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ON THE PRICING OF CONTINGENT CLAIMS AND THE MODIGLIANI–MILLER THEOREM*

Robert C. MERTON

Sloan School of Management, M.I.T., Cambridge, MA 02139, U.S.A.

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A general formula is derived for the price of a security whose value under specified conditions is a known function of the value of another security. Although the formula can be derived using the arbitrage technique of Black and Scholes, the alternative approach of continuous-time portfolio strategies is used instead. This alternative derivation allows the resolution of some controversies surrounding the Black and Scholes methodology. Specifically, it is demonstrated that the derived pricing formula must be continuous with continuous first derivatives, and that there is not a 'pre-selection bias' in the choice of independent variables used in the formula. Finally, the alternative derivation provides a direct proof of the Modigliani–Miller theorem even when there is a positive probability of bankruptcy.

1. Introduction

The theory of portfolio selection in continuous-time has as its foundation two assumptions: (1) the capital markets are assumed to be open at all times, and therefore, economic agents have the opportunity to trade continuously and (2) the stochastic processes generating the state variables can be described by diffusion processes with continuous sample paths.[1] If these assumptions are accepted, then the continuous-time model can be used to derive equilibrium security prices.[2] The pricing formulas derived by this method will in general require as minimum inputs estimates of the price of risk, the covariance of the security's cash flows with the market, and the expected cash flows. These numbers are difficult to estimate. However, it is not always necessary to have these numbers to price a security.

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[1]For references to the mathematics of diffusion processes and their applications in economics, see the bibliographies in Merton (1971) and (1973b).

[2]See Merton (1973a) and (1975).

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In a seminal paper, Black and Scholes (1973) used the continuous-time analysis to derive a formula for pricing common stock call options.[3] Although their derivation uses the same assumptions and analytical tools used in the continuous-time portfolio analysis, the resulting formula expressed in terms of the price of the underlying stock does not require as inputs expected returns, expected cash flows, the price of risk, or the covariance of the returns with the market. In effect, all these variables are implicit in the stock's price. Because expected returns and market covariances are not part of the inputs, the Black–Scholes evaluation formula is robust with respect to a reasonable amount of heterogeneity of expectations among investors, and because the required inputs are for the most part observable, the formula is testable. All of this has created substantial interest in extending their analysis to the evaluation of other types of securities.

The essential reason that the Black–Scholes pricing formula requires so little information as inputs is that the call option is a security whose value on a specified future date is uniquely determined by the price of another security (the stock). As such, a call option is an example of a *contingent claim*. While call options are very specialized financial instruments, Black and Scholes and others[4] recognized that the same analysis could be applied to the pricing of corporate liabilities generally where such liabilities were viewed as claims whose values were contingent on the value of the firm. Moreover, whenever a security's return structure is such that it can be described as a contingent claim, the same technique is applicable.

In section 2 of this paper, I derive a general formula for the price of a security whose value under specified conditions is a known function of the value of another security. Although the formula can be derived using the arbitrage technique employed by Merton (1974) to derive the price of risky debt, an alternative approach is used to demonstrate that the resulting formula will obtain even if institutional restrictions prohibit arbitrage.

Because the formula is often used to evaluate corporate liabilities as a function of the value of the firm, it is important to know conditions under which the value of the firm will not be affected by the form of its capital structure. In section 3, the Modigliani–Miller Theorem (1958) that the value of the firm is invariant to its capital structure is extended to the case where there is a positive probability of bankruptcy.

[3]A call option gives its owner the right to buy a specified number of shares of a given stock at a specified price (the 'exercise price') on or before a specified date (the 'expiration date').

[4]The literature based on the Black–Scholes analysis has expanded so rapidly that rather than attempt to list individual published articles and works-in-progress, I refer the reader to a survey article by Smith (1976).

2. A general derivation of a contingent claim price

To develop the contingent-claim pricing model, I make the following assumptions:

(A.1) *'Frictionless markets'*

There are no transactions costs or taxes. Trading takes place continuously in time. Borrowing and shortselling are allowed without restriction. The borrowing rate equals the lending rate.

(A.2) *Riskless asset*

There is a riskless asset whose rate of return per unit time is known and constant over time. Denote this return rate by r.

(A.3) *Asset #1*

There is a risky asset whose value at any point in time is denoted by $V(t)$. The dynamics of the stochastic process generating $V(\ )$ over time is assumed to be describable by a diffusion process with a formal stochastic differential equation representation of

$$\mathrm{d}V = [\alpha V - D_1(V, t)]\mathrm{d}t + \sigma V \mathrm{d}Z,$$

where α = instantaneous expected rate of return on the asset per unit time; σ^2 = instantaneous variance per unit time of the rate of return; $D_1(V, t)$ = instantaneous payout to the owners of the asset per unit time; $\mathrm{d}Z \equiv$ standard Wiener process; α can be generated by a stochastic process of a quite general type, and σ^2 is restricted to be at most a function of V and t.

(A.4) *Asset #2*

There is a second risky asset whose value at any date t is denoted by $W(t)$ with the following properties: For $0 \leq t < T$, its owners will receive an instantaneous payout per unit time, $D_2(V, t)$. For any $t(0 \leq t < T)$, if $V(t) = \overline{V}(t)$, then the value of the second asset is given by: $W(t) = f[\overline{V}(t), t]$, where f is a known function. For any $t(0 \leq t < T)$, if $V(t) = \underline{V}(t)$, then the value of the second asset is given by $[\underline{V}(t) < \overline{V}(t)]$: $W(t) = g[\underline{V}(t), t]$, where g is a known function. For $t = T$, the value of the second asset is given by: $W(T) = h[V(T)]$. Asset #2 will be called a contingent claim, contingent on the value of Asset #1.

(A.5) *Investor preferences and expectations*

It is assumed that investors prefer more to less. It is assumed that investors agree upon σ^2, but it is *not* assumed that they necessarily agree on α.

(A.6) *Other*

There can be as many or as few other assets or securities as one likes. Market prices need not be general equilibrium prices. The constant interest rate and most of the 'frictionless' market assumptions are not essential to the development of the model but are chosen for expositional convenience. The critical assumptions are continuous-trading opportunities and the dynamics description for Asset #1.

If it is assumed that the value of Asset #2 can be written as a twice-continuously differentiable function of the price of Asset #1 and time, then the pricing formula for Asset #2 can be derived by the same procedure used in Merton (1974, pp. 451–453) to derive the value of risky debt. If $W(t) = F[V(t), t]$ for $0 \leq t \leq T$ and for $\underline{V}(t) \leq V(t) \leq \bar{V}(t)$, then to avoid arbitrage, F must satisfy the linear partial differential equation

$$0 = \tfrac{1}{2}\sigma^2 V^2 F_{11} + [rV - D_1]F_1 - rF + F_2 + D_2, \tag{1}$$

where subscripts on F denote partial derivatives with respect to its two explicit arguments, V and t. Inspection of (1) shows that in addition to V and t, F will depend on σ^2 and r. However, F does *not* depend on the expected return on Asset #1, α, and it does *not* depend on the characteristics of other assets available in the economy. Moreover, investors' preferences do not enter the equation either.

To solve (1), boundary conditions must be specified. From (A.4), we have that

$$F[\bar{V}(t), t] = f[\bar{V}(t), t], \tag{2a}$$

$$F[\underline{V}(t), t] = g[\underline{V}(t), t], \tag{2b}$$

$$F[V, T] = h[V]. \tag{2c}$$

While the functions f, g, and h are required to solve for F, they are generally deducible from the terms of the specific contingent claim being priced. For example, the original case examined by Black and Scholes is a common stock call option with an exercise price of E dollars and an expiration date of T. If V is the value of the underlying stock, then the boundary conditions can be written as

$$F/V \leq 1 \quad \text{as} \quad V \to \infty, \tag{3a}$$

$$F[0, t] = 0, \tag{3b}$$

$$F[V, T] = \text{Max}\,[0, V - E], \tag{3c}$$

where (3a) is a regularity condition which replaces the usual boundary condition when $\bar{V}(t) = \infty$. Both (3a) and (3b) follow from limited liability and from the easy-to-prove condition that the underlying stock is always more valuable than the option. (3c) follows from the terms of the call option which establish the exact price relationship between the stock and option on the expiration date.[5]

Hence, (1) together with (2a)–(2c) provide the general equation for pricing contingent claims. Moreover, if the contingent claim is priced according to (1) and (2), then it follows that there is no opportunity for intertemporal arbitrage. I.e., the relative prices (W, V, r) are intertemporally consistent.

Suppose there exists a twice-continuously differentiable solution to (1) and (2). Because the derivation of (1) used the *assumption* that the pricing function satisfies this condition, it is possible that some other solution exists which does not satisfy this differentiability condition. Indeed, in discussing the Black–Scholes solution to the call option case, Smith[6] points out that there are an infinite number of continuous solutions to eqs. (1) and (3) which have discontinuous derivatives at only one interior point although the Black–Scholes solution is the only solution with continuous derivatives. He goes on to state 'the economics of the option pricing problem would suggest that the solution be continuous, but there is no obvious argument that it be differentiable everywhere'.

The following alternative derivation is a direct proof that if a twice-continuously differential solution to (1) and (2) exists, then to rule out arbitrage, it must be the pricing function.

Let F be the formal twice-continuously differentiable solution to eq. (1) with boundary conditions (2). Consider the continuous-time portfolio strategy where the investor allocates the fraction $w(t)$ of his portfolio to Asset #1 and $[1-w(t)]$ to the riskless asset. Moreover, let the investor make net 'withdrawals' per unit time (for example, for consumption) of $C(t)$. If $C(t)$ and $w(t)$ are right-continuous functions and $P(t)$ denotes the value of the investor's portfolio, then I have shown elsewhere[7] that the dynamics for the value of the portfolio, P, will satisfy the stochastic differential equation

$$\mathrm{d}P = \{[w(\alpha-r)+r]P-C\}\mathrm{d}t+w\sigma P\mathrm{d}Z. \tag{4}$$

Suppose we pick the particular portfolio strategy with

$$w(t) = F_1[V, t]V(t)/P(t), \tag{5}$$

where F_1 is the partial derivative of F with respect to V, and the 'consumption'

[5]In some cases, either $\bar{V}(t)$ or $\underline{V}(t)$ must be determined simultaneously with the solution of eq. (1) for F. Two examples are the American call and put options on a dividend-paying stock with the potential for early exercise. In such cases, there is usually an additional boundary condition imposed on the derivative of F which allows just enough 'over-specification' to determine $\bar{V}$. See Merton (1973b, pp. 173–174) for discussion. The structural definition of Asset #2 can be easily adjusted to include these cases.

[6]See Smith (1976, p. 23, footnote 21).

[7]See Merton (1971, p. 379).

strategy,

$$C(t) = D_2(V, t). \tag{6}$$

By construction, F_1 is continuously-differentiable, and hence is a right-continuous function. Substituting from (5) and (6) into (4), we have that

$$\mathrm{d}P = F_1\mathrm{d}V + \{F_1(D_1 - rV) + rP - D_2\}\mathrm{d}t, \tag{7}$$

where $\mathrm{d}V$ is given in (A.3).

Since F is twice-continuously differentiable, we can use Ito's Lemma[8] to express the stochastic process for F as

$$\mathrm{d}F = [\tfrac{1}{2}\sigma^2 V^2 F_{11} + (\alpha V - D_1)F_1 + F_2]\mathrm{d}t + F_1\sigma V\mathrm{d}Z. \tag{8}$$

But F satisfies eq. (1). Hence, we can rewrite (8) as

$$\mathrm{d}F = F_1\mathrm{d}V + \{F_1(D_1 - rV) + rF - D_2\}\mathrm{d}t. \tag{9}$$

Let $Q(t) \equiv P(t) - F[V(t), t]$. Then, from (7) and (9), we have that

$$\begin{aligned} \mathrm{d}Q &= \mathrm{d}P - \mathrm{d}F \\ &= r(P - F)\mathrm{d}t \\ &= rQ\mathrm{d}t. \end{aligned} \tag{10}$$

But, (10) is a non-stochastic differential equation with solution

$$Q(t) = Q(0)\mathrm{e}^{rt}, \tag{11}$$

for any time t and where $Q(0) \equiv P(0) - F[V(0), 0]$. Suppose the initial amount invested in the portfolio, $P(0)$, is chosen equal to $F[V(0), 0]$. Then from (11) we have that

$$P(t) = F[V(t), t]. \tag{12}$$

By construction, the value of Asset #2, $W(t)$, will equal F at the boundaries $V(t)$ and $\bar{V}(t)$ and at the termination date T. Hence, from (12), the constructed portfolio's value, $P(t)$, will equal $W(t)$ at the boundaries. Moreover, the interim 'payments' or withdrawals available to the portfolio strategy, $D_2[V(t), t]$, are identical to the interim payments made to Asset #2.

[8] See Merton (1971) for a discussion of Itô's Lemma and stochastic differential equations.

Therefore, if $W(t) > P(t)$, then the investor could short-sell Asset #2; proceed with the prescribed portfolio strategy including all interim payments; and be guaranteed a positive return on zero investment. I.e., there would be an arbitrage opportunity. If $W(t) < P(t)$, then the investor could essentially 'short-sell' the prescribed portfolio strategy; use the proceeds to buy Asset #2; and again be guaranteed a positive return on zero investment. If institutional restrictions prohibit arbitrage,[9] then a similar argument can be applied using the principle that no security should be priced so as to 'dominate' another security.[10] Hence, $W(t)$ must equal $F[V(t), t]$.

While this method of proof may appear to be very close to the original derivation, unlike the original derivation, it does not *assume* that the dynamics of Asset #2 can be described by an Ito process, and therefore, it does not assume that Asset #2 has a smooth pricing function. Indeed, the portfolio stategy described by (5) and (6) involves only combinations of Asset #1 and the riskless asset, and therefore, does not even require that Asset #2 exists! The connection between the portfolio strategy and Asset #2 is that if Asset #2 exists, then the price of Asset #2 must equal $F[V(t), t]$, or else, there will be an opportunity for intertemporal arbitrage.

Not only does this alternative derivation provide the 'obvious argument' why such pricing functions must be differentiable everywhere, but it also can be used to resolve other issues that have been raised about results derived using this type of analysis. In the next section, two of the more important issues are resolved.

3. On the Modigliani–Miller theorem with bankruptcy

In an earlier paper (1974, p. 460), I proved that in the absence of bankruptcy costs and corporate taxes, the Modigliani–Miller theorem (1958) obtains even in the presence of bankruptcy. In a comment on this earlier paper, Long (1974) has asserted that my method of proof was 'logically incoherent'. Rather than debate over the original proof's validity, the method of derivation used in the previous section provides an immediate alternative proof.

Let there be a firm with two corporate liabilities: (1) a single homogeneous debt issue and (2) equity. The debt issue is promised a continuous coupon payment per unit time, C, which continues until either the maturity date of the bond, T, or until the total assets of the firm reach zero. The firm is prohibited by the debt indenture from issuing additional debt or paying dividends. At the maturity date, there is a promised principal payment of B to the debtholders. In the event the payment is not made, the firm is defaulted to the debtholders, and the equityholders receive nothing. If $S(t)$ denotes the value of the firm's equity and

[9] One example would be restrictions on short-sales.

[10] See Merton (1973b, p. 143) and Smith (1976, p. 7) for a discussion of 'dominance' in this context.

$D(t)$ the value of the firm's debt, then the value of the (levered) firm, $V_L(t)$, is identically equal to $S(t)+D(t)$. Moreover, in the event that the total assets of the firm reach zero, $V_L(t) = S(t) = D(t) = 0$ by limited liability. Also, by limited liability, $D(t)/V_L(t) \leq 1$.

Consider a second firm with initial assets and an investment policy identical to those of the levered firm. However, the second firm is all-equity financed with total value equal to $V(t)$. To ensure the identical investment policy including scale, it follows from the well-known accounting identity that the net payout policy of the second firm must be the same as for the first firm. Hence, let the second firm have a dividend policy that pays dividends of C per unit time until either date T or until the value of its total assets reach zero (i.e., $V = 0$). Let the dynamics of the firm's value be as posited in (A.3) where $D_1(V, t) = C$ for $V > 0$ and $D_1 = 0$ for $V = 0$.

Let $F[V, t]$ be the formal twice-continuously differentiable solution to eq. (1) subject to the boundary conditions: $F[0, t] = 0$; $F[V, t]/V \leq 1$; and $F[V(T), T] = \text{Min}\,[V(T), B]$. Consider the dynamic portfolio strategy of investing in the all-equity firm and the riskless asset according to the 'rules' (5) and (6) of section 2 where $C(t)$ is taken equal to C. If the total initial amount invested in the portfolio, $P(0)$, is equal to $F[V(0), 0]$, then from (12), $P(t) = F[V(t), t]$.

Because both the levered firm and the all-equity firm have identical investment policies including scale, it follows that $V(t) = 0$ if and only if $V_L(t) = 0$. And it also follows that on the maturity date T, $V_L(T) = V(T)$.

By the indenture conditions on the levered firm's debt, $D(T) = \text{Min}\,[V_L(T), B]$. But since $V(T) = V_L(T)$ and $P(T) = F[V(T), T]$, it follows that $P(T) = D(T)$. Moreover, since $V_L(t) = 0$ if and only if $V(t) = 0$, it follows that $P(t) = F[0, t] = D(t) = 0$ in that event.

Thus, by following the prescribed portfolio strategy, one would receive interim payments exactly equal to those on the debt of the levered firm. Moreover, on a specified future date, T, the value of the portfolio will equal the value of the debt. Hence, to avoid arbitrage or dominance, $P(t) = D(t)$.

The proof for equity follows along similar lines. Let $f[V, t]$ be the formal solution to eq. (1) subject to the boundary conditions: $f[0, t] = , f[V, t]/V \leq 1$; and $f[V(T), T] = \text{Max}\,[0, V(T)-B]$. Consider the dynamic portfolio strategy of investing in the all-equity firm and the riskless asset according to the 'rules' (5) and (6) of section 2 where $C(t)$ is now taken equal to zero. If the total initial amount invested in this portfolio, $p(0)$, is equal to $f[V(0), 0]$, then from (12), $p(t) = f[V(t), t]$.

As with debt, if $V(t) = 0$, then $p(t) = S(t) = 0$, and at the maturity date, $p(T) = \text{Max}\,[0, V(T)-B] = S(T)$.

Thus, by following this prescribed portfolio strategy, one would receive the same interim payments as those on the equity of the levered firm. On the maturity date, the value of the portfolio will equal the value of the levered firm's equity. Therefore, to avoid arbitrage or dominance, $p(t) = S(t)$.

If one were to combine both portfolio strategies, then the resulting interim payments would be C per unit time with a value at the maturity date of $V(T)$. I.e., both strategies together are the same as holding the equity of the unlevered firm. Hence, $f[V(t), t]+F[V(t), t] = V(t)$. But it was shown that $f[V(t), t]+F[V(t), t] = S(t)+D(t) \equiv V_L(t)$. Therefore, $V_L(t) = V(t)$, and the proof is completed.

While the proof was presented in the traditional context of a firm with a single debt issue, the proof goes through in essentially the same fashion for multiple debt issues or for 'hybrid securities' such as convertible bonds, preferred stock, or warrants.[11]

In his comment on my earlier paper Long (1974, p. 485) claims that the original derivation builds into the model that risky debt can only depend on the 'prespecified explanatory variables'. His point is that in fact, bond prices could depend on 'the price of beer'; 'the value of the market portfolio'; or 'the rate of inflation', but by assuming that the bond price depends only on the value of the firm, the market rate of interest, the volatility of the market value of the firm, and time until maturity, the derived model price rules out such additional dependencies. The derivation in section 2 did not assume that the value of Asset #2 depends only on these prespecified variables. The assumptions used are only the stated ones (A.1)–(A.6). Hence, given the current values of Asset #1, the only way that the price of beer, the market portfolio, or the rate of inflation can affect the price of Asset #2 is if they affect σ^2, r, or the boundary conditions. While it could be argued that *in fact*, σ^2 and r may depend on these other variables, such an argument would simply be a criticism of assumptions (A.2) and (A.3), and not of the derivation itself.

[11]In more complicated bond indentures, the restrictions may be in terms of accounting variables rather than market values. In such cases, the analysis requires that these accounting variables can be written as functions of the market values.

References

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–659.

Long, J.B., 1974, Discussion, Journal of Finance 29, 485–488.

Merton, R.C., 1971, Optimum consumption and portfolio rules in a continuous-time model, Journal of Economic Theory 3, 373–413.

Merton, R.C., 1973a, An intertemporal capital asset pricing model, Econometrica 41, 867–887.

Merton, R.C., 1973b, Theory of rational option pricing, Bell Journal of Economics and Management Science 4, 141–183.

Merton, R.C., 1974, On the pricing of corporate debt: The risk structure of interest rates, Journal of Finance 29, 449–470.

Merton, R.C., 1975, Theory of finance from the perspective of continuous time, Journal of Financial and Quantitative Analysis, 659–674.

Merton, R.C., 1976, Continuous-time portfolio theory and the pricing of contingent claims, Working Paper no. 881–76 (Sloan School of Management, M.I.T., Cambridge, MA).

Miller, M. and F. Modigliani, 1958, The cost of capital, corporation finance and the theory of investment, American Economic Review 48.

Smith, C.W., Jr., 1976, Option pricing: A review, Journal of Financial Economics 3, 3–51.