1}\right] = (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Sigma \left(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A}\right).$$

Thus, A-specialist demand for stock A is

$$b_{A,t} = \tau \frac{E_t\left[rx_{A,t+1}\right]}{Var_t\left[rx_{A,t+1}\right]} = \left[\tau \frac{\bar{c}_A - r\alpha_{A0}}{(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Sigma \left(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A}\right)}\right] + \left[\tau \frac{(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Gamma - (1+r)\, \boldsymbol{\alpha}'_{A1}}{(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Sigma \left(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A}\right)}\right] \mathbf{x}_t.$$

The relevant expressions for stock B are analogous.

k-period excess return for stock A are given by

$$\begin{aligned}
rx_{A,t\to t+k} &= p_{A,t+k} + \sum_{j=1}^{k}(1+r)^{k-j} c_{A,t+j} - (1+r)^k p_{A,t} \\
&= \left[\left(1-(1+r)^k\right)\left(\alpha_{A0} - \frac{\bar{c}_A}{r}\right)\right] + (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{x}_{t+k} + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_A}\mathbf{x}_{t+j} - (1+r)^k\boldsymbol{\alpha}'_{A1}\mathbf{x}_t
\end{aligned}$$

Thus, we have[11]

$$E\left[rx_{A,t\to t+k}\right] = \left[\left(1-(1+r)^k\right)\left(\alpha_{A0} - \frac{\bar{c}_A}{r}\right)\right] + \left[(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_A}\boldsymbol{\Gamma}^j - (1+r)^k\boldsymbol{\alpha}'_{A1}\right]\mathbf{x}_t$$

We have

$$\begin{aligned}
V_A^{(k)} &= Var_t\left[(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{x}_{t+k} + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_A}\mathbf{x}_{t+j}\right] \\
&= (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{C}_{[t+k,t+k]}\left(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A}\right) \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(1+r)^{2k-i-j}\mathbf{e}'_{c_A}\mathbf{C}_{[t+i,t+j]}\mathbf{e}_{c_A} \\
&\quad + 2\sum_{j=1}^{k}(1+r)^{k-j}\mathbf{e}'_{c_A}\mathbf{C}_{[t+j,t+k]}\left(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A}\right)
\end{aligned}$$

where $Cov\left[\mathbf{x}_{t+i}, \mathbf{x}_{t+j}|\mathbf{x}_t\right] = \mathbf{C}_{[t+i,t+j]} = \sum_{s=1}^{min\{i,j\}}\left[\boldsymbol{\Gamma}^{i-s}\right]\boldsymbol{\Sigma}\left[\boldsymbol{\Gamma}^{j-s}\right]'$. The relevant expressions for asset B are analogous.

[11] As in our baseline model, it is straightforward to confirm that the equilibrium expected returns and investor holdings of stocks A and B do not depend on fundamentals $c_{A,t}$ and $c_{B,t}$.

Finally, we have

$$\begin{aligned} C_{AB}^{(k)} &= Cov_t \begin{bmatrix} (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{x}_{t+k} + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_A}' \mathbf{x}_{t+j}, \\ (\boldsymbol{\alpha}_{B1} + \mathbf{e}_{c_B})' \mathbf{x}_{t+k} + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_B}' \mathbf{x}_{t+j} \end{bmatrix} \\ &= \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (1+r)^{2k-i-j} \mathbf{e}_{c_A}' \mathbf{C}_{[t+i,t+j]} \mathbf{e}_{c_B} \\ &\quad + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_A}' \mathbf{C}_{[t+j,t+k]} (\boldsymbol{\alpha}_{B1} + \mathbf{e}_{c_B}) \\ &\quad + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_B}' \mathbf{C}_{[t+j,t+k]} (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_B}) \\ &\quad + (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \mathbf{C}_{[t+k,t+k]} (\boldsymbol{\alpha}_{B1} + \mathbf{e}_{c_B}) \end{aligned}$$

Slow-moving generalist demand is

$$\begin{bmatrix} d_{A,t} \\ d_{B,t} \end{bmatrix} = \frac{\tau}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2} \begin{bmatrix} V_B^{(k)} E[rx_{A,t \to t+k}] - C_{AB}^{(k)} E[rx_{B,t \to t+k}] \\ V_A^{(k)} E[rx_{B,t \to t+k}] - C_{AB}^{(k)} E[rx_{A,t \to t+k}] \end{bmatrix}.$$

Thus, given our conjectures, slow-moving generalists demands will indeed take a linear form. Specifically, we have

$$\delta_{A0} = \tau \left(1 - (1+r)^k\right) \frac{V_B^{(k)} (\alpha_{A0} - \bar{c}_A/r) - C_{AB}^{(k)} (\alpha_{B0} - \bar{c}_B/r)}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2},$$

$$\delta_{B0} = \tau \left(1 - (1+r)^k\right) \frac{V_A^{(k)} (\alpha_{B0} - \bar{c}_B/r) - C_{AB}^{(k)} (\alpha_{A0} - \bar{c}_A/r)}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2},$$

$$\boldsymbol{\delta}_{A1}' = \tau \frac{\begin{pmatrix} V_B^{(k)} \left[(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_A}' \boldsymbol{\Gamma}^j - (1+r)^k \boldsymbol{\alpha}_{A1}' \right] \\ -C_{AB}^{(k)} \left[(\boldsymbol{\alpha}_{B1} + \mathbf{e}_{c_B})' \boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_B}' \boldsymbol{\Gamma}^j - (1+r)^k \boldsymbol{\alpha}_{B1}' \right] \end{pmatrix}}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2},$$

and

$$\boldsymbol{\delta}_{B1}' = \tau \frac{\begin{pmatrix} V_A^{(k)} \left[(\boldsymbol{\alpha}_{B1} + \mathbf{e}_{c_B})' \boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_B}' \boldsymbol{\Gamma}^j - (1+r)^k \boldsymbol{\alpha}_{B1}' \right] \\ -C_{AB}^{(k)} \left[(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1} (1+r)^{k-j} \mathbf{e}_{c_A}' \boldsymbol{\Gamma}^j - (1+r)^k \boldsymbol{\alpha}_{A1}' \right] \end{pmatrix}}{V_A^{(k)} V_B^{(k)} - (C_{AB}^{(k)})^2}.$$

The market-clearing condition for market A is

$$\overbrace{(1 - q_A - q_B) k^{-1} d_{A,t} + q_A b_{A,t}}^{\text{Active demand}} = \overbrace{s_{A,t} - (1 - q_A - q_B) \left(k^{-1} \sum_{i=1}^{k-1} d_{A,t-i} \right)}^{\text{Active supply}}.$$

Letting $V_A^{(1)} \equiv Var_t [rx_{A,t+1}] = (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Sigma (\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})$ denote the equilibrium variance of 1-period excess returns on asset A, active demand is

$$\begin{aligned} &(1 - q_A - q_B) k^{-1} d_{A,t} + q_A b_{A,t} \\ &= \left[(1 - q_A - q_B) k^{-1} \delta_{A0} + q_A \tau \frac{(\bar{c}_A - r\alpha_{A0})}{V_A^{(1)}} \right] \\ &+ \left[(1 - q_A - q_B) k^{-1} \boldsymbol{\delta}_{A1}' + q_A \tau \frac{(\boldsymbol{\alpha}_{A1} + \mathbf{e}_{c_A})' \Gamma - (1+r) \boldsymbol{\alpha}_{A1}'}{V_A^{(1)}} \right] \mathbf{x}_t \end{aligned}$$

Active supply is

$$
\begin{aligned}
& s_{A,t} - (1-q_A-q_B)k^{-1}\sum\nolimits_{i=1}^{k-1} d_{A,t-i} \\
&= s_{A,t} - (1-q_A-q_B)\frac{(k-1)}{k}\delta_{A0} - (1-q_A-q_B)k^{-1}\sum\nolimits_{i=1}^{k-1}\left(d_{A,t-i}-\delta_{A0}\right) \\
&= \left[\bar{s}_A - (1-q_A-q_B)\frac{(k-1)}{k}\delta_{A0}\right] + \left[\left(\mathbf{e}_{s_A} - (1-q_A-q_B)k^{-1}\mathbf{1}_{(A)}\right)'\right]\mathbf{x}_t
\end{aligned}
$$

Matching constants terms, we obtain

$$
\alpha_{A0} = \frac{\bar{c}_A}{r} - \frac{1}{r}\frac{V_A^{(1)}}{q_A\tau}\left(\bar{s}_A - (1-q_A-q_B)\delta_{A0}\right)
$$

Matching slope coefficients, we have

$$
\boldsymbol{\alpha}_{A1} = \left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\Gamma'\mathbf{e}_{c_A} - \frac{V_A^{(1)}}{q_A\tau}\left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\left[\mathbf{e}_{s_A} - (1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)\right].
$$

Thus, equilibrium prices in market A are given by

$$
\begin{aligned}
p_{A,t} = & \overbrace{\left\{\frac{\bar{c}_A}{r} + \frac{\rho_{c_A}}{1+r-\rho_{c_A}}\left(c_{A,t}-\bar{c}_A\right)\right\}}^{\text{Risk-neutral discounted future cash flows}} \\
& - \overbrace{\left[\left(q_A\tau\right)^{-1}V_A^{(1)}\left(\frac{\bar{s}_A - (1-q_A-q_B)\,\delta_{A0}}{r}\right)\right]}^{\text{Unconditional risk premia}} \\
& - \overbrace{\left[\left(q_A\tau\right)^{-1}V_A^{(1)}\left(\begin{array}{c}\frac{1}{1+r-\rho_{s_A}}\left(s_{A,t}-\bar{s}_A\right) \\ -\left(1-q_A-q_B\right)k^{-1}\sum_{i=0}^{\infty}\left(1+r\right)^{-(i+1)}E_t[\sum_{j=0}^{k-1}(d_{A,t+i-j}-\delta_{A0})]\end{array}\right)\right]}^{\text{Conditional risk premia}}
\end{aligned}
$$

The prices for market B takes a similar form.

Thus, an equilibrium is a solution to the following system of equations.

$$
\boldsymbol{\alpha}_{A1} = \left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\Gamma'\mathbf{e}_{c_A} - \frac{V_A^{(1)}}{q_A\tau}\left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\left[\mathbf{e}_{s_A} - (1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(A)}+\boldsymbol{\delta}_{A1}\right)\right].
$$

$$
\boldsymbol{\alpha}_{B1} = \left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\Gamma'\mathbf{e}_{c_B} - \frac{V_B^{(1)}}{q_B\tau}\left[(1+r)\,\mathbf{I}-\boldsymbol{\Gamma}'\right]^{-1}\left[\mathbf{e}_{s_B} - (1-q_A-q_B)k^{-1}\left(\mathbf{1}_{(B)}+\boldsymbol{\delta}_{B1}\right)\right]
$$

$$
\boldsymbol{\delta}'_{A1} = \tau\frac{\left(\begin{array}{c} V_B^{(k)}\left[\left(\boldsymbol{\alpha}_{A1}+\mathbf{e}_{c_A}\right)'\boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_A}\boldsymbol{\Gamma}^j - (1+r)^k\boldsymbol{\alpha}'_{A1}\right] \\ -C_{AB}^{(k)}\left[\left(\boldsymbol{\alpha}_{B1}+\mathbf{e}_{c_B}\right)'\boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_B}\boldsymbol{\Gamma}^j - (1+r)^k\boldsymbol{\alpha}'_{B1}\right]\end{array}\right)}{V_A^{(k)}V_B^{(k)} - (C_{AB}^{(k)})^2},
$$

and

$$
\boldsymbol{\delta}'_{B1} = \tau\frac{\left(\begin{array}{c} V_A^{(k)}\left[\left(\boldsymbol{\alpha}_{B1}+\mathbf{e}_{c_B}\right)'\boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_B}\boldsymbol{\Gamma}^j - (1+r)^k\boldsymbol{\alpha}'_{B1}\right] \\ -C_{AB}^{(k)}\left[\left(\boldsymbol{\alpha}_{A1}+\mathbf{e}_{c_A}\right)'\boldsymbol{\Gamma}^k + \sum_{j=1}^{k-1}(1+r)^{k-j}\mathbf{e}'_{c_A}\boldsymbol{\Gamma}^j - (1+r)^k\boldsymbol{\alpha}'_{A1}\right]\end{array}\right)}{V_A^{(k)}V_B^{(k)} - (C_{AB}^{(k)})^2}.
$$

In summary, each step in the set up and solution for this two-stock model corresponds exactly to a step in our baseline model of two fixed income assets. Therefore, we conclude that our model's key insights are not in any way specific to a particular set of (fixed income) asset classes.

D.2 Two-asset version of the Duffie (2010) model

In this section, we consider a multi-asset version of Duffie's (2010) model of slow-moving capital. This model is similar to our baseline model except that there is now just one class of fast-moving investors with mass q who can buy *both* assets A and B and a mass $(1-q)$ of slow-moving investors who can gradually adjust their holdings of both assets A and B. In other words, unlike our model, the multi-asset Duffie (2010) does not feature partial market segmentation.

Specifically, the demand of fast-moving investors is

$$\mathbf{b}_t = \tau[\mathbf{V}^{(1)}]^{-1} E_t\left[\mathbf{rx}_{t+1}\right]$$

where

$$\mathbf{V}^{(1)} = \begin{bmatrix} \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \boldsymbol{\Sigma} \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right) & \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \boldsymbol{\Sigma} \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right) \\ \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \boldsymbol{\Sigma} \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right) & \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \boldsymbol{\Sigma} \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right) \end{bmatrix}$$

and

$$E_t\left[\mathbf{rx}_{t+1}\right] = \begin{bmatrix} (\alpha_{A0} - \overline{r}) + \left(\frac{1}{1-\theta_A}\boldsymbol{\alpha}_{A1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_A}{1-\theta_A}\boldsymbol{\alpha}_{A1}\right)' \boldsymbol{\Gamma}\mathbf{x}_t \\ (\alpha_{B0} - \overline{r} - \overline{z}) + \left(\frac{1}{1-\theta_B}\boldsymbol{\alpha}_{B1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta_B}{1-\theta_B}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \boldsymbol{\Gamma}\mathbf{x}_t \end{bmatrix}.$$

Slow-moving investors are exactly as in our baseline model. Specifically, their demands take the form

$$\mathbf{d}_t = \tau\left[\mathbf{V}^{(k)}\right]^{-1} E_t\left[\mathbf{rx}_{t\to t+k}\right] = \mathbf{d}_0 + \mathbf{D}\mathbf{x}_t$$

where $\mathbf{d}_0 = [\delta_{A0}, \delta_{B0}]'$ and $\mathbf{D} = [\boldsymbol{\delta}_{A0}, \boldsymbol{\delta}_{B0}]'$.

The vector of market clearing conditions is

$$q\mathbf{b}_t + (1-q)k^{-1}\mathbf{d}_t = \mathbf{s}_t - (1-q)(k^{-1}\sum_{i=1}^{k-1}\mathbf{d}_{t-i}).$$

Letting $\mathbf{s}_t = \overline{\mathbf{s}} + \mathbf{S_s}\mathbf{x}_t$ and $\sum_{i=1}^{k-1}\mathbf{d}_{t-i} = (k-1)\,\mathbf{d}_0 + \mathbf{S_d}\mathbf{x}_t$ where $\mathbf{S_s}$ and $\mathbf{S_d}$ are appropriate selection matrices, we can rewrite these market-clearing conditions as

$$q\mathbf{b}_t = [\overline{\mathbf{s}} - (1-q)\mathbf{d}_0] + \left[\mathbf{S_s} - (1-q)k^{-1}\left(\mathbf{S_d} + \mathbf{D}\right)\right]\mathbf{x}_t.$$

Thus, making used of the fact that $\mathbf{b}_t = \tau\left[\mathbf{V}^{(1)}\right]^{-1} E_t\left[\mathbf{rx}_{t+1}\right]$, we have

$$E_t\left[\mathbf{rx}_{t+1}\right] = q^{-1}\tau^{-1}\mathbf{V}^{(1)}\left[\overline{\mathbf{s}} - (1-q)\mathbf{d}_0\right] + q^{-1}\tau\mathbf{V}^{(1)}\left[\mathbf{S_s} - (1-q)k^{-1}\left(\mathbf{S_d} + \mathbf{D}\right)\right]\mathbf{x}_t.$$

We can write the equilibrium conditions defining $\boldsymbol{\alpha}'_{A1}$ and $\boldsymbol{\alpha}'_{B1}$ compactly if we assume that $\theta_A = \theta_B = \theta$. (It is easy to relax this restriction at the loss of some notational convenience.) Specifically, write

$$\mathbf{y} = \mathbf{a}_0 + \mathbf{A}\mathbf{x}_t$$

where

$$\mathbf{a}_0 = \begin{bmatrix}\alpha_{A0} \\ \alpha_{B0}\end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix}\boldsymbol{\alpha}'_{A1} \\ \boldsymbol{\alpha}'_{B1}\end{bmatrix}.$$

Since

$$\begin{bmatrix}E_t\left[rx_{A,t+1}\right] \\ E_t\left[rx_{B,t+1}\right]\end{bmatrix} = \begin{bmatrix} (\alpha_{A0} - \overline{r}) + \left(\frac{1}{1-\theta}\boldsymbol{\alpha}_{A1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta}{1-\theta}\boldsymbol{\alpha}_{A1}\right)' \boldsymbol{\Gamma}\mathbf{x}_t \\ (\alpha_{B0} - \overline{r} - \overline{z}) + \left(\frac{1}{1-\theta}\boldsymbol{\alpha}_{B1} - \mathbf{e}_r\right)' \mathbf{x}_t - \left(\frac{\theta}{1-\theta}\boldsymbol{\alpha}_{B1} + \mathbf{e}_z\right)' \boldsymbol{\Gamma}\mathbf{x}_t \end{bmatrix},$$

we can write

$$E_t\left[\mathbf{rx}_{t+1}\right] = \mathbf{a}_0 - \mathbf{1}\overline{r} - \mathbf{1}_2\overline{z} - \mathbf{E}_r\mathbf{x}_t - \mathbf{E}_z\Gamma\mathbf{x}_t + \frac{1}{1-\theta}\mathbf{A}\left(\mathbf{I}-\theta\boldsymbol{\Gamma}\right)\mathbf{x}_t$$

where

$$\mathbf{1} = \begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{1}_2 = \begin{bmatrix}0\\1\end{bmatrix},\ \mathbf{E}_r = \begin{bmatrix}\mathbf{e}_r'\\ \mathbf{e}_r'\end{bmatrix},\ \text{and } \mathbf{E}_z = \begin{bmatrix}\mathbf{0}'\\ \mathbf{e}_z'\end{bmatrix}$$

Thus, we can write the market clearing conditions as

$$\begin{aligned}
& \left[\mathbf{a}_0 - \mathbf{1}\overline{r} - \mathbf{1}_2\overline{z}\right] + \left[(1-\theta)^{-1}\mathbf{A}\left(\mathbf{I}-\theta\boldsymbol{\Gamma}\right) - \mathbf{E}_r - \mathbf{E}_z\Gamma\right]\mathbf{x}_t && (84)\\
= \;& q^{-1}\tau^{-1}\mathbf{V}^{(1)}\left[\overline{\mathbf{s}} - (1-q)\mathbf{d}_0\right] + q^{-1}\tau\mathbf{V}^{(1)}\left[\mathbf{S}_{\mathbf{s}} - (1-q)k^{-1}\left(\mathbf{S}_{\mathbf{d}} + \mathbf{D}\right)\right]\mathbf{x}_t. && (85)
\end{aligned}$$

Matching the constant terms, we obtain

$$\mathbf{a}_0 - \mathbf{1}\overline{r} - \mathbf{1}_2\overline{z} = q^{-1}\tau^{-1}\mathbf{V}^{(1)}\left[\overline{\mathbf{s}} - (1-q)\mathbf{d}_0\right]. \tag{86}$$

Matching the terms on $\mathbf{x}_t$, we obtain

$$\begin{aligned}
\mathbf{A} \;=\;& (1-\theta)\,\mathbf{E}_r\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1} + (1-\theta)\,\mathbf{E}_z\Gamma\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1} && (87)\\
& + (1-\theta)\,q^{-1}\tau^{-1}\mathbf{V}^{(1)}\left[\mathbf{S}_{\mathbf{s}} - (1-q)k^{-1}\left(\mathbf{S}_{\mathbf{d}} + \mathbf{D}\right)\right]\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1}. && (88)
\end{aligned}$$

Thus, an equilibrium of the multi-asset Duffie (2010) model solves (87), (49), and (50).

We can compare these equilibrium conditions for a multi-asset Duffie (2010) model featuring only slow-moving capital to our model which features both slow-moving capital and partial segmentation—i.e., a solution to (47), (48), (49), and (50). Specifically, when $\theta_A = \theta_B = \theta$ we can write (47) and (48) in the same compact notation used above. Specifically, the price coefficients in our model satisfy

$$\begin{aligned}
\mathbf{A} \;=\;& (1-\theta)\,\mathbf{E}_r\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1} + (1-\theta)\,\mathbf{E}_z\Gamma\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1}\\
& + (1-\theta)\,\tau^{-1}\left[diag(\mathbf{q})\right]^{-1}\left[diag(diag\left(\mathbf{V}^{(1)}\right))\right]\left[\mathbf{S}_{\mathbf{s}} - (1-q_A-q_B)\,k^{-1}\left(\mathbf{S}_{\mathbf{d}} + \mathbf{D}\right)\right]\left[\mathbf{I}-\theta\boldsymbol{\Gamma}\right]^{-1}.
\end{aligned}$$

and the price constants satisfy

$$\mathbf{a}_0 - \mathbf{1}\overline{r} - \mathbf{1}_2\overline{z} = \tau^{-1}\left[diag(\mathbf{q})\right]^{-1}\left[diag(diag\left(\mathbf{V}^{(1)}\right))\right]\left[\overline{\mathbf{s}} - (1-q_A-q_B)\,\mathbf{d}_0\right].$$

where

$$diag\left(\mathbf{q}\right) = \begin{bmatrix}q_A & 0\\ 0 & q_B\end{bmatrix} \text{ and } diag(diag\left(\mathbf{V}^{(1)}\right)) = \begin{bmatrix}V_A^{(1)} & 0\\ 0 & V_B^{(1)}\end{bmatrix}.$$

The equilibrium conditions take a similar mathematical form, but there is a key difference. By replacing the non-diagonal matrix $q^{-1}\mathbf{V}^{(1)}$ with the diagonal matrix $\left[diag(\mathbf{q})\right]^{-1}\left[diag(diag\left(\mathbf{V}^{(1)}\right))\right]$, our model adds a second critical asset pricing friction, namely partial segmentation.

D.3 Version of our model with multiple risky assets in each market

We now explore how the main results of our model carry over to a more complex setting in which there are multiple risky assets trading in each market. Subject to some mild conditions which guarantee that cross-market arbitrage remains risky, we show that the intuitions from the two risky asset model carry over to this richer setting. Specifically, we show that a A-market-specific pricing model prices all assets in the A market and that a different B-market-specific pricing model prices all assets in the B market. These two market-specific pricing models are linked over time by the cross-market arbitrage activities of slow-moving asset allocators, who take steps to equalize the way that risk is priced by specialists in the two markets. Much like before, the degree of market integration depends on the risks faced by cross-market arbitrageurs.

D.3.1 Markets and assets

Suppose there are N bonds in market A, denoted A_1, A_2, ..., A_N. As above, we assume that the A-market bonds are default–free and are only exposed to interest rate risk.[12] However, the bonds have different durations. Specifically, asset A_n has duration D_{A_n}. As in equation (67), the log excess return on bond A_n over the short-term interest rate from time t to $t+1$ is

$$rx_{A_n,t+1} \approx \overbrace{\frac{1}{1-\theta_{A_n}}}^{D_{A_n}} y_{A_n,t} - \overbrace{\frac{\theta_{A_n}}{1-\theta_{A_n}}}^{D_{A_n}-1} y_{A_n,t+1} - r_t, \tag{89}$$

where $\theta_{A_n} = 1 - 1/D_{A_n}$.

We also assume that there are N defaultable bonds in market B, denoted B_1, B_2, ..., B_N. The return on bond B_n from time t to $t+1$ takes the form

$$1 + R_{B_n,t+1} = (1 - Z_{t+1})^{\psi_{B_n}} (1 - U_{B_n,t+1}) \frac{(\delta_{B_N} P_{B_n,t+1} + C_{B_n})}{P_{B_n,t}}, \tag{90}$$

where Z_{t+1} is a default process common to all bonds in the B market, ψ_{B_n} is the exposure of perpetuity B_n to this systematic default factor, and $U_{B_n,t+1}$ is an idiosyncratic default process that is specific to bond B_n. Therefore, the log excess return on bond B_n from time t to $t+1$ is

$$rx_{B_n,t+1} \approx \overbrace{\frac{1}{1-\theta_{B_n}}}^{D_{B_n}} y_{B_n,t} - \overbrace{\frac{\theta_{B_n}}{1-\theta_{B_n}}}^{D_{B_n}-1} y_{B_n,t+1} - \psi_{B_n} z_{t+1} - u_{B_n,t+1} - r_t, \tag{91}$$

where $\theta_{B_n} = 1 - 1/D_{B_n}$ and $u_{B_n,t+1} = -\ln(1 - U_{B_n,t+1})$. Given this formulation for default losses it is perhaps most natural to think of bond B_n as corresponding to a *portfolio* of defaultable bonds, albeit one that *imperfectly diversified* and is therefore exposed to *idiosyncratic* default losses. For instance, one could think of B_n as representing a portfolio of all bonds in a certain industry with some specified credit rating and some specified maturity.

We assume that the processes for the short rate r_t and for the common default process z_t are as in the main text. We assume that idiosyncratic default process for bond B_n follows

$$u_{B_n,t+1} = \overline{u}_{B_n} + \rho_{u_{B_n}} (u_{B_n,t} - \overline{u}_{B_n}) + \varepsilon_{u_{B_n},t+1}. \tag{92}$$

We assume the net supplies that investors must hold in the A assets are

$$\mathbf{s}_{A,t} = \mathbf{s}_{A0} + \mathbf{s}_{A1} \times s_{A,t} \tag{93}$$

where $s_{A,t}$ follows

$$s_{A,t+1} = \rho_{s_A} s_{A,t} + \varepsilon_{s_A,t+1}. \tag{94}$$

[12] In order to have perpetuities with different durations, we introduce a set of "geometrically decaying perpetuities." Specifically, consider a perpetuity that promises to pay a decaying stream C, δC, $\delta^2 C$, $\delta^3 C$, ... where $(1-\delta) \in [0,1]$ denotes the geometric decay rate. Thus, $\delta = 0$ corresponds to 1-period debt and $\delta = 1$ is a consol bond. Assuming a yield of $Y_{L,t}$, the price of this security is $P_{L,t} = \sum_{j=1}^{\infty} (1 + Y_{L,t})^{-j} \delta^{j-1} C = C/(1 + Y_{L,t} - \delta)$ which implies a Macaulay duration of $-\partial p_{L,t}/\partial y_{L,t} = (1 + Y_{L,t})/(1 + Y_{L,t} - \delta)$. Suppose the perpetuity's price is $\overline{P}_L = 1$ and its yield is $\overline{Y}_L$ in the steady-state. This implies a coupon of $C = 1 - \delta + \overline{Y}_L$ and a steady-state duration of $-\partial p_{L,t}/\partial y_{L,t} = (C + \delta)/C$ which is increasing in δ.

Using the same steps as above, the log return on the decaying perpetuity from t to $t+1$ is approximately $r_{L,t+1} = \log[(\delta P_{L,t+1} + C)/P_{L,t}] \approx (1-\theta)^{-1} y_{L,t} - \theta (1-\theta)^{-1} y_{L,t+1}$. where $\theta = \delta/(\delta + \exp(c - \overline{p}_L))$. Since the steady-state price is par, we have $\theta = \delta/(\delta + C)$. Thus, bond duration is $-\partial p_{L,t}/\partial y_{L,t} = (1-\theta)^{-1} = (\delta + C)/C$ which corresponds to the Macaulay duration when the perpetuity is trading at par. Thus, we assume that security A_n has a geometric decay rate if δ_{A_n} and a coupon of $C_{A_n} = 1 - \delta_{A_n} + \overline{Y}_{A_n}$, implying a duration of $D_{A_n} = (1 - \theta_{A_n})^{-1} = \left(1 + \overline{Y}_{A_n}\right)/\left(1 + \overline{Y}_{A_n} - \delta_{A_n}\right)$.

Similarly, the net supplies that investors must hold in the B assets are

$$\mathbf{s}_{B,t} = \mathbf{s}_{B0} + \mathbf{s}_{B1} \times s_{B,t},$$

where where $s_{B,t}$ follows

$$s_{B,t+1} = \rho_{s_B} s_{B,t} + \varepsilon_{s_B,t+1}. \tag{95}$$

We assume that $\varepsilon_{r,t+1}$, $\varepsilon_{z,t+1}$, $\varepsilon_{s_A,t+1}$, $\varepsilon_{s_B,t+1}$, $\varepsilon_{u_{B_1},t+1}$, $\varepsilon_{u_{B_2},t+1}$, ..., $\varepsilon_{u_{B_N},t+1}$ are mutually orthogonal.

D.3.2 Market participants

As above, there are three-types of investors, each with risk tolerance τ. A-specialists are present in mass q_A, B-specialists are present in mass q_B, and generalists are present in mass $(1 - q_A - q_B)$.

Fast-moving A-specialists are free to adjust their holdings of all assets in the A market (and the riskless short-term asset) each period, but cannot hold the B assets. Let $b_{A_n,t}$ denote the demand of A specialists for asset A_n and let $\mathbf{b}_{A,t}$ denote the $N \times 1$ vector of their holdings of each of the N assets in market A. Collecting the excess returns on these N assets in a vector, the excess return on A-specialists portfolio is thus $rx_{A_t,t+1} = (\mathbf{b}_{A,t})'\mathbf{rx}_{A,t+1}$.

A-specialists have mean-variance preferences over 1-period portfolio returns and solve

$$\begin{aligned} & \max_{\mathbf{b}_{A,t}} \left\{ E_t \left[rx_{A_t,t+1} \right] - (2\tau)^{-1} Var_t \left[rx_{A_t,t+1} \right] \right\} \\ = & \max_{\mathbf{b}_{A,t}} \left\{ \mathbf{b}'_{A,t} E_t \left[\mathbf{rx}_{A,t+1} \right] - (2\tau)^{-1} \mathbf{b}'_{A,t} Var_t \left[\mathbf{rx}_{A,t+1} \right] \mathbf{b}_{A,t} \right\}. \end{aligned}$$

Thus, the demands of A-specialists are given by

$$\mathbf{b}_{A,t} = \tau \left(Var_t \left[\mathbf{rx}_{A,t+1} \right] \right)^{-1} E_t \left[\mathbf{rx}_{A,t+1} \right].$$

Since this implies $Var_t \left[rx_{A_t,t+1} \right] = \tau E_t \left[rx_{A_t,t+1} \right]$, we have

$$E_t \left[\mathbf{rx}_{A,t+1} \right] = \tau^{-1} Var_t \left[\mathbf{rx}_{A,t+1} \right] \mathbf{b}_{A,t} = \boldsymbol{\beta}_t \left[\mathbf{rx}_{A,t+1}, rx_{A_t,t+1} \right] E_t \left[rx_{A_t,t+1} \right]$$

where $\boldsymbol{\beta}_t \left[\mathbf{rx}_{A,t+1}, rx_{A_t,t+1} \right] = Cov_t \left[\mathbf{rx}_{A,t+1}, rx_{A_t,t+1} \right] / Var_t \left[rx_{A_t,t+1} \right]$. Thus, the 1-period returns on all A-market assets will be priced by a local conditional pricing model that is specific to the A-market—i.e., where the relevant "market portfolio" is the time t portfolio of A-market specialists, $rx_{A_t,t+1} = (\mathbf{b}_{A,t})'\mathbf{rx}_{A,t+1}$.

Since a symmetric analysis hold for the N assets in market B, we will have two conditional pricing models: one that prices all the assets in market A and another that prices all the assets in market B. The key question is how these two conditional pricing models will be linked together in equilibrium by the cross-market arbitrage activities of generalist—i.e., where A- and B-specialists will price exposures to common risk factors in the same way. As above, slow-moving generalists are present in mass $1 - q_A - q_B$. Fraction $1/k$ of generalists investors are active each period and choose the portfolios of assets from the A and B markets that they will hold over the following k periods.

D.3.3 Solution

For all securities $X \in \{A_1, A_2, ...A_N, B_1, B_2..., B_2\}$, we conjecture that equilibrium yields take the form

$$y_{X,t} = \alpha_{X\_0} + \boldsymbol{\alpha}'_{X\_1} \mathbf{x}_t \tag{96}$$

and that generalist demands are of the form

$$d_{X,t} = \delta_{X\_0} + \boldsymbol{\delta}'_{X\_1} \mathbf{x}_t. \tag{97}$$

Now the state vector, $\mathbf{x}_t$, will depend on the granularity of portfolio choices available to slow-moving generalists.

Building on the arguments made above, it is trivial to determine the way that yields respond to changes in the short rate r_t, the common default process z_t, and the security-specific default processes $u_{B_n,t}$. Specifically, we can show that show that

$$\alpha_{r,X} = \frac{1-\theta_X}{1-\rho_r \theta_X}. \tag{98}$$

For securities in the B market we have

$$\alpha_{z,B_n} = \psi_{B_n} \rho_z \frac{1-\theta_{B_n}}{1-\rho_z \theta_{B_n}}. \tag{99}$$

and

$$\alpha_{u,B_n} = \frac{1-\theta_{B_n}}{1-\rho_{u_{B_n}} \theta_{B_n}} \rho_{u_{B_n}}. \tag{100}$$

As above the key question concerns how yields respond to shifts in asset supply—i.e., the values of $\alpha_{s_A,X}$ and $\alpha_{s_B,X}$ as well as whatever inactive generalist holdings that are included in the state vector.

Clearing market A Let $\varphi_{r,A_n} = -\frac{\theta_{A_n}}{1-\theta_{A_n}} \alpha_{r,A_n} = -\frac{\theta_{A_n}}{1-\theta_{A_n}\rho_r}$, $\varphi_{s_A,A_n} = -\frac{\theta_{A_n}}{1-\theta_{A_n}} \alpha_{s_A,A_n}$, $\varphi_{s_B,A_n} = -\frac{\theta_{A_n}}{1-\theta_{A_n}} \alpha_{s_B,A_n}$, so that

$$rx_{A_n,t+1} - E_t\left[rx_{A_n,t++1}\right] = \varphi_{r,A_n}\varepsilon_{r,t+1} + \varphi_{s_A,A_n}\varepsilon_{s_A,t+1} + \varphi_{s_B,A_n}\varepsilon_{s_B,t+1}. \tag{101}$$

Thus, we have

$$Var_t\left[\mathbf{rx}_{A,t+1}\right] = \boldsymbol{\varphi}_{r,A}\boldsymbol{\varphi}'_{r,A}\sigma_r^2 + \boldsymbol{\varphi}_{s_A,A}\boldsymbol{\varphi}'_{s_A,A}\sigma_{s_A}^2 + \boldsymbol{\varphi}_{s_B,A}\boldsymbol{\varphi}'_{s_B,A}\sigma_{s_B}^2. \tag{102}$$

Market clearing requires

$$q_A \mathbf{b}_{A,t} + (1-q_A-q_B)\, k^{-1}\mathbf{d}_{A,t} = \mathbf{s}_{A0} + \mathbf{s}_{A1} s_{A,t} - (1-q_A-q_B)\, k^{-1} \textstyle\sum_{i=1}^{k-1} \mathbf{d}_{A,t-i}. \tag{103}$$

Substituting this into $E_t\left[\mathbf{rx}_{A,t+1}\right] = \tau^{-1} Var_t\left[\mathbf{rx}_{A,t+1}\right]\mathbf{b}_{A,t}$ and grouping terms, then implies

$$\begin{aligned} E_t\left[\mathbf{rx}_{A,t+1}\right] &= \sum_{x\in\{r,s_A,s_B\}} \left[\boldsymbol{\varphi}_{x,A}\sigma_x^2 \left(q_A\tau\right)^{-1} \boldsymbol{\varphi}'_{x,A}\right] \left(\mathbf{s}_{A0} - (1-q_A-q_B)\,\boldsymbol{\delta}_{A0}\right) \\ &+ \sum_{x\in\{r,s_A,s_B\}} \left[\boldsymbol{\varphi}_{x,A}\sigma_x^2 \left(q_A\tau\right)^{-1} \boldsymbol{\varphi}'_{x,A}\right] \left(\mathbf{s}_{A1} s_{A,t} - (1-q_A-q_B)\, k^{-1} \textstyle\sum_{i=0}^{k-1} \left(\mathbf{d}_{A,t-i} - \boldsymbol{\delta}_{A0}\right)\right) \end{aligned} \tag{104}$$

which can be written as

$$E_t\left[\mathbf{rx}_{A,t+1}\right] = \boldsymbol{\Lambda}_{A0} + \boldsymbol{\Lambda}_{A1}\mathbf{x}_t. \tag{105}$$

The yield of security A_n is given by

$$\begin{aligned} y_{A_n,t} &= (1-\theta_{A_n}) \textstyle\sum_{i=0}^{\infty} \theta_{A_n}^i E_t[r_{t+i} + rx_{A,t+1}] \\ &= \overbrace{\left\{\overline{r} + \left(\frac{1-\theta_{A_n}}{1-\rho_r\theta_{A_n}}\right)(r_t - \overline{r})\right\}}^{\text{Expected future short rates}} \\ &\quad + \overbrace{\left[\sum_{x\in\{r,s_A,s_B\}} \left[\varphi_{x,A_n}\sigma_x^2 \left(q_A\tau\right)^{-1} \boldsymbol{\varphi}'_{x,A}\right] \left(\mathbf{s}_{A0} - (1-q_A-q_B)\,\boldsymbol{\delta}_{A0}\right)\right]}^{\text{Unconditional risk premia} = \lambda_{A0_n}} \\ &\quad + \overbrace{\boldsymbol{\lambda}'_{A1_n} (1-\theta_{A_n}) \left[\mathbf{I} - \theta_{A_n}\boldsymbol{\Gamma}\right]^{-1} \mathbf{x}_t]}^{\text{Conditional term premia}} \end{aligned} \tag{106}$$

where λ_{A0_n} is the n^{th} element of $\mathbf{\Lambda}_{A0}$ and $\boldsymbol{\lambda}'_{A1_n}$ is the n^{th} row of $\mathbf{\Lambda}_{A1}$.

Clearing market B Let $\varphi_{r,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{r,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}\rho_r}$, $\varphi_{z,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{z,B_n} - \psi_{B_n} = -\frac{\psi_{B_n}}{1-\rho_z\theta_{B_n}}$, $\varphi_{s_A,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{s_A,B_n}$, $\varphi_{s_B,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{s_B,B_n}$, $\varphi_{u,B_n} = -\frac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{u,B_n} - 1 = \frac{1}{1-\rho_{u_{B_n}}\theta_{B_n}}$ so that

$$rx_{B_n,t+1} - E_t\left[rx_{B_n,t+1}\right] = \varphi_{r,B_n}\varepsilon_{r,t+1} + \varphi_{z,B_n}\varepsilon_{z,t+1} + \varphi_{s_A,B_n}\varepsilon_{s_A,t+1} + \varphi_{s_B,B_n}\varepsilon_{s_B,t+1} + \varphi_{u,B_n}\varepsilon_{u_{B_n},t+1} \quad (107)$$

Thus, we have

$$Var_t\left[\mathbf{rx}_{B,t+1}\right] = \boldsymbol{\varphi}_{r,B}\boldsymbol{\varphi}'_{r,B}\sigma_r^2 + \boldsymbol{\varphi}_{z,B}\boldsymbol{\varphi}'_{z,B}\sigma_z^2 + \boldsymbol{\varphi}_{s_A,B}\boldsymbol{\varphi}'_{s_A,B}\sigma_{s_A}^2 + \boldsymbol{\varphi}_{s_B,B}\boldsymbol{\varphi}'_{s_B,B}\sigma_{s_B}^2 + diag\left[\varphi_{u,B_n}^2\sigma_{u_{B_n}}^2\right] \quad (108)$$

Market clearing requires

$$q_B\mathbf{b}_{B,t} + (1-q_A-q_B)\,k^{-1}\mathbf{d}_{B,t} = \mathbf{s}_{B0} + \mathbf{s}_{B1}s_{B,t} - (1-q_A-q_B)\,k^{-1}\textstyle\sum_{i=1}^{k-1}\mathbf{d}_{B,t-i} \quad (109)$$

Substituting this into $E_t\left[\mathbf{rx}_{B,t+1}\right] = \tau^{-1}Var_t\left[\mathbf{rx}_{B,t+1}\right]\mathbf{b}^*_{B,t}$ and grouping terms, then implies

$$\begin{aligned} E_t\left[\mathbf{rx}_{B,t+1}\right] &= \sum_{x\in\{r,z,s_A,s_B\}} \boldsymbol{\varphi}_{x,B}\sigma_x^2\left(q_B\tau\right)^{-1}\boldsymbol{\varphi}'_{x,B}\left(\mathbf{s}_{B0} - (1-q_A-q_B)\,\boldsymbol{\delta}_{B0}\right) \quad (110)\\ &\quad + \sum_{x\in\{r,z,s_A,s_B\}} \boldsymbol{\varphi}_{x,B}\sigma_x^2\left(q_B\tau\right)^{-1}\boldsymbol{\varphi}'_{x,B}\left(\mathbf{s}_{B1}s_{B,t} - (1-q_A-q_B)\,k^{-1}\textstyle\sum_{i=0}^{k-1}\left(\mathbf{d}_{B,t-i} - \boldsymbol{\delta}_{B0}\right)\right)\\ &\quad + \left(q_B\tau\right)^{-1} diag\left[\varphi_{u,B_n}^2\sigma_{u_{B_n}}^2\right]\left(\begin{array}{c} \mathbf{s}_{B0} - (1-q_A-q_B)\,\boldsymbol{\delta}_{B0} + \mathbf{s}_{B1}s_{B,t} \\ -(1-q_A-q_B)\,k^{-1}\sum_{i=0}^{k-1}\left(\mathbf{d}_{B,t-i} - \boldsymbol{\delta}_{B0}\right)\end{array}\right) \end{aligned}$$

which can be written as $E_t\left[\mathbf{rx}_{B,t+1}\right] = \mathbf{\Lambda}_{B0} + \mathbf{\Lambda}_{B1}\mathbf{x}_t$. Assume that $\lim_{N\to\infty}\sum_{n=1}^N q_{B_n,t} \to Q_{B,t}$, then as $N\to\infty$ the last term will vanish because idiosyncratic risks will not be priced in equilibrium. To see this, note that the n^{th} element of this vector is $(q_B\tau)^{-1}\left(\varphi_{u,B_n}\right)^2\sigma_{u_{B_n}}^2 b^*_{B_n,t}$. If we assume that $\lim_{N\to\infty}\sum_{n=1}^N q_{B_n,t} \to Q_{B,t}$, then under regularity conditions we will have $-c/N \le b^*_{B_n,t} \le c/N$ for some c, implying that $\lim_{N\to\infty}\left(\varphi_{u,B_n}\right)^2\sigma_{u_{B_n}}^2 b^*_{B_n,t} = 0$ for all n.

The yield of security B_n is given by

$$\begin{aligned} y_{B_n,t} &= (1-\theta_{A_n})\textstyle\sum_{i=0}^{\infty}\theta_{A_n}^i E_t[r_{t+i} + rx_{B_n,t+1} + \psi_{B_n}z_{t+i+1} + u_{n,t+i+1}] \quad (111)\\ &= \overbrace{\left\{\overline{r} + \left(\frac{1-\theta_{B_n}}{1-\rho_r\theta_{B_n}}\right)(r_t - \overline{r})\right\}}^{\text{Expected future short rates}}\\ &\quad \overbrace{+\psi_{B_n}\left\{\overline{z} + \frac{1-\theta_{B_n}}{1-\rho_z\theta_{B_n}}\rho_z\left(z_t - \overline{z}\right)\right\} + \left\{\overline{u}_{B_n} + \frac{1-\theta_{B_n}}{1-\rho_{u_{B_n}}\theta_{B_n}}\rho_{u_{B_n}}\left(u_{B_n,t} - \overline{u}_{B_n}\right)\right\}}^{\text{Expected future default losses}}\\ &\quad \overbrace{+\left[\sum_{x\in\{r,z,s_A,s_B\}}\varphi_{x,B_n}\sigma_x^2\left(q_B\tau\right)^{-1}\boldsymbol{\varphi}'_{x,B}\left(\mathbf{s}_{B0} - (1-q_A-q_B)\,\boldsymbol{\delta}_{B0}\right)\right]}^{\text{Unconditional risk premia} = \lambda_{B0_n}}\\ &\quad \overbrace{+\boldsymbol{\lambda}'_{B1_n}\left(1-\theta_{B_n}\right)\left[\mathbf{I} - \theta_{B_n}\mathbf{\Gamma}\right]^{-1}\mathbf{x}_t}^{\text{Conditional term premia}}], \end{aligned}$$

where λ_{B0_n} is the n^{th} element of $\mathbf{\Lambda}_{B0}$ and $\boldsymbol{\lambda}'_{B1_n}$ is the n^{th} row of $\mathbf{\Lambda}_{B1}$.

D.3.4 Factor model representation of returns

In this subsection, we explore the factor model representation of returns in this multi-asset setting. Specifically, we need to know the dimensionality of the underlying factor model representation of returns.

In the absence of idiosyncratic risk, it is clear that 1-period returns follow a 4-factor linear model of the form

$$rx_{A_n,t+1} - E_t\left[rx_{A_n,t++1}\right] = \varphi_{r,A_n}\varepsilon_{r,t+1} + \varphi_{s_A,A_n}\varepsilon_{s_A,t+1} + \varphi_{s_B,A_n}\varepsilon_{s_B,t+1} \tag{112}$$

and

$$rx_{B_n,t+1} - E_t\left[rx_{B_n,t+1}\right] = \varphi_{r,B_n}\varepsilon_{r,t+1} + \varphi_{z,B_n}\varepsilon_{z,t+1} + \varphi_{s_A,B_n}\varepsilon_{s_A,t+1} + \varphi_{s_B,B_n}\varepsilon_{s_B,t+1}. \tag{113}$$

However, even in the absence of idiosyncratic risk, k-period returns will not follow a 4-factor model. Instead, k-period returns satisfy a $2(k+1)$-factor model. To see this, note that the unanticipated return on security A_n is

$$\begin{aligned}
&\textstyle\sum_{i=1}^{k} rx_{A_n,t+i} - E_t[\sum_{i=1}^{k} rx_{A_n,t+i}] && (114)\\
=\;& (\boldsymbol{\alpha}_{A_n1} - \mathbf{e}_r)' \textstyle\sum_{j=1}^{k-1}\left(\mathbf{x}_{t+j} - E_t\left[\mathbf{x}_{t+j}\right]\right) - \frac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}'_{A_n1}\left(\mathbf{x}_{t+k} - E_t\left[\mathbf{x}_{t+k}\right]\right)\\
=\;& (\boldsymbol{\alpha}_{A_n1} - \mathbf{e}_r)' \textstyle\sum_{j=1}^{k-1}\sum_{i=1}^{j}\boldsymbol{\Gamma}^{j-i}\boldsymbol{\epsilon}_{t+i} - \frac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}'_{A_n1}\sum_{i=1}^{k}\boldsymbol{\Gamma}^{k-i}\boldsymbol{\epsilon}_{t+i}\\
=\;& \textstyle\sum_{i=1}^{k-1}\left[(\boldsymbol{\alpha}_{A_n1} - \mathbf{e}_r)'\left[\mathbf{I}-\boldsymbol{\Gamma}\right]^{-1}\left[\mathbf{I}-\boldsymbol{\Gamma}^{k-i}\right] - \frac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}'_{A_n1}\boldsymbol{\Gamma}^{k-i}\right]\boldsymbol{\epsilon}_{t+i} - \frac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}'_{A_n1}\boldsymbol{\epsilon}_{t+k}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^{k} rx_{B_n,t+i} - E_t\left[\sum_{i=1}^{k} rx_{B_n,t+i}\right] && (115)\\
&= \left(\boldsymbol{\alpha}_{B_n1} - \mathbf{e}_r - \psi_{B_n}\mathbf{e}_z\right)' \textstyle\sum_{j=1}^{k-1}\left(\mathbf{x}_{t+j} - E_t\left[\mathbf{x}_{t+j}\right]\right) - \left(\frac{\theta_{B_n}}{1-\theta_{B_n}}\boldsymbol{\alpha}_{B_n1} + \psi_{B_n}\mathbf{e}_z\right)'\left(\mathbf{x}_{t+k} - E_t\left[\mathbf{x}_{t+k}\right]\right)\\
&= \textstyle\sum_{i=1}^{k-1}\left[\left(\boldsymbol{\alpha}_{B_n1} - \mathbf{e}_r - \psi_{B_n}\mathbf{e}_z\right)'\left[\mathbf{I}-\boldsymbol{\Gamma}\right]^{-1}\left[\mathbf{I}-\boldsymbol{\Gamma}^{k-i}\right] - \left(\frac{\theta_{B_n}}{1-\theta_{B_n}}\boldsymbol{\alpha}_{B_n1} + \psi_{B_n}\mathbf{e}_z\right)'\boldsymbol{\Gamma}^{k-i}\right]\boldsymbol{\epsilon}_{t+i}\\
&- \left(\frac{\theta_{B_n}}{1-\theta_{B_n}}\boldsymbol{\alpha}_{B_n1} + \psi_{B_n}\mathbf{e}_z\right)'\boldsymbol{\epsilon}_{t+k}
\end{aligned}$$

Since generalists demand does not depend on shocks to r_t, the part of unexpected returns due to interest rate shocks for any security X is

$$\begin{aligned}
&\textstyle\sum_{i=1}^{k-1}\left[\left(\alpha_{r,X} - 1\right)\frac{1-\rho_r^{k-i}}{1-\rho_r} - \frac{\theta_X}{1-\theta_X}\alpha_{r,X}\rho_r^{k-i}\right]\varepsilon_{r,t+i} && (116)\\
&-\frac{\theta_X}{1-\theta_X}\alpha_{r,X}\varepsilon_{r,t+k}\\
=\;& \textstyle\sum_{i=1}^{k-1}\left[\left(\frac{1-\theta_X}{1-\rho_r\theta_X} - 1\right)\frac{1-\rho_r^{k-i}}{1-\rho_r} - \frac{\theta_X}{1-\theta_X}\frac{1-\theta_X}{1-\rho_r\theta_X}\rho_r^{k-i}\right]\varepsilon_{r,t+i} - \frac{\theta_X}{1-\theta_X}\frac{1-\theta_X}{1-\rho_r\theta_X}\varepsilon_{r,t+k}\\
=\;& -\frac{\theta_X}{1-\rho_r\theta_X}\textstyle\sum_{i=1}^{k}\varepsilon_{r,t+i}\\
=\;& \varphi_{r,X}\textstyle\sum_{i=1}^{k}\varepsilon_{r,t+i},
\end{aligned}$$

where the second line follows by setting $\alpha_{r,X} = (1-\theta_X)/(1-\rho_r\theta_X)$ and the third line follows from simplifying. Similarly, the part of unexpected returns due to shocks to the common default factor z_t for any security n in market B is

$$\begin{aligned}
& \textstyle\sum_{i=1}^{k-1}\left[\left(\alpha_{z,B_n}-\psi_{B_n}\right)\dfrac{1-\rho_z^{k-i}}{1-\rho_z}-\left(\dfrac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{z,B_n}+\psi_{B_n}\right)\rho_z^{k-i}\right]\varepsilon_{z,t+i} \\
& -\left(\dfrac{\theta_{B_n}}{1-\theta_{B_n}}\alpha_{z,B_n}+\psi_{B_n}\right)\varepsilon_{z,t+k} \\
= & \textstyle\sum_{i=1}^{k-1}\left[\left(\psi_{B_n}\rho_z\dfrac{1-\theta_{B_n}}{1-\rho_z\theta_{B_n}}-\psi_{B_n}\right)\dfrac{1-\rho_z^{k-i}}{1-\rho_z}-\left(\dfrac{\theta_{B_n}}{1-\theta_{B_n}}\psi_{B_n}\rho_z\dfrac{1-\theta_{B_n}}{1-\rho_z\theta_{B_n}}+\psi_{B_n}\right)\rho_z^{k-i}\right]\varepsilon_{z,t+i} \\
& -\left(\dfrac{\theta_{B_n}}{1-\theta_{B_n}}\psi_{B_n}\rho_z\dfrac{1-\theta_{B_n}}{1-\rho_z\theta_{B_n}}+\psi_{B_n}\right)\varepsilon \\
= & -\dfrac{\psi_{B_n}}{1-\rho_z\theta_{B_n}}\textstyle\sum_{i=1}^{k}\varepsilon_{z,t+i} \\
= & \varphi_{z,B_n}\textstyle\sum_{i=1}^{k}\varepsilon_{z,t+i},
\end{aligned} \tag{117}$$

where the second line follows by setting $\alpha_{z,B_n}=\psi_{B_n}\rho_z\left(1-\theta_{B_n}\right)/\left(1-\rho_z\theta_{B_n}\right)$ and the third line follows from simplifying. This is a natural result: since shocks to fundamentals are uncorrelated in our model and generate no return predictability, the k-period returns on any asset load linearly on the sum of k one-period shocks to fundamentals.

However, a similar reduction does not hold for shocks to asset supply. Consider the part of unexpected returns on A_n due to shocks to $s_{A,t}$

$$\begin{aligned}
& \textstyle\sum_{i=1}^{k-1}\left(\boldsymbol{\alpha}_{A_n1}'\left[\mathbf{I}-\boldsymbol{\Gamma}\right]^{-1}\left[\mathbf{I}-\boldsymbol{\Gamma}^{k-i}\right]-\dfrac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}_{A_n1}'\boldsymbol{\Gamma}^{k-i}\right)\mathbf{e}_{s_A}\varepsilon_{s_A,t+i} \\
& -\dfrac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\alpha}_{A_n1}'\mathbf{e}_{s_A}\varepsilon_{s_A,t+k} \\
= & \textstyle\sum_{i=1}^{k-1}\left[\boldsymbol{\alpha}_{A_n1}'\left(\left[\mathbf{I}-\boldsymbol{\Gamma}\right]^{-1}\left[\mathbf{I}-\boldsymbol{\Gamma}^{k-i}\right]-\dfrac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\Gamma}^{k-i}\right)\mathbf{e}_{s_A}\right]\varepsilon_{s_A,t+i}+\varphi_{s_A,A_n}\varepsilon_{s_A,t+k}.
\end{aligned} \tag{118}$$

Unlike the case of shocks to fundamentals, $\left[\boldsymbol{\alpha}_{A_n1}'\left(\left[\mathbf{I}-\boldsymbol{\Gamma}\right]^{-1}\left[\mathbf{I}-\boldsymbol{\Gamma}^{k-i}\right]-\frac{\theta_{A_n}}{1-\theta_{A_n}}\boldsymbol{\Gamma}^{k-i}\right)\mathbf{e}_{s_A}\right]\neq\varphi_{s_A,A_n}$, so this does **not** reduce to $\varphi_{s_A,A_n}\sum_{i=1}^{k}\varepsilon_{s_A,t+i}$. Instead, the k-period returns on securities with different durations will have different exposures to the $\{\varepsilon_{s_A,t+i}\}_{i=1}^{k}$.

D.3.5 The risk of cross-market arbitrage

With multiple assets, the key question concerns the risks that generalists face when they undertake cross-market arbitrage. Note that assets in market A are exposed to exogenous shocks to three state variables: r_{t+1}, $s_{A,t+1}$, and $s_{B,t+1}$. Assets in market B are exposed to exogenous shocks to r_{t+1}, $s_{A,t+1}$, and $s_{B,t+1}$ as well as exogenous shocks to z_{t+1}. In addition, each asset B_n is potentially exposed to idiosyncratic shocks to $u_{B_n,t+1}$.

An interesting complication arises if generalists are able to freely choose their holdings of the N assets in market A and the N assets in market B. In this case, it may be possible to use A assets to construct a "factor-mimicking portfolio" that is only exposed to shocks to r_{t+1} and to construct a similar factor-mimicking portfolio using only B assets. If this is possible then, unless the risks associated with shocks to r_{t+1} are being priced the same way by A- and B- specialists at each date, generalists will have a riskless arbitrage opportunity.

In general, it is *possible* to construct (nearly-perfectly) factor-mimicking portfolios if both A and B markets contain many redundant assets. But even if it is *possible* to construct factor-mimicking portfolios in both A and B markets, this may not be feasible for slow-moving generalists. For instance, generalists may lack the expertise to construct these complicated (long-short) mimicking portfolios or may face institutional frictions that make this infeasible. We can distinguish between at least three cases:

1. **Case 1:** It is possible to construct factor-mimicking portfolios in both markets A and B for each of the common risk factors.
2. **Case 2:** It is not possible to construct factor-mimicking portfolios in this way.
3. **Case 3:** It is possible to construct factor-mimicking portfolio in this way, but generalists are not capable of doing so: generalists function as coarse asset-allocators as opposed to granular cross-market arbitrageurs.

We discuss each of these three cases in greater detail.

Case 1: It is possible to construct factor-mimicking portfolios To begin, suppose that the B assets are not exposed to idiosyncratic shocks. As explained above, k-period returns in this case will satisfy a linear factor model with $2(1+k)$ independent factors (recall that the model's state vector, $\mathbf{x}_t$, contains $2(1+k)$ elements). Specifically, there are 2 factors that correspond to innovations to fundamentals, r_t and z_t, and k factors corresponding to innovations to each of the supply factors at different horizons.[13]

Suppose generalists can freely choose positions in all $2N$ assets where $N \geq 2(1+k)$ and all assets are non-redundant (their factor loadings must be linearly independent). In this case, it will be possible to construct factor-mimicking portfolios for shocks to r_t, z_t, $s_{A,t}$, and shocks to $s_{B,t}$ using only A assets and using only B assets.[14] As a result, the active generalists will work to perfectly integrate factor pricing between the A and B markets—i.e., A- and B-specialists will charge the same factor risk prices. Intuitively, because cross asset-class arbitrage is riskless in this case, all agents will charge in the same risk factor prices—i.e., the two markets will perfectly integrated in the short run (conditionally) and the long run (unconditionally). Of course, because generalists are slow-moving, the risk factor prices that prevail in the short-run will be subject to slow-moving capital effects, e.g., risk factor prices will overreact to shocks to the supply of that risk factor.

Even if the B market assets are subject to idiosyncratic default shocks, this outcome will obtain in the limit where we hold constant the total supply of A and B assets but allow $N \to \infty$. In this case, investors' portfolios will become arbitrarily granular, implying that it will be easy for generalist investors to diversify away these idiosyncratic shocks when constructing factor-mimicking portfolios. This is essentially the intuition behind Ross's (1976) Arbitrage Pricing Theory. Thus, if generalists can freely choose positions in all $2N$ assets, Case 1 will be a good approximation in the case when N is large relative to the number of common risk factors in the A and B markets.

Case 2: It is not possible to construct factor-mimicking portfolios Next suppose generalists can freely choose positions in all $2N$ assets. However, suppose that N is not large, so it is not possible to construct accurate factor-mimicking portfolios. If the B assets are not exposed to idiosyncratic shocks, cross-market arbitrage remains risky for generalists so long as $N < 2(1+k)$. And, assuming that the B assets are exposed to idiosyncratic shocks, cross-market arbitrage will remain risky even when $N \geq 2(1+k)$. Specifically, the N assets in market B are exposed to $N + 2(k+1)$ risk factors—$2(k+1)$ that are common and N that are asset-specific. A factor-mimicking portfolio is a set of N unknown positions in the B assets that must satisfy $N + 2(k+1)$ linear equations. In general, there is no such solution. And, if N is small and idiosyncratic volatility is large, then any portfolio will do a poor job of mimicking each common factor. As a result, cross-market arbitrage will remain risky for generalists, so cross-market integration will be imperfect,

[13] Specifically, k-period returns for all securities in markets A and B will load linearly on $\sum_{i=1}^{k} \varepsilon_{r,t+i}$ and k-period returns on all B-market securities will load linearly on $\sum_{i=1}^{k} \varepsilon_{z,t+i}$. However, the k-period returns on securities with different durations will load differently on supply shocks in the near versus distant future. Thus, all k-period returns satisfy a linear factor model in $(\sum_{i=1}^{k} \varepsilon_{r,t+i}, \sum_{i=1}^{k} \varepsilon_{z,t+i}, \{\varepsilon_{s_A,t+i}\}_{i=1}^{k}, \{\varepsilon_{s_B,t+i}\}_{i=1}^{k})$.

[14] If $N = 2(k+1)$, there will be a unique set of factor-mimicking portfolios using B market securities. If $N > 2(k+1)$, there will be multiple possible factor-mimicking portfolios.

both in the short-run and the long-run—i.e., A- and B-specialists will price common risks differently. This case will be empirically relevant, if the "idiosyncratic risks" are not asset specific, but are shared by a large subset of assets in a market (e.g., industry default factors). In this case, cross-market arbitrage will always entail some amount of "basis risk," rendering it risky for generalists.

Case 3: Generalists are unable to construct factor-mimicking portfolios Finally, suppose it is possible to construct accurate factor-mimicking portfolios. However, suppose that generalists cannot freely choose positions in all $2N$ assets. For instance, generalists may lack the expertise to construct the complicated factor-mimicking portfolios or may face institutional frictions that make this infeasible. In this case, generalists function as coarse asset-allocators and not granular cross-market arbitrageurs: their degrees of freedom for within market asset allocation are less than the number of common risk factors $(2(1+k))$. Under these conditions, the markets for A and B will again by imperfectly integrated in both the short- and long-run, because cross-market arbitrage is risky for generalists with granular portfolios of this sort.

To give a concrete example, we might suppose that generalists only have one-degree of freedom within each market: how much to allocate to a baseline portfolio in each market and cannot vary their allocation at all within markets. Specifically, one could assume that $\mathbf{d}_{A,t} = \mathbf{s}_{A0} \times d_{A,t}$ and $\mathbf{d}_{B,t} = \mathbf{s}_{B0} \times d_{B,t}$, so that choosing different values of $d_{A,t}$ and $d_{B,t}$ only moves the baseline portfolios, $\mathbf{s}_{A0}$ and $\mathbf{s}_{B0}$, up and down.

The bottom line is that, under plausible conditions, cross-market arbitrage will expose generalists to risk. As a result, the key insights of our simpler two asset model will carry over to the more general case where multiple assets trade in two partially segmented markets.