Appendix- Omitted Proofs [NOT FOR PUBLICATION]

**Proof of Proposition 1:** The planner’s date-1 problem is given by Eq. (11). Differentiating with respect to $B_{1,2}$ yields the first order condition

$$-\beta(B_{0,1} - \beta B_{1,2}) + \beta(B_{1,2} + B_{0,2}) = 0.$$  \hspace{1cm} (A1)

The second order condition is $\beta(1 + \beta) > 0$. The solution to (A1) is then

$$B_{1,2} = (B_{0,1} - B_{0,2}) / (1 + \beta),$$  \hspace{1cm} (A2)

which implies that

$$\tau_1 = \tau_2 = (B_{0,1} + B_{0,2} \beta) / (1 + \beta).$$  \hspace{1cm} (A3)

Consider the problem at $t = 0$ where $\tau_0 = G - B_{0,1} - B_{0,2}$. Substituting (A3) into Eq. (8), yields

$$\min_{(B_{0,1}, B_{0,2})} \left[ \frac{1}{2} (G - B_{0,1} - B_{0,2})^2 + E \left[ \frac{(B_{0,1} + \beta B_{0,2})^2}{1 + \beta} \right] \right],$$  \hspace{1cm} (A4)

which is equivalent to

$$\min_{(S, D)} \left[ \frac{1}{2} (G - D)^2 + \frac{1}{2} E \left[ \frac{(SD + (1 - S) D \beta)^2}{1 + \beta} \right] \right],$$  \hspace{1cm} (A5)

where we have made the change of variables to $D = B_{0,1} + B_{0,2}$, and $S = B_{0,1} / D$.

We first differentiate (A5) with respect to $S$, yielding

$$D^2 E \left[ \frac{1 - \beta}{1 + \beta} \left( \beta + S(1 - \beta) \right) \right] = 0.$$  \hspace{1cm} (A6)

We note that $S^* = 1/2$ is the solution to this the first order condition, since

$$E \left[ \frac{1 - \beta}{1 + \beta} \left( \beta + \frac{1}{2} (1 - \beta) \right) \right] = E \left[ \frac{1}{2} (1 - \beta) \right] = 0.$$  \hspace{1cm} (A7)

Noting that

$$E \left[ \frac{\beta (1 - \beta)}{1 + \beta} \right] + \frac{1}{2} E \left[ \frac{(1 - \beta)^2}{1 + \beta} \right] = 0,$$  \hspace{1cm} (A8)

and defining $b = E[(1 - \beta)^2 / (1 + \beta)]$, we can rewrite (A6) as

$$D^2 b (S - 1/2) = 0.$$  \hspace{1cm} (A9)
We now solve for $D$, the level of debt. Note that

$$E \left[ \frac{(S' + (1-S') \beta)^2}{1 + \beta} \right] = E \left[ \frac{(\frac{1}{2} + \frac{1}{2} \beta)^2}{1 + \beta} \right] = \frac{1}{4} E[1 + \beta] = \frac{1}{2}. \quad (A10)$$

Optimal $D$ thus satisfies

$$\min_D \left[ \frac{1}{2} (G - D)^2 + \frac{1}{4} D^2 \right], \quad (A11)$$

which has first order condition

$$-(G-D) + D/2 = 0 \Rightarrow D^* = 2G/3. \quad (A12)$$

Using these facts, we can show that

$$\frac{1}{2} E \left[ \frac{(SD + (1-S)D \beta)^2}{1 + \beta} \right] = \frac{D^2}{2} \left( b(S - 1/2)^2 + 1/2 \right). \quad (A13)$$

One can confirm that the second order conditions are satisfied at this solution. Moreover, as demonstrated below, the objective is globally convex in $B_{0,1}$ and $B_{0,2}$, so the solution is unique.\(^1\)

Allowing the government to issue risky securities

We now show that, in the absence of money demand, these results continue to hold if we allow for arbitrary risky securities whose payouts are possibly contingent on the realization of $\beta$. Specifically, we now allow the government to issue face value $B_R$ of risky securities with payoff $X_R(\beta)$ at $t = 2$. We assume that these securities are fairly priced by households with price $P_R = E[\beta X_R(\beta)]$. The government’s budget constraint becomes

$$t = 0: G = \tau_0 + B_{0,1} + B_{0,2} + B_R P_R$$
$$t = 1: B_{0,1} = \tau_1 + B_{1,2} P_{1,2}$$
$$t = 2: B_{1,2} + B_{0,2} + B_R X_R(\beta) = \tau_2$$

As above, we work backwards from $t = 1$. The planner’s date-1 problem is

$$\min_{B_{0,2}} \left[ \frac{1}{2} (\tau_1^2 + \beta \tau_2^2) \right] = \min_{B_{0,2}} \left[ \frac{1}{2} (B_{0,1} - B_{1,2} \beta)^2 + \frac{1}{2} \beta (B_{1,2} + B_{0,2} + B_R X_R(\beta))^2 \right]. \quad (A15)$$

\(^1\) While the objective may not be globally convex in $S$ and $D$, global convexity in $B_{0,1}$ and $B_{0,2}$ shows the solution is unique.
Taking first order conditions with respect to $B_{1,2}$ yields
\[ B_{1,2} = (B_{0,1} - B_{0,2} - B_R X_R(\beta)) / (1 + \beta), \] (A16)
which implies $\tau_1 = \tau_2 = (B_{0,1} + B_{0,2}\beta + B_R \beta X_R(\beta)) / (1 + \beta)$.

Consider the problem at $t = 0$ where $\tau_0 = G - B_{0,1} - B_{0,2} - B_R P_R$:
\[ \min_{\{B_{0,1}, B_{0,2}, B_R\}} \left[ \frac{1}{2} (G - B_{0,1} - B_{0,2} - B_R P_R)^2 + E \left[ \frac{(B_{0,1} + B_{0,2}\beta + B_R \beta X_R(\beta))^2}{1 + \beta} \right] \right]. \] (A17)

The first order conditions are
\[ -(G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{1}{1 + \beta} (B_{0,1} + B_{0,2}\beta + B_R \beta X_R(\beta)) \right] = 0, \]
\[ -(G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{\beta}{1 + \beta} (B_{0,1} + B_{0,2}\beta + B_R \beta X_R(\beta)) \right] = 0, \] (A18)
\[ -P_R (G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{\beta X_R(\beta)}{1 + \beta} (B_{0,1} + B_{0,2}\beta + B_R \beta X_R(\beta)) \right] = 0. \]

Since $E[\beta] = 1$ and $P_R = E[\beta X_R(\beta)]$, it is easy to see that $B_{0,1} = B_{0,2} = G / 3$ and $B_R = 0$ satisfies these three conditions for an arbitrary risky security.

We now show that the objective function is globally convex in its three arguments, showing that this is the unique solution to the planner’s problem. Specifically, the Hessian is
\[ H = \begin{bmatrix} 1 & 1 & P_R \\ 1 & 1 & P_R \\ P_R & P_R & P_R^2 \end{bmatrix} + \begin{bmatrix} E[(1 + \beta)^{-1}] & E[(1 + \beta)^{-1} \beta] & E[(1 + \beta)^{-1} \beta X_R] \\ E[(1 + \beta)^{-1} \beta] & E[(1 + \beta)^{-1} \beta^2] & E[(1 + \beta)^{-1} \beta^2 X_R] \\ E[(1 + \beta)^{-1} \beta X_R] & E[(1 + \beta)^{-1} \beta^2 X_R] & E[(1 + \beta)^{-1} \beta^2 X_R^2] \end{bmatrix} \] (A19)

The first matrix is positive semi-definite with eigenvalues of $2 + P_R^2 > 0$ and 0 (multiplicity 2). Let
\[ E^*[Z] = \frac{E[(1 + \beta)^{-1} Z]}{E[(1 + \beta)^{-1}]} \] (A20)
denote the expectation with respect to the $(1 + \beta)^{-1}$ twisted probability measure and note that the second term can be written as
\[ E[(1 + \beta)^{-1}] \cdot E^* \begin{bmatrix} 1 \\ \beta \\ \beta X_R \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \beta X_R \end{bmatrix} \] (A21)
which is positive definite, assuming that $1$, $\beta$, and $\beta X_R$ are linearly independent. This shows that the objective function is globally convex for an arbitrary $X_R$ and, hence, that the unique optimum is $B_{r,1}^* = B_{0,2}^* = G/3$ and $B_{r}^* = 0$.\footnote{The matrix is positive semi-definite if these three variables are linearly dependent. Specifically, if $X_R = c$, a constant, the security is equivalent to 2-period riskless bonds. In this case, all solutions with $B_{0,2} + cB_R = G/3$ are equivalent, so while $B_{0,2} + cB_R$ is determined, neither $B_{0,2}$ nor $B_R$ is determined. Similarly, if $X_R = c/\beta$ so that $\beta X_R = c$, the security is equivalent to 1-period riskless from an ultimate tax-perspective. Of course, these are simply different ways of implementing perfect tax-smoothing, so these two indeterminate cases do not alter our substantive conclusion.}

**Proof of Proposition 2:** The planner solves

$$\min_{S,D} \left[ \frac{1}{2}(G - D)^2 + \frac{D^2}{2} \left( b(S - 1/2)^2 + 1/2 \right) - \gamma f(SD) \right].$$

(A22)

The first order conditions for $S$ and $D$ are

$$0 = D^2 b(S - 1/2) - D\gamma f'(SD),$$

(A23)

and

$$0 = -(G - D) + D \left( b(S - 1/2)^2 + 1/2 \right) - S\gamma f'(SD).$$

(A24)

The solution takes the form

$S^* = \frac{1}{2} + \frac{\gamma f'(S^* D^*)}{D^* b}$

$D^* = \frac{2}{3} G + \frac{\gamma f'(S^* D^*)}{3}$.

(A25)

Note that the Hessian evaluated at the solution in (A25) is

$$H = \begin{bmatrix}
D^2 b - D^2 \gamma f''(SD) & \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) \\
\gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) & b(S - 1/2)^2 + \frac{3}{2} - \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right)^2 \gamma f''(SD)
\end{bmatrix},$$

(A26)

with determinant $\det(H) = 3bD^2 - 2 - \gamma f''(SD)(3/2 + b/4) > 0$, so this is a minimum. Furthermore, so long as $f''(\cdot) \leq 0$, the objective is globally convex in $B_{0,1}$ and $B_{0,2}$ and the solution is unique.

We now derive the comparative statics. Consider the impact of $\gamma$ on $S^*$ and $D^*$:
Since $D^* = 2G/3 + \gamma f'(S^*D^*)/3$, we have $3D^* > \gamma f'(S^*D^*)$ since $G > 0$. Therefore, we have $\partial S^* / \partial \gamma > 0$ and $\partial D^* / \partial \gamma > 0$.

We next examine the impact of $b$ on $S^*$ and $D^*$:

$$
\begin{align*}
\begin{bmatrix}
\partial S^* / \partial b \\
\partial D^* / \partial b \\
\end{bmatrix}
&= \begin{bmatrix}
D^2b - D^2\gamma f''(SD) & \gamma f'(SD) - D\left(\frac{1}{2} + \frac{2\gamma f'(SD)}{D^2b}\right)\gamma f''(SD)
\end{bmatrix}^{-1}
\begin{bmatrix}
Df'(SD) \\
Sf'(SD) \\
\end{bmatrix}

&= \frac{2f'(SD)}{6b - \gamma(6 - b)f''(SD)} \begin{bmatrix}
\frac{\gamma f''(SD)}{D^2} \\
\frac{\gamma^2 f''(SD)}{6b} \\
\end{bmatrix}.
\end{align*}
\tag{A27}
$$

Thus, $\partial S^* / \partial b < 0$ and $\partial D^* / \partial b \geq 0$.

Last, the impact of $G$ is given by:

$$
\begin{align*}
\begin{bmatrix}
\partial S^* / \partial G \\
\partial D^* / \partial G \\
\end{bmatrix}
&= \begin{bmatrix}
D^2b - D^2\gamma f''(SD) & \gamma f'(SD) - D\left(\frac{1}{2} + \frac{2\gamma f'(SD)}{D^2b}\right)\gamma f''(SD)
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
1 \\
\end{bmatrix}

&= \frac{2}{6b - \gamma(6 - b)f''(SD)} \begin{bmatrix}
\frac{\gamma f''(SD)bD - 2bf'(SD) + 2\gamma f'(SD)f''(SD)}{2b - 2f''(SD)\gamma} \\
\end{bmatrix}.
\tag{A29}
\end{align*}
$$

Thus, $\partial S^* / \partial G < 0$ and $\partial D^* / \partial G > 0$.

**Proof of Proposition 3:** For simplicity, assume that all government debt is sold to foreign investors, including both short-term and long-term debt. Since all debt is sold to foreign investors, domestic household consumption is given by:

$$
\begin{align*}
C_0 &= 1 - \tau_0 - (1/2)\tau_0^2 \\
C_1 &= 1 - \tau_1 - (1/2)\tau_1^2 \\
C_2 &= 1 - \tau_2 - (1/2)\tau_2^2.
\end{align*}
\tag{A30}
$$
(Compare (A30) with equation (7) in the text.) Substituting the government’s budget constraint in (6) into (A30) we obtain

\[
\begin{align*}
C_0 &= 1 - (1/2)r_0^2 - G + (B_{0,1}P_{0,1} + B_{0,2}P_{0,2}) \\
C_1 &= 1 - (1/2)r_1^2 + (B_{1,2}P_{1,2} - B_{0,1}) \\
C_2 &= 1 - (1/2)r_2^2 - (B_{1,2} + B_{0,2}).
\end{align*}
\]

(A31)

As before, domestic consumption is impacted by distortionary tax costs, but relative to (8) there are additional terms which reflect net foreign borrowing in each period.

Assuming that \( P_{0,1} = 1 + v'(B_{0,1}) \), \( P_{0,2} = 1 \), and \( P_{1,2} = \beta \) (i.e., foreign investors have the same preference shock as domestic households), it is easy to see that

\[
U = C_0 + E[C_1 + \beta C_2] = 3 - G - \frac{1}{2}\left[r_0^2 + E[r_1^2] + E[\beta r_2^2]\right] + B_{0,1}v'(B_{0,1}).
\]

(A32)

Thus, dropping constants, the nationalistic planner’s problem can be rewritten as

\[
\begin{align*}
\min_{S,D} & \left[\frac{1}{2}(G - D - R(DS))^2 + \frac{D^2}{2}\left(b\left(S - \frac{1}{2}\right)^2 + \frac{1}{2}\right) - R(DS)\right],
\end{align*}
\]

(A33)

where \( R(M) = v'(M)M \) denotes seignorage revenue and we assume \( R'(M) > 0 \) and \( R''(M) \leq 0 \).

The first order conditions for \( S \) and \( D \) are

\[
0 = -(1 + G - D - R(DS))DR'(SD) + D^2b(S - 1/2),
\]

(A34)

and

\[
0 = -(G - D - R(DS))S + (1 + G - D - R(DS))SR'(SD) + D\left(b\left(S - 1/2\right)^2 + \frac{1}{2}\right).
\]

(A35)

Solving (A34) and (A35) shows that the solution takes the form

\[
S^* = \frac{1 + R'(D^*S^*)}{2} \frac{(G - R(D^*S^*) + 3)}{b\left(R'(D^*S^*)\right)(G - R(D^*S^*) + 3) + 2(G - R(D^*S^*) - R'(D^*S^*))}
\]

\[
D^* = \frac{\left(2 + R'(D^*S^*)\right)(G - R(D^*S^*) + 3) + 2(G - R(D^*S^*) - R'(D^*S^*))}{3 + R'(D^*S^*)}.
\]

(A36)

We now derive the comparative statics. Consider the impact of \( b \) on \( S^* \) and \( D^* \). We have:

\[
\begin{align*}
\frac{\partial S^*}{\partial b} &= \left[\frac{(DR')^2 - (1 + G - D - R)D^2R^* + D^2b}{(1 + SR')DR' - (1 + G - D - R)DSR^* + Db(S - 1/2)}\right]^{-1}\left[D^2(S - 1/2)\right] \\
\frac{\partial D^*}{\partial b} &= \frac{2(S - 1/2)}{6b + 4bR' + (2 + b)(R')^2 - R'(6 + b)(1 + G - D - R)} \left[-\left(R' + 2SR' + S(R')^2 - SR'(1 + G - D - R) + 3\right)\right]^{-1}\left[D^2(S - 1/2)\right].
\end{align*}
\]
Since $S^* > 1/2$, $(1 + G - D - R) = (1 + r_0) > 0$, $R' > 0$, and $R^* < 0$, we have $\partial S^* / \partial b < 0$ and $\partial D^* / \partial b > 0$.

The impact of $G$ is given by:

$$\begin{bmatrix} \partial S^* / \partial G \\ \partial D^* / \partial G \end{bmatrix} = \begin{bmatrix} \left(DR'\right)^2 - \left(1 + G - D - R\right)D^2R^* + D^2b & (1 + SR')DR' - \left(1 + G - D - R\right)DSR^* + Db(S - 1/2) \\ (1 + SR')DR' - \left(1 + G - D - R\right)DSR^* + Db(S - 1/2) & (1 + SR')^2 - (1 + G - D - R)S^2R^* + b(S - 1/2)^2 + 1/2 \end{bmatrix} \begin{bmatrix} DR' \\ (1 + SR') \end{bmatrix}$$

where the second line follows by using (A36) to simplify the resulting expression. Thus, $\partial S^* / \partial G < 0$ and $\partial D^* / \partial G > 0$.

**Proof of Propositions 4 and 5:** We solve the second-best problem. The first-best problem can be seen as a special case of the second-best problem which is obtained by setting $\phi = 1$. We start with the planner’s objective function

$$U_{SOCIAL} = E[g(K) - K] + \nu(M_0) - \frac{1}{2}[\tau_0^2 + E[\tau_t^2] + E[\beta\tau_t^2]]. \quad (A37)$$

We plug into this the reaction function implicitly defined by equation (21) in the text, $M^*_p(M_G, \phi)$:

$$U_{SOCIAL} = p[g(W - W] + (1 - p)\left[g(W - M^*_p(SD, \phi)) - (W - M^*_p(SD, \phi))\right] + \nu(M^*_p(SD, \phi) + SD) - \frac{1}{2}(G - D)^2 - \frac{D^2}{2} \left(b\left(S - \frac{1}{2}\right)^2 + \frac{1}{2}\right). \quad (A38)$$

The first order condition for $S^{***}$ is

$$0 = -(1 - p)(g'(W - M^*_p) - 1)D \frac{\partial M^*_p}{\partial M_G} + \nu'(SD + M^*_p)D\left(1 + \frac{\partial M^*_p}{\partial M_G}\right) - D^2b(S - 1/2) \quad (A39)$$

$$= (1 - p)(\phi - 1)g'(W - M^*_p) \frac{\partial M^*_p}{\partial M_G} + \nu'(SD + M^*_p)D - D^2b(S - 1/2).$$

Where the second line follows from the fact that $\nu'(M^*_p + SD) = (1 - p)(\phi g'(W - M^*_p) - 1)$.

Rearranging and dividing by $D$, we obtain

$$Db(S - 1/2) = (1 - p)(\phi - 1)g'(W - M^*_p) \frac{\partial M^*_p}{\partial M_G} + \nu'(SD + M^*_p), \quad (A40)$$

which is equation (25) in the text. Note that the first-best solution given in equation (21) obtains as a special case of (A40) by setting $\phi = 1$. The first order condition for $D^{***}$ can be written as
0 = (1 - p)(\phi - 1)g'(W - M^*_p)^2 + v'(SD + M^*_p)S + (G - D) - D\left(b(S - 1/2)^2 + 1/2\right). \quad (A41)

We later use the fact that the first order conditions for \(S^{***}\) and \(D^{***}\) imply

\[ G = D^{***}\left(3/2 - b(S^{***} - 1/2)/2\right). \quad (A42) \]

Letting

\[ \lambda_p = \frac{\partial M'_p}{\partial M_g} = -\frac{v^*(M_g + M^*_p(M,\phi))}{v^*(M_g + M^*_p(M,\phi)) + (1 - p)\phi g^*(W - M^*_p(M,\phi))} < 0 \]

(recall that \(-1 < \lambda_p < 0\)) denote the crowding out effect of short-term government issuance and

\[ \Psi = (1 - p)(1 - \phi)g''(\cdot)\lambda_p^2 + (1 - p)(\phi - 1)g'(\cdot) \left(\frac{\partial \lambda_p}{\partial M_g} + \frac{\partial \lambda_p}{\partial M_p} \lambda_p\right) + v''(\cdot)(1 + \lambda_p). \quad (A44) \]

The Hessian for this problem at the solution defined by (A39) and (A41) is

\[
H = \begin{bmatrix}
\Psi D^2 - bD^2 & \Psi SD - Db(S - 1/2) \\
\Psi SD - Db(S - 1/2) & \Psi S^2 - b(S - 1/2)^2 - 3/2
\end{bmatrix}
\]

\( (A45) \)

We assume that \(\Psi < 0\) at \(S = S^{***}\) and \(D = D^{***}\). This ensures that the second order conditions are satisfied since this implies \( \det(H) = D^2[(3/2)b - (3/2)\Psi - (1/4)b\Psi] > 0\). As above, if \(\Psi < 0\), the objective will be globally concave in \(B_{0,1}\) and \(B_{0,2}\), ensuring uniqueness.

We now examine the comparative statics with respect to \(\phi\). Differentiate the first order condition for \(S\) with respect to \(\phi\) to obtain:

\[
(1 - p)g'(\cdot)\lambda_p - (1 - p)(\phi - 1)g''(\cdot)\lambda_p^2 \frac{\partial M'_p}{\partial \phi} + (1 - p)(\phi - 1)g'(\cdot) \left(\frac{\partial \lambda_p}{\partial M_g} + \frac{\partial \lambda_p}{\partial M_p} \lambda_p\right) + v''(\cdot)\frac{\partial M'_p}{\partial \phi} D. \quad (A46)
\]

Noting that

\[
\frac{\partial M'_p}{\partial \phi} = \frac{(1 - p)g'(\cdot)}{v'(\cdot) + (1 - p)\phi g^*(\cdot)} < 0;
\]
\[
\frac{\partial \lambda_p}{\partial \phi} = \frac{(1 - p)v''(\cdot)g^*(\cdot)}{[v''(\cdot) + (1 - p)\phi g^*(\cdot)]^2} > 0
\]

\[
\frac{\partial \lambda_p}{\partial M_p} = -(1 - p)\phi \frac{v''(\cdot)g^*(\cdot) + v'(\cdot)g''(\cdot)}{[v''(\cdot) + (1 - p)\phi g^*(\cdot)]^2},
\]

which imply \((1 - p)g'(\cdot)\lambda_p + v''(\cdot)(\partial M'_p / \partial \phi) = 0\), the expression in (A46) simplifies to
\[-(1-p)(\phi-1)g''(\phi)\lambda_p \frac{\partial M'_r}{\partial \phi} + (1-p)(\phi-1)g'(\phi) \left[ \frac{\partial \lambda_p}{\partial \phi} \right. + \left. \frac{\partial \lambda_p}{\partial M'_r} \frac{\partial M'_r}{\partial \phi} \right] \]
\[= \frac{(\phi-1)g'(\phi)(1-p)^2}{[v''(\phi) + (1-p)\phi g''(\phi)]^2} \left(2v''(\phi)g''(\phi) + (1-p)\phi g'(\phi) + v''(\phi)g''(\phi) \right) \] (A48)

The expression in (A48) will be negative when \( \phi < 1 \) so long as \( g''(\phi) \) and \( v''(\phi) \) are not too large which we assume is the case.\(^3\)\(^4\)

Combining all of this we have
\[\left[ \frac{\partial S''''}{\partial \phi} \right] = \left( \frac{(1-\phi)g'(\phi)(1-p)^i}{[v''(\phi) + (1-p)\phi g''(\phi)]^2} \right) \left(2v''(\phi)g''(\phi) + (1-p)\phi g'(\phi) + v''(\phi)g''(\phi) \right) \times \]
\[\left[ \begin{array}{c} \Psi D^3 - bD^2 \\ \Psi SD - Db(S-1/2) \\ \Psi SD - Db(S-1/2) - 3/2 \end{array} \right] = \left[ D \right] \]
\[= \frac{1}{\det(H)} \left[ \frac{(1-\phi)g'(\phi)(1-p)^i}{[v''(\phi) + (1-p)\phi g''(\phi)]^2} \right] \left(2v''(\phi)g''(\phi) + (1-p)\phi g'(\phi) + v''(\phi)g''(\phi) \right) \] (A49)

where we have made use of the fact that \( G = D'''' \left( 3/2 - b(S'''' - 1/2) / 2 \right) \) from (A42). Thus, \( \partial S'''' / \partial \phi < 0 \) and \( \partial D'''' / \partial \phi < 0 \) so long as \( \phi < 1 \) and \( g''(\phi) \) and \( v''(\phi) \) are not too large (this also implies that \( M''''_G = D'''' S'''' \) is decreasing in \( \phi \)).

\(^3\) If \( g''(\phi) \) and \( v''(\phi) \) were too large, then \( \partial S / \partial \phi = (\partial S / \partial \phi) + (\partial S / \partial M_r)(\partial M_r / \partial \phi) \), the total derivative of \( \lambda_r \) with respect to \( \phi \), would be a large negative number. In this case, as \( \phi \) declined, \( |\lambda_r| \) would decline significantly (since \( \lambda_p < 0 \), \( \lambda_r \) would rise), greatly reducing the crowding-out benefit from issuing short-term government debt. If this force were strong enough, it could outweigh the direct effect, \(-(1-p)(\phi-1)g''(\phi)\lambda_r \), which reflects the fact that \( M_r \) rises as \( \phi \) falls, exacerbating the under-investment problem in the bad state. However, note that \( \partial \lambda_r / \partial \phi > 0 \) which reflects the fact that, holding \( M_r \) and \( M_G \) fixed, private money creation becomes more not less sensitive to the money premium as \( \phi \) declines because firms more severely underweight its costs. Thus, if \( \partial \lambda_r / \partial M_r \) is not too large (i.e. the functions are well approximated locally by quadratics, so \( g''(\phi) \) and \( v''(\phi) \) are small), then we will have \( \partial S / \partial \phi > 0 \), implying that \( \partial S'''' / \partial \phi < 0 \) and \( \partial D'''' / \partial \phi < 0 \).

\(^4\) The second order conditions for \( S'''' \) and \( D'''' \) also depend on \( g''(\phi) \) and \( v''(\phi) \) through \( \Psi \) which we assume is negative. Specifically, one can show that \( (\partial \lambda_r / \partial M_r) + (\partial \lambda_r / \partial M_G)\lambda_r \) is increasing in \( v''(\phi) \) and decreasing in \( g''(\phi) \). Therefore, \( g''(\phi) \) cannot be too large if the second order conditions for \( S'''' \) and \( D'''' \) are to hold. Specifically, if \( g''(\phi) \) is too large, a rise in \( M_r \) would significantly raise \( \lambda_r \), implying an increasing as opposed to diminishing crowding out benefit from issuing more short-term debt.
Finally, note that

$$\frac{\delta}{\delta \phi} [M_p^{**}(\phi, M_G^{**}(\phi))] = \frac{\partial M_p^{**}}{\partial \phi} + \frac{\partial M_G^{**}}{\partial \phi} \frac{\partial M_p^{**}}{\partial \phi} > \frac{\partial M_p^{**}}{\partial \phi}$$

$$\frac{\delta}{\delta \phi} [M_p^{**}(\phi, M_G^{**}(\phi)) + M_G^{**}(\phi)] = \frac{\partial M_p^{**}}{\partial \phi} + \frac{\partial M_G^{**}}{\partial \phi} \left(1 + \frac{\partial M_G^{**}}{\partial \phi}\right) < \frac{\partial M_p^{**}}{\partial \phi} < 0.$$ (A50)

Thus, the increase in private money following a decline in \( \phi \) is smaller when the government recognizes the “crowding out” benefit of short-term bills. However, the total increase in public plus private short-term debt is greater than in the absence of such a policy because each dollar of additional short-term government debt crowds out less than one dollar of short-term private debt. Finally, \( \delta[M_p^{**}(\phi, M_G^{**}(\phi))] / \delta \phi < 0 \), so long as \( g''(\cdot) \) and \( v''(\cdot) \) are not too large and \( \phi \) is not too small (e.g. if \( g''(\cdot) = v''(\cdot) = 0 \) and \( \phi > 1/2 \)). Since the first best solution obtains when \( \phi = 1 \), the second-best solution involves a larger quantity of government bills and more private money creation.

**Proof of Proposition 6:** Let \( M_p^*(M_g, \theta_p, \phi) \) denote the solution to equation (27) repeated here:

\[ v'(M_p^* + M_g) = \theta_p + (1-p)(\phi g'(W - M_p^*)) - 1. \] (A51)

It follows that

\[ \frac{\partial M_p^*}{\partial M_g} = \lambda_p = -\frac{v''(M_p^* + M_g)}{v'(M_p^* + M_g) + (1-p)\phi g''(W - M_p^*)} < 0 \]

\[ \frac{\partial M_p^*}{\partial \theta_p} = \eta_p = \frac{1}{v'(M_p^* + M_g) + (1-p)\phi g''(W - M_p^*)} < 0 \] (A52)

\[ \frac{\partial M_p^*}{\partial \phi} = \frac{(1-p)g'(W - M_p^*)}{v'(M_p^* + M_g) + (1-p)\phi g''(W - M_p^*)} < 0, \]

with \(-1 < \lambda_p < 0\). To get the second best solution, we rewrite the planner’s objective function in equation (28) as

\[ U_{SOCIAL} = p[g(W) - W] + (1-p)[g(W - M_p^*(SD, \theta_p)) - (W - M_p^*(SD, \theta_p))] + v(M_p^*(SD, \theta_p) + SD) - \frac{1}{2}(G - D)^2 - \frac{D^2}{2} \left( b \left( \frac{S - 1}{2} \right)^2 + 1 \right) - \frac{Y}{2} \theta_p^p. \] (A53)
The planner now has three control variables: $S$, $D$, and $\theta_p$.

We first compute optimal taxes. The planner’s first order condition for $\theta_p$ is

$$
0 = \frac{\partial M_p^*}{\partial \theta_p} [v'(M_p^*(SD, \theta_p, \phi) + SD) - (1 - p)g'(W - M_p^*(SD, \theta_p, \phi) - 1)] - \gamma \theta_p
$$

$$
= \frac{\partial M_p^*}{\partial \theta_p} [\theta_p + (1 - p)(\phi - 1)g'(W - M_p^*)] - \gamma \theta_p.
$$

where the second line uses (A51). This implies that

$$
\theta_p^{***} = \frac{\left| \frac{\partial M_p^*}{\partial \theta_p} \right|}{\frac{\partial M_p^*}{\partial \theta_p} + \gamma}(1 - p)(1 - \phi)g'(W - M_p^*) \leq (1 - p)(1 - \phi)g'(W - M_p^*).
$$

(A55)

Thus, we have $\theta_p^{***} > 0$ if $\phi < 1$ and $\gamma$ is finite. When $\gamma = 0$, we obtain

$$
\theta_p^{***} = (1 - p)(1 - \phi)g'(W - M_p^*),
$$

(A56)

which implies that

$$
v'(M_p^* + M_g) = \theta_p^{***} + (1 - p)(\phi g'(W - M_p^*) - 1) = (1 - p)(g'(W - M_p^*) - 1).
$$

(A57)

This is the same as the condition defining the first-best level of optimal private money $M_p^{**}$.

However, with positive deadweight costs, the optimal tax is only a fraction of the tax that makes the banks fully internalize the fire-sale externality (i.e., $(1 - p)(1 - \phi)g'(W - M_p^*)$). This fraction is higher if $\left| \frac{\partial M_p^*}{\partial \theta_p} \right|$ is larger or if the deadweight costs are smaller.

We next compute optimal government debt maturity. The first order condition for $S$ is the same as before:

$$
0 = \left[ -(1 - p)(g'(W - M_p^*) - 1) \frac{\partial M_p^*}{\partial M_g} + v'(SD + M_p^*) \left( 1 + \frac{\partial M_p^*}{\partial M_g} \right) \right] D - D^2 b (S - 1/2)
$$

(A58)

Rearranging and dividing by $D$, we obtain equation (29) in the text:
This manipulation uses the market clearing condition (A51) for $M_p^*$ to substitute out for $v'$ and then uses the optimal expression for $\theta_p$ from (A55). The second term above, which reflects the crowding-out benefits of issuing additional short-term government debt, is positive so long as $\phi < 1$ and $\Upsilon > 0$.

We now turn to the comparative statics calculations for the planner’s problem with two tools.

The first order conditions for $S$, $D$, and $\theta_p$ can be written as:

$$0 = [v'(M_p^* + SD)(1 + \lambda_p) - (1 - p)(g'(W - M_p^*) - 1)\lambda_p]D - D^2 b(S - 1 / 2)$$

$$0 = [v'(M_p^* + SD)(1 + \lambda_p) - (1 - p)(g'(W - M_p^*) - 1)\lambda_p]S + (G - D) - D b(S - 1 / 2)^2 + 1 / 2$$

$$0 = \eta_p[v'(M_p^* + SD) - (1 - p)(g'(W - M_p^*) - 1)] - \Upsilon \theta_p$$

The Hessian for this problem takes the form

$$H = \begin{bmatrix}
\Psi D^2 - bD^2 & \Psi SD - bD(S - 1 / 2) & \Phi D \\
\Psi SD - bD(S - 1 / 2) & \Psi S^2 - b(S - 1 / 2)^2 - 3 / 2 & \Phi S \\
\Phi D & \Phi S & \Xi - \Upsilon
\end{bmatrix},$$

where

$$\Psi = v^\nu(\cdot)(1 + \lambda_p)^2 + (1 - p)g^\nu(\cdot)(\lambda_p)^2 = (1 - p)v^\nu(\cdot)g^\nu(\cdot) - \frac{v^\nu(\cdot) + (1 - p)\phi g^\nu(\cdot)}{v^\nu(\cdot) + (1 - p)\phi g^\nu(\cdot)} < 0$$

$$\Xi = (\eta_p)^2[v^\nu(\cdot) + (1 - p)g^\nu(\cdot)] - \frac{v^\nu(\cdot) + (1 - p)g^\nu(\cdot)}{v^\nu(\cdot) + (1 - p)\phi g^\nu(\cdot)} < 0$$

$$\Phi = \eta_p\lambda_p[v^\nu(\cdot) + (1 - p)g^\nu(\cdot)] = -v^\nu(\cdot)\frac{v^\nu(\cdot) + (1 - p)g^\nu(\cdot)}{v^\nu(\cdot) + (1 - p)\phi g^\nu(\cdot)} < 0.$$
so long as the relevant third derivatives are not too large in magnitude. We assume that \( \det(H) < 0 \) by the second order condition for the planner’s problem.

The comparative statics with respect to \( \gamma \) follow from

\[
\begin{bmatrix}
\partial S^{***} / \partial \gamma \\
\partial D^{***} / \partial \gamma \\
\partial \theta_{\phi r}^{***} / \partial \gamma 
\end{bmatrix} = -
\begin{bmatrix}
\Psi D^2 - bD^2 & \Psi SD - bD(S - 1/2) & \Phi D \\
\Psi SD - bD(S - 1/2) & \Psi S^2 - b(S - 1/2)^2 - 3/2 & \Phi S \\
\Phi D & \Phi S & \Xi - \gamma
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
0 \\
-\theta_r
\end{bmatrix}
\]

(A63)

where we have used the fact that \( G = D^{***} \left( 3/2 - b(S^{***} - 1/2)/2 \right) \) by (A60) to simplify the resulting expression. Thus, we have \( \partial S^{***} / \partial \gamma > 0, \partial D^{***} / \partial \gamma > 0 \), and \( \partial \theta_{\phi r}^{***} / \partial \gamma < 0 \).

The comparative statics with respect to \( b \) follow from

\[
\begin{bmatrix}
\partial S^{***} / \partial b \\
\partial D^{***} / \partial b \\
\partial \theta_{\phi r}^{***} / \partial b 
\end{bmatrix} = -
\begin{bmatrix}
\Psi D^2 - bD^2 & \Psi SD - bD(S - 1/2) & \Phi D \\
\Psi SD - bD(S - 1/2) & \Psi S^2 - b(S - 1/2)^2 - 3/2 & \Phi S \\
\Phi D & \Phi S & \Xi - \gamma
\end{bmatrix}^{-1}
\begin{bmatrix}
-D^2(S - 1/2) \\
-D(S - 1/2)^2 \\
0
\end{bmatrix}
\]

(A64)

Making the natural regularity assumption that \( \Psi \Xi > \Phi^2 \) so the social returns to limiting private money creation are concave in \( M_g \) and \( \theta_r \) (a sufficient condition is that \( (1 - p) \phi g^"(\cdot) < v^"(\cdot) \)) and noting that \( S^{***} > 1/2 \), we have \( \partial S^{***} / \partial b < 0, \partial D^{***} / \partial b > 0 \), and \( \partial \theta_{\phi r}^{***} / \partial b > 0 \).

Turning to the comparative statics with respect to \( \phi \), note that \( \partial \lambda_p / \partial \phi > 0 \) and \( \partial \eta_p / \partial \phi > 0 \) (i.e., the efficacy of both crowding out and regulation rise as \( \phi \) falls). Next let
\[
\Theta = [v'(\cdot) - (1 - p)(g'(\cdot) - 1)] \frac{\partial \lambda_p}{\partial \phi} + [v''(\cdot)(1 + \lambda_p) + (1 - p)g''(\cdot)\lambda_p] \frac{\partial M_p'}{\partial \phi} \\
= [v'(\cdot) - (1 - p)(g'(\cdot) - 1) - (1 - p)(1 - \phi)g'(\cdot)] \frac{(1 - p)g''(\cdot)v''(\cdot)}{(v''(\cdot) + (1 - p)g'(\cdot))} < 0
\] 

(A65)

(the term is square brackets is negative for \( \phi < 1 \)) which reflects the lower social returns to crowding out when the externality becomes less severe (i.e., when \( \phi \) rises). Similarly, let

\[
\Gamma = [v'(\cdot) - (1 - p)(g'(\cdot) - 1)] \frac{\partial \eta_p}{\partial \phi} + \eta_p[v''(\cdot) + (1 - p)g''(\cdot)] \frac{\partial M_p'}{\partial \phi} < 0,
\] 

(A66)

(the term is square brackets is negative for \( \phi < 1 \)), which reflects the lower social returns to direct regulation when \( \phi \) rises. We then have

\[
\begin{bmatrix}
\frac{\partial S^{***}}{\partial \phi} \\
\frac{\partial D^{***}}{\partial \phi} \\
\frac{\partial \theta^{***}_p}{\partial \phi}
\end{bmatrix}
= -\begin{bmatrix}
\Psi D^2 - bD^2 & \Psi SD - bD(S - 1/2) & \Phi D \\
\Psi SD - bD(S - 1/2) & \Psi S^2 - b(S - 1/2)^2 - 3/2 & \Phi S \\
\Phi D & \Phi S & \Xi - \Gamma
\end{bmatrix}^{-1} \begin{bmatrix}
\Theta D \\
\Theta S \\
\Gamma
\end{bmatrix}
\]

\[
= \frac{-1}{\det(H)} \begin{bmatrix}
((\Gamma\Phi - \Theta\Xi) + \Theta\Psi)G \\
\frac{1}{2}((\Gamma\Phi - \Theta\Xi) + \Theta\Psi)bD^2 \\
(\Theta\Phi - \Gamma\Psi)\frac{1}{4}D^2(6 + b) + \frac{3}{2}bD^2\Gamma
\end{bmatrix}.
\]

(A67)

Since crowding out and regulation are substitutes from the perspective of limiting private money creation (i.e., \( \Phi < 0 \)), both \((\Gamma\Phi - \Theta\Xi)\) and \((\Theta\Phi - \Gamma\Psi)\) are ambiguous because they are the difference of two positive terms. However, if either (i) \(|\Phi|\) is small relative to both \(|\Xi|\) and \(|\Psi|\) or (ii) both \(b\) and \(\Psi\) are sufficiently large, then we have \(\frac{\partial S^{***}}{\partial \phi} < 0\), \(\frac{\partial D^{***}}{\partial \phi} < 0\), and \(\frac{\partial \theta^{***}_p}{\partial \phi} < 0\).

**Proof of Proposition 7:** We now extend the model by adding an additional period (i.e., the dates of the model are now \(t = 0, 1, 2, 3\)) and by allowing short-term debt to generate monetary services at the interim dates \((t = 1\) and \(t = 2\)) in addition to the initial date. (For simplicity, we do not allow for private money creation.) This extension serves two purposes. First, it shows that our results are not driven by the simplifying assumption that households only enjoy money services at time 0. Secondly, the extension allows us to investigate how the hedging opportunities afforded by multiples maturities alter the tax-smoothing costs faced by the government.
The government finances a one-time expenditure $G$ at date 0 by issuing short-term (1-period) bonds $B_{0,1}$, medium-term (2-period) bonds $B_{0,2}$, and long-term (3-period) bonds $B_{0,3}$ to households and by levying distortionary taxes, $\tau_0$. At time 1, the government must repay any maturing debt by levying taxes and issuing new short- and long-term bonds. At time 2, the government repays maturing debt by levying taxes and issuing new short-term bonds. All debt maturing at time 3 must be repaid by levying taxes. Thus, the sequence of government budget constraints is given by:

\begin{align}
  t = 0: & \quad G = \tau_0 + B_{0,1}P_{0,1} + B_{0,2}P_{0,2} + B_{0,3}P_{0,3} \\
  t = 1: & \quad B_{0,1} = \tau_1 + B_{1,2}P_{1,2} + B_{1,3}P_{1,3} \\
  t = 2: & \quad B_{0,2} + B_{1,2} = \tau_2 + B_{2,3}P_{2,3} \\
  t = 3: & \quad B_{0,3} + B_{1,3} + B_{2,3} = \tau_3,
\end{align}

where $P_{0,1}$, $P_{0,2}$, and $P_{0,3}$ denote the prices of short-, medium-, and long-term bonds issued at date 0, $P_{1,2}$ and $P_{1,3}$ denote the uncertain prices of short- and long-term bonds issued at date 1, and $P_{2,3}$ is uncertain price of short-term bonds issued at date 2.

There are now two uncertain interest rates. At time 1, households learn $\beta_1$ which pins down the short rate between periods 1 and 2. At time 2, households learn $\beta_2$ which determines the short rate between periods 2 and 3. However, at time 1, households also update their expectations of $\beta_2$ based on the realization of $\beta_1$. Specifically, households learn $\delta_1 = E[\beta_2 | \beta_1]$ at time 1, but $\beta_2 = \delta_1 \epsilon_2$ is only realized at time 2 where $E[\epsilon_2 | \beta_1, \delta_1] = 1$. Note that $1/\delta_1$ is simply the gross forward short-term interest rate at time 1. Thus, there are now effectively three interest rate shocks: the realization of $\beta_1$ at time 1, the “news” about $\beta_2$ at time 1, and the ultimate realization of $\beta_2$ at time 2. If all shifts in the yield curve are parallel, then $\delta_1 \equiv \beta_1$ so, in that case, there are only two non-degenerate shocks. Without loss of generality, we assume the term structure is initially flat: $E[\beta_1] = E[\beta_1 \beta_2] = 1$. However, we allow $\beta_1$ and $\delta_1$ to be correlated. Since $1 = E[\beta_1 \delta_1] = Cov[\beta_1, \delta_1] + E[\beta_1]E[\delta_1]$, we have $Cov[\beta_1, \delta_1] = 1 - E[\delta_1]$.

By the above assumptions, the prices of 1-, 2-, and 3-period bonds are all equal to 1 at time 0: $P_{0,1} = P_{0,2} = P_{0,3} = 1$. At time 1, the price of 1-period bonds maturing at time 2 is $P_{1,2} = \beta_1$ and the price of 2-period bonds maturing at time 3 is $P_{1,3} = \beta_1 \delta_1$. Finally, at time 2, the price of 1-period bonds maturing at time 3 is $P_{2,3} = \beta_2 = \delta_1 \epsilon_2$.

Household enjoy monetary services at time 0, 1, and 2 based on the total stock of outstanding 1-period bonds at each date. For simplicity, we work with linear money utility in this extension, so
households obtain money utility \( v(M_t) = \gamma \cdot M_t \) at time \( t \) where \( M_t \) is the total stock of outstanding 1-period bonds at \( t \). Thus, we have \( M_0 = B_{0,1}, M_1 = B_{0,2} + B_{1,2}, \) and \( M_2 = B_{0,3} + B_{1,3} + B_{2,3} \).

*The government’s time-2 problem:* To solve the model, we work backwards from time 2, taking time-1 and time-2 issuance as given. Using (A68), the expressions for prices, and the expression for \( M_2 \), the government’s problem at time 2 is:

\[
\begin{align*}
\min_{\beta_2, \gamma} & \left[ \frac{1}{2} (\tau_2^2 + \beta_2 \tau_3^2) - \gamma \cdot M_2 \right] \\
& = \min_{\beta_2, \gamma} \left[ \frac{1}{2} (B_{0,2} + B_{1,2} - B_{2,3} \beta_2)^2 + \frac{1}{2} \beta_2 (B_{0,3} + B_{1,3} + B_{2,3})^2 - \gamma \cdot (B_{0,3} + B_{1,3} + B_{2,3}) \right].
\end{align*}
\]  

(A69)

The solution to (A69) is then

\[
B_{2,3}^* = \frac{(B_{0,2} + B_{1,2}) - (B_{0,3} + B_{1,3})}{1 + \beta_2} + \frac{\gamma}{\beta_2 (1 + \beta_2)},
\]  

(A70)

which is the natural generalization of (A2). The first term in (A70) implements perfect tax smoothing between times 2 and 3 and the second component is the optimal deviation from perfect smoothing: the government tilts toward short-term debt because households derive monetary services from short-term debt at time 2 (i.e., \( \gamma > 0 \)). This solution implies that

\[
\tau_2^* = \frac{(B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})}{1 + \beta_2} + \frac{\gamma}{\beta_2 (1 + \beta_2)},
\]  

(A71)

\[
\tau_3^* = \frac{(B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})}{1 + \beta_2} + \frac{\gamma}{\beta_2 (1 + \beta_2)}.
\]

The present value of taxes is \( \tau_2^* + \beta_2 \tau_3^* = (B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3}) \) independently of \( \gamma \). Thus, the government responds to \( \gamma > 0 \) by taxing a bit less an \( t = 2 \) and a bit more at \( t = 3 \) in order to create additional money services at \( t = 2 \). Algebra shows that the minimized objective function is given by

\[
V_2(B_{0,2} + B_{1,2}, B_{0,3} + B_{1,3}, \beta_2) = \min_{\beta_2, \gamma} \left[ \frac{1}{2} (\tau_2^2 + \beta_2 \tau_3^2) - \gamma \cdot M_2 \right]
\]

\[
= \frac{1}{2} \left( (B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3}) \right)^2 - \gamma \cdot \left( (B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3}) \right) - \frac{\gamma^2}{2} \frac{1}{\beta_2 (1 + \beta_2)}.
\]  

(A72)

We can omit the final term when we move backwards to time 1 since this term is independent of the government’s prior debt maturity choices.

*The government’s time-1 problem:* Now consider the government’s problem at time 1 taking time 0 issuance as given. Recall that at time 1 agents learn \( \beta_1 \) and \( \delta_1 = E[\beta_2 | \beta_1] \), but that \( \beta_2 = \delta_1 \varepsilon_2 \) is still uncertain as of time 1 since \( \varepsilon_2 \) is not realized until time 2. The government’s time 1 problem is:
The first order conditions for $B_{1,2}$ and $B_{1,3}$ are

$$0 = -\beta_1(B_{0,1} - B_{1,2}\beta_1 - B_{1,3}\beta_1\delta_1) - \gamma + \beta_1 E\left[\frac{((B_{0,2} + B_{1,2}) + \delta_1\varepsilon_2(B_{0,3} + B_{1,3}))}{1 + \delta_1\varepsilon_2}\right] - \gamma\cdot\frac{1}{1 + \delta_1\varepsilon_2}\ | \delta_1\right]$$

(A74)

$$0 = -\beta_1\delta_1(B_{0,1} - B_{1,2}\beta_1 - B_{1,3}\beta_1\delta_1) + \beta_1 E\left[\delta_1\varepsilon_2\frac{((B_{0,2} + B_{1,2}) + \delta_1\varepsilon_2(B_{0,3} + B_{1,3}))}{1 + \delta_1\varepsilon_2}\right] - \gamma\cdot\frac{1}{1 + \delta_1\varepsilon_2}\ | \delta_1\right].$$

We can rewrite this system as

$$\begin{bmatrix}
\beta_1 + w_1 & \beta_1\delta_1 + x_1 \\
\beta_1\delta_1 + x_1 & \beta_1\delta_1^2 + y_1
\end{bmatrix} = \begin{bmatrix}
\gamma(1/\beta_1 + w_1) + B_{0,1} - w_1B_{0,2} - x_1B_{0,3} \\
\gamma x_1 + \delta_1B_{0,1} - x_1B_{0,2} - y_1B_{0,3}
\end{bmatrix},$$

(A75)

where

$$w_1 \equiv E\left[\frac{1}{1 + \delta_1\varepsilon_2}\ | \delta_1\right], \ x_1 \equiv E\left[\frac{\delta_1\varepsilon_2}{1 + \delta_1\varepsilon_2}\ | \delta_1\right], \text{ and } y_1 \equiv E\left[\frac{(\delta_1\varepsilon_2)^2}{1 + \delta_1\varepsilon_2}\ | \delta_1\right].$$

(A76)

are random variables that are functions of the realized value of $\delta_1$.

It is straightforward to verify that the perfect tax-smoothing “consol” bond solution extends to the 4-period model when $\gamma = 0$. This in turn implies that $B^*_{1,2} = B^*_{1,3} = 0$ for all realizations of $\beta_1$ and $\delta_1$ when $\gamma = 0$ and $B_{0,1} = B_{0,2} = B_{0,3}$ which implies

$$\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 - w_1 - x_1 \\
\delta_1 - x_1 - y_1
\end{bmatrix} \Rightarrow x_1 = 1 - w_1 \text{ and } y_1 = w_1 - (1 - \delta_1).$$

(A77)

Thus, optimal issuance is given by

$$\begin{bmatrix}
B^*_{1,2} \\
B^*_{1,3}
\end{bmatrix} = \begin{bmatrix}
\beta_1 + w_1 & \beta_1\delta_1 + 1 - w_1 \\
\beta_1\delta_1 + 1 - w_1 & \beta_1\delta_1^2 + 1 - \delta_1
\end{bmatrix}^{-1} \begin{bmatrix}
\gamma(1/\beta_1 + w_1) + B_{0,1} - w_1B_{0,2} - (1 - w_1)B_{0,3} \\
\gamma(1 - w_1) + \delta_1B_{0,1} - (1 - w_1)B_{0,2} - (w_1 - (1 - \delta_1))B_{0,3}
\end{bmatrix},$$

$$= \frac{1}{1 + \beta_1 + \beta_1\delta_1} \begin{bmatrix}
(B_{0,1} - B_{0,2}) - \beta_1\delta_1(B_{0,2} - B_{0,3}) \\
(B_{0,1} - B_{0,3}) + \beta_1(B_{0,2} - B_{0,3})
\end{bmatrix} + \frac{\gamma}{\beta_1(1 + \beta_1 + \beta_1\delta_1)(w_1 + \delta_1w_1 - 1)} \begin{bmatrix}
w_1(1 + \beta_1(1 + \beta_1\delta_1)(1 + \delta_1)) + (\beta_1(\delta_1^2 - 1) + \delta_1(1 - \beta_1^2) - 1) \\
w_1(1 - \beta_1^2(1 + \delta_1)) + (\beta_1^2 - \beta_1\delta_1 - 1)
\end{bmatrix},$$

(A78)
and optimal time 1 taxes \( r_1^* = B_{0,1} - B_{1,2}^* \beta_1 - B_{1,3}^* \beta_1 \delta_1 \) are

\[
r_1^* = \frac{B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}}{1 + \beta_1 + \beta_1 \delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1}.
\] (A79)

In both (A78) and (A79), the second term reflects the government’s optimal response to money demand whereas the first term reflects a pure tax-smoothing motive (e.g., the first term in (A79) is the constant tax rate such that the present value of taxes equals the present value of future debt obligations). These tax-smoothing and money-creation motives decouple neatly in this model because the government has the option to re-optimize at time 2.\(^5\) Finally, tedious algebra shows that the minimized time-1 objective function takes the form:

\[
V_1(B_{0,1}, B_{0,2}, B_{0,3}, \beta_1, \delta_1) = \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} \tau_1^2 - \gamma \cdot M_0 + \beta_1 \cdot E \left[ V_2(B_{0,2} + B_{1,2}, B_{0,3} + B_{1,3}, \beta_2) \right] \right] 
= \frac{1}{2} \frac{(B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3})^2}{1 + \beta_1 + \beta_1 \delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3})
\]

For simplicity, we have omitted a term from (A80) that does not depend on initial maturity choices.

**The government’s time-0 problem:** Let \( D = B_{0,1} + B_{0,2} + B_{0,3} \) denote the total amount of debt issued at time 0 and let \( S = B_{0,1}/D \) and \( L = B_{0,3}/D \) denote the fraction of debt that is short- and long-term. Straightforward manipulations allow us to successively rewrite the time 0 problem as:

\[
\min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} \tau_1^2 - \gamma \cdot M_0 + \beta_1 \cdot E \left[ V_1(B_{0,1}, B_{0,2}, B_{0,3}, \beta_1, \delta_1) \right] \right]
= \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} (G - B_{0,1} - B_{0,2} - B_{0,3})^2 - \gamma \cdot B_{0,1} \right]
\]

\[
+ \beta_1 \cdot E \left[ \frac{1}{2} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3})^2 - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right]
\]

\[
= \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} (G - D)^2 - \gamma D \left((1 + A_4)S + A_2(1 - S - L) + A_4L \right) \right]
+ \frac{1}{2} \beta_1 \cdot E \left[ \tau_1^2 - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right]
\]

\[
= \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} (G - D)^2 - \gamma D \left(1 + A_4 \right) \left((1 + A_4)S + A_2(1 - S - L) + A_4L \right) \right]
+ \frac{1}{2} \beta_1 \cdot E \left[ \tau_1^2 - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right]
\]

\[
= \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} (G - D)^2 - \gamma D \left((1 + A_4)S + A_2(1 - S - L) + A_4L \right) \right]
+ \frac{1}{2} \beta_1 \cdot E \left[ \tau_1^2 - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right]
\]

\[
= \min_{b_{0,1}, b_{0,2}, b_{0,3}} \left[ \frac{1}{2} (G - D)^2 - \gamma D \left((1 + A_4)S + A_2(1 - S - L) + A_4L \right) \right]
+ \frac{1}{2} \beta_1 \cdot E \left[ \tau_1^2 - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right]
\]

where

\(^5\) In addition, the linear specification for money utility implies that the marginal monetary services from each additional unit of short-term debt do not depend on the existing stock of short-term debt. With non-linear money utility the tax-smoothing and money demand motives would interact in indirect ways because the marginal money benefit would depend on past issuance and, hence, on past interest rate shocks.
reflect the dispersion of $\beta$, the co-movement between $\beta$ and $\delta$, and the dispersion of $\delta$, respectively. (Note that equation (A81) is the natural generalization of equation (A22).) For instance, $c > 0$ means that spot ($\beta$) and forward short rates ($\delta$) tend to move together at time 1. Naturally, higher level of $b$ and $d$ reflect greater uncertainty about spot and forward short rates. Furthermore

$$A_1 = E \left[ \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad A_2 = E \left[ \frac{\beta_1 (1 + \beta_1)}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad \text{and} \quad A_3 = E \left[ \frac{\beta_1 \delta_1 (1 + \beta_1)}{1 + \beta_1 + \beta_1 \delta_1} \right] \quad (A83)$$

reflect the expected present value of future monetary services associated with issuing an additional unit of short-, medium-, and long-term debt at time 0, respectively.

**The first order conditions for $D$, $S$, and $L$ are**

$$0 = -(G - D) - \gamma ((1 + A_1)S + A_2 (1 - S - L) + A_3 L) + D \left( \frac{1}{3} + b \left( S - \frac{1}{3} \right)^2 - 2c \left( S - \frac{1}{3} \right) \left( L - \frac{1}{3} \right) + d \left( L - \frac{1}{3} \right)^2 \right).$$

$$0 = -\gamma D(1 + A_1 - A_2) + D^2 \left( b \left( S - \frac{1}{3} \right) - c \left( L - \frac{1}{3} \right) \right)$$

$$0 = -\gamma D(A_3 - A_2) + D^2 \left( -c \left( S - \frac{1}{3} \right) + d \left( L - \frac{1}{3} \right) \right) \quad (A84)$$

Under the assumption that the second order condition for this problem is satisfied, it is straightforward to show that $bd > c^2$. Assuming that $b > 0$ and $d > 0$, the solution takes the form

$$D^* = \frac{3}{4} \left( G + \gamma \left[ \frac{1}{3} (1 + A_1) + \frac{1}{3} A_2 + \frac{1}{3} A_3 \right] \right)$$

$$S^* = \frac{1}{3} + \gamma \frac{d + c (A_3 - A_2) + d (A_1 - A_2)}{D^*(bd - c^2)} \quad \text{(A85)}$$

$$L^* = \frac{1}{3} + \gamma \frac{c + b (A_1 - A_2) + c (A_1 - A_2)}{D^*(bd - c^2)}.$$

The average duration of debt issued at time 0 is

$$DUR = 1 \cdot S^* + 2 \cdot (1 - S^* - L^*) + 3 \cdot L^* = 2 + \frac{\gamma}{D^*} \frac{(c - d)(1 + A_1 - A_2) + (b - c)(A_1 - A_2)}{bd - c^2}. \quad (A86)$$

Thus, in the absence of monetary services ($\gamma = 0$) we again obtain the perfect tax-smoothing outcome ($r^*_0 = r^*_1 = r^*_2 = r^*_3 = G/4$) which is implemented by issuing a “consol” bond:
\( D' = (3/4)G \) and \( S' = L' = 1/3 \). However, with positive monetary services \((\gamma > 0)\) the government issues more short-term debt in order to satisfy household money demand: \( S' > 1/3 \).

It is easy to show that \( A_3 < A_2 < A_1 \) so long as \( \text{Cov}[\beta_1, \delta_1] > 0 \). However, in general, \( A_1 \approx A_2 \approx A_3 \) for almost any plausible parameterization of the two interest rate shocks. As a result, allowing for interim monetary services has a modest effect on the choice of \( D \), but has little if any effect on the choice of \( S \) and \( L \)—i.e., on the optimal maturity structure of the debt. To simplify the analysis, we apply the approximation that \( A_1 \approx A_2 \approx A_3 \), and obtain:

\[
S^* = \frac{1}{3} + \gamma \frac{1}{D'(b - c^2 / d)} \quad \text{and} \quad L^* = \frac{1}{3} + \gamma \frac{c / d}{D'(b - c^2 / d)}. \tag{A87}
\]

Equation (A87) shows that \( \partial S^* / \partial b < 0 \), \( \partial S^* / \partial d < 0 \), \( \partial S^* / \partial c < 0 \), \( \partial L^* / \partial b < 0 \), \( \partial L^* / \partial d < 0 \), and \( \partial L^* / \partial c > 0 \). Thus, the government issues more short-term debt when uncertainty about spot or forward rates is lower (i.e., \( b \) or \( d \) is lower). However, a larger absolute correlation between these two shocks enables the government to better hedge its interest rate exposure and ultimately take on more roll-over risk (i.e., a larger value of \( c \) is associated with a higher value of \( S \)). In the natural case where \( c > 0 \), this is accomplished via a “barbell” strategy in which the government issues lots of short- and long-term debt at time 0, but little if any medium-term debt. Indeed the government may even choose to lend on an intermediate-dated basis (i.e., we may have \( 1 - S - L < 0 \)).

The intuition is that this barbell strategy enables the government to hedge the roll-over risk that is created by deviating from the “consol” solution by issuing larger amounts short-term debt at time 0. To better see the intuition, note that time 1 taxes are

\[
r^*_1 = D \frac{S + \beta_i (1 - S - L) + \beta_i \delta_i L}{1 + \beta_i + \beta_i \delta_i} - \gamma \cdot \frac{1 + \beta_i}{1 + \beta_i + \beta_i \delta_i}. \tag{A88}
\]

so that

---

6 The expressions given in (A87) obtain exactly under the assumption that households derive no monetary services from short-term debt at time 1 and 2—i.e., they only derive utility from monetary services at time 0.
\[
\frac{\partial \tau^*_1}{\partial \beta_1} = D \frac{\delta (L - S) - 2(S - (1 - L)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} - \gamma \cdot \frac{\delta_i}{(1 + \beta_1 + \beta_1 \delta_1)^2}, \\
\frac{\partial \tau^*_1}{\partial \delta_1} = D \frac{\beta_1 (L - S) + 2 \beta_1 (L - (1 - S)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} + \gamma \cdot \frac{\beta_i (1 + \beta_1)}{(1 + \beta_1 + \beta_1 \delta_1)^2}.
\] (A89)

In an attempt to satisfy money demand at time 0, they government will choose \( S > L \) and \( S > (1 - L)/2 \) which implies that \( \frac{\partial \tau^*_1}{\partial \beta_1} < 0 \). In other words, the need to roll-over short-term debt means that taxes will be high when the short-term interest rate is high at time 1 (i.e., when \( \beta_1 \) is low). Thus, choosing a high value of \( S \) naturally exposes the government budget and hence taxes to \( \beta_1 \) shocks. What about the government’s exposure to \( \delta_1 \) shocks? Equation (A89) shows that by pursuing a barbell strategy in which \( L > (1 - S)/2 \) the government can reduce the exposure of taxes to \( \delta_1 \) or even create an offsetting exposure such that \( \frac{\partial \tau^*_1}{\partial \delta_1} > 0 \). When the correlation between \( \beta_1 \) and \( \delta_1 \) is high, this barbell strategy allows the government to hedge the exposure of time 1 taxes to interest rate shocks. This hedging strategy lowers the tax-smoothing costs associated with issuing additional short-term debt at time 0 which explains why \( S^* \) is increasing in \( |c| \).

How is this strategy implemented in terms of time 1 issuance? We have

\[
\begin{pmatrix}
B^*_1,1 \\
B^*_1,2 \\
B^*_1,3
\end{pmatrix} = \frac{D}{1 + \beta_1 + \beta_1 \delta_1} \begin{pmatrix}
2(S - (1 - L)/2) + 2 \beta_1 \delta_1 (L - (1 - S)/2) \\
(S - L) - 2 \beta_1 (L - (1 - S)/2)
\end{pmatrix} \\
+ \frac{\gamma}{\beta_i (1 + \beta_1 + \beta_1 \delta_1) (w_1 + \delta_i w_1 - 1)} \begin{pmatrix}
w_i (1 + \beta_1 (1 + \beta_1 \delta_1) (1 + \delta_1)) + \left( \beta_i (\delta^2_i - 1) + \delta_i (1 - \beta^2_i) - 1 \right) \\
w_i (1 - \beta^2_i (1 + \delta_1)) + \left( \beta^2_i - \beta_i \delta_i - 1 \right)
\end{pmatrix}. 
\] (A93)

Ignoring the terms that depend on \( \gamma \), we have

\[
\begin{pmatrix}
\frac{\partial B^*_1,1}{\partial \beta_1} \\
\frac{\partial B^*_1,2}{\partial \beta_1} \\
\frac{\partial B^*_1,3}{\partial \beta_1}
\end{pmatrix} = \frac{D - \delta_i (S - L) - 2(S - (1 - L)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} \begin{pmatrix}1 \\
1 \\
1
\end{pmatrix} \\
\begin{pmatrix}
\frac{\partial B^*_1,1}{\partial \delta_1} \\
\frac{\partial B^*_1,2}{\partial \delta_1} \\
\frac{\partial B^*_1,3}{\partial \delta_1}
\end{pmatrix} = \beta_i D \frac{-(S - L) + 2 \beta_1 (L - (1 - S)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} \begin{pmatrix}1 \\
1 \\
1
\end{pmatrix}. 
\] (A94)

Thus, the government must issue more of both maturities when \( \beta_1 \) is low (i.e., when short rates are high at time 1). How does issuance vary with shocks to \( \delta_1 \)? Since \( S > L \), there is the natural tendency for issuance to fall with \( \delta_1 \)—i.e., issuance is also higher when \( \delta_1 \) is lower (forward rates are higher) at time 1. However, to the extent that \( L > (1 - S)/2 \)—i.e., if the government is using a barbell strategy—issuance can actually increase in \( \delta_1 \). Intuitively, the government needs to issue more to repurchase long-term bonds when long-term bond prices are high (i.e., \( \delta_1 \) is high). Thus, by pursuing
a barbell strategy at time 0, the government can partially hedge its time 1 issuance and hence time taxes against interest rate shocks.

**Numerical example:** Here we explore a simple numerical example. We assume that:

- \( \Pr(\beta_1 = 1 - \sigma_\beta) = \Pr(\beta_1 = 1 + \sigma_\beta) = 1/2 \), so that \( \sigma_\beta \) is the standard deviation of \( \beta_1 \).
- \( \Pr(\delta_1 = \delta - \sigma_\delta) = \Pr(\delta_1 = \delta + \sigma_\delta) = 1/2 \), so that \( \sigma_\delta \) is the standard deviation of \( \delta_1 \).
- \( \Pr(\varepsilon_2 = 1 - \sigma_\varepsilon) = \Pr(\varepsilon_2 = 1 + \sigma_\varepsilon) = 1/2 \), so that \( \sigma_\varepsilon \) is the standard deviation of \( \varepsilon_2 \).
- \( \rho \) is the correlation between \( \beta_1 \) and \( \delta_1 \)—i.e., \( \Pr(\delta_1 = \delta + \sigma_\delta | \beta_1 = 1 + \sigma_\beta) = (1 + \rho)/2 \). Note that \( \delta = 1 - \rho \sigma_\beta \sigma_\delta \) since \( E[\beta_1 \delta_1] = 1 \) and that an increase in \( \rho \) also raises \( c \) in (A82).

Given these assumptions it is straightforward to calculate the model parameters given in (A82) and (A83) and then to compute the optimal debt structure given in (A85) or (A87) which imposes the approximation that \( A_i \approx A_2 \approx A_3 \). In the following table, we compute the optimal values of \( S \), \( L \), and \( DUR \) varying the level of money demand, \( \gamma \), and the correlation between the shocks to \( \beta_1 \) and \( \delta_1 \), \( \rho \).

Table A1 below computes the optimal values of \( S^*, L^*, \) and \( DUR \) for various values of \( \gamma \) and \( \rho \) using equations (A88) based on the parameters defined in (A82) and (A83). Computations based on (A87) yield nearly identical results. The computations in Table A1 assume that \( G = 1 \) and that \( \sigma_\beta = \sigma_\delta = \sigma_\varepsilon = 30\% \). The table shows that \( S^* \) is increasing in both \( \gamma \) and \( \rho \), \( L^* \) is increasing in both \( \gamma \) and \( \rho \) for \( \rho > 0 \) (\( L^* \) is decreasing in \( \gamma \) for \( \rho = 0 \) which translates into a tiny negative value for \( c \)); and \( DUR \) is increasing in \( \rho \) for \( \gamma > 0 \).
Table A1. Numerical Example. This table computes the optimal values of $S^*$, $L^*$, and $DUR$ for various values of $\gamma$ and $\rho$ using equations (A88) based on the parameters defined in (A82) and (A83). Computations based on (A87) yield nearly identical results. The computations in Table A1 assume that $G = 1$ and that $\sigma_\gamma = \sigma_\delta = \sigma_\epsilon = 30\%$. The table shows that $S^*$ is increasing in both $\gamma$ and $\rho$. $L^*$ is increasing in both $\gamma$ and $\rho$ for $\rho > 0$ ($L^*$ is decreasing in $\gamma$ for $\rho = 0$ which translates into a tiny negative value for $c$; and $DUR$ is increasing in $\rho$ for $\gamma > 0$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.0%</th>
<th>33.0%</th>
<th>66.0%</th>
<th>80.0%</th>
<th>90.0%</th>
<th>95.0%</th>
<th>99.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
</tr>
<tr>
<td>0.01%</td>
<td>33.76%</td>
<td>33.79%</td>
<td>34.00%</td>
<td>34.34%</td>
<td>35.16%</td>
<td>36.82%</td>
<td>50.10%</td>
</tr>
<tr>
<td>0.02%</td>
<td>34.19%</td>
<td>34.25%</td>
<td>34.67%</td>
<td>35.34%</td>
<td>36.99%</td>
<td>40.30%</td>
<td>66.86%</td>
</tr>
<tr>
<td>0.03%</td>
<td>34.61%</td>
<td>34.71%</td>
<td>35.34%</td>
<td>36.34%</td>
<td>38.81%</td>
<td>43.78%</td>
<td>83.62%</td>
</tr>
<tr>
<td>0.04%</td>
<td>35.04%</td>
<td>35.17%</td>
<td>36.01%</td>
<td>37.34%</td>
<td>40.64%</td>
<td>47.27%</td>
<td>100.38%</td>
</tr>
<tr>
<td>0.05%</td>
<td>35.47%</td>
<td>35.63%</td>
<td>36.68%</td>
<td>38.34%</td>
<td>42.46%</td>
<td>50.75%</td>
<td>117.13%</td>
</tr>
<tr>
<td>0.06%</td>
<td>35.90%</td>
<td>36.09%</td>
<td>37.35%</td>
<td>39.35%</td>
<td>44.29%</td>
<td>54.23%</td>
<td>133.88%</td>
</tr>
<tr>
<td>0.07%</td>
<td>36.32%</td>
<td>36.55%</td>
<td>38.02%</td>
<td>40.35%</td>
<td>46.11%</td>
<td>57.71%</td>
<td>150.62%</td>
</tr>
<tr>
<td>0.08%</td>
<td>36.75%</td>
<td>37.01%</td>
<td>38.69%</td>
<td>41.35%</td>
<td>47.94%</td>
<td>61.19%</td>
<td>167.37%</td>
</tr>
<tr>
<td>0.09%</td>
<td>37.18%</td>
<td>37.47%</td>
<td>39.36%</td>
<td>42.35%</td>
<td>49.76%</td>
<td>64.67%</td>
<td>184.10%</td>
</tr>
<tr>
<td>0.10%</td>
<td>37.60%</td>
<td>37.93%</td>
<td>40.03%</td>
<td>43.35%</td>
<td>51.58%</td>
<td>68.15%</td>
<td>200.84%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.0%</th>
<th>33.0%</th>
<th>66.0%</th>
<th>80.0%</th>
<th>90.0%</th>
<th>95.0%</th>
<th>99.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
</tr>
<tr>
<td>0.01%</td>
<td>33.32%</td>
<td>33.46%</td>
<td>33.76%</td>
<td>34.14%</td>
<td>35.06%</td>
<td>36.89%</td>
<td>51.49%</td>
</tr>
<tr>
<td>0.02%</td>
<td>33.30%</td>
<td>33.58%</td>
<td>34.18%</td>
<td>34.95%</td>
<td>36.78%</td>
<td>40.44%</td>
<td>69.64%</td>
</tr>
<tr>
<td>0.03%</td>
<td>33.28%</td>
<td>33.71%</td>
<td>34.60%</td>
<td>35.75%</td>
<td>38.51%</td>
<td>43.99%</td>
<td>87.78%</td>
</tr>
<tr>
<td>0.04%</td>
<td>33.26%</td>
<td>33.83%</td>
<td>35.02%</td>
<td>35.95%</td>
<td>38.56%</td>
<td>40.23%</td>
<td>105.92%</td>
</tr>
<tr>
<td>0.05%</td>
<td>33.25%</td>
<td>33.96%</td>
<td>35.44%</td>
<td>37.37%</td>
<td>41.96%</td>
<td>51.09%</td>
<td>124.06%</td>
</tr>
<tr>
<td>0.06%</td>
<td>33.23%</td>
<td>34.09%</td>
<td>35.86%</td>
<td>38.17%</td>
<td>43.68%</td>
<td>54.64%</td>
<td>142.20%</td>
</tr>
<tr>
<td>0.07%</td>
<td>33.21%</td>
<td>34.21%</td>
<td>36.28%</td>
<td>38.98%</td>
<td>45.41%</td>
<td>58.19%</td>
<td>160.33%</td>
</tr>
<tr>
<td>0.08%</td>
<td>33.19%</td>
<td>34.34%</td>
<td>36.70%</td>
<td>39.79%</td>
<td>47.13%</td>
<td>61.74%</td>
<td>178.45%</td>
</tr>
<tr>
<td>0.09%</td>
<td>33.18%</td>
<td>34.46%</td>
<td>37.13%</td>
<td>40.59%</td>
<td>49.76%</td>
<td>64.67%</td>
<td>196.58%</td>
</tr>
<tr>
<td>0.10%</td>
<td>33.16%</td>
<td>34.59%</td>
<td>37.55%</td>
<td>41.40%</td>
<td>51.58%</td>
<td>68.15%</td>
<td>214.70%</td>
</tr>
</tbody>
</table>

$S^* = \text{Initial Short-Term Share}$

$L^* = \text{Initial Long-term Share}$

$DUR = \text{Average Duration}$