On the Role of the Wiener Process in Finance Theory and Practice: The Case of Replicating Portfolios

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1. Introduction

The core of finance theory is the study of the behavior of economic agents in allocating and deploying their resources, both spatially and across time, in an uncertain environment. Time and uncertainty are the central elements that influence financial behavior. The complexity of their interaction brings intrinsic excitement to the study of finance, since its sophisticated analytical tools are often required to capture the effects of this interaction. Indeed, the mathematical models of modern finance contain some beautiful applications of probability and optimization theory. But, of course, all that is beautiful in science need not also be practical. And surely, not all that is practical in science is beautiful. Here we have both. With all their seemingly abstruse mathematical complexity, the models of finance theory have nevertheless had a direct and significant influence on finance practice. Although not unique, this conjoining of intrinsic intellectual interest with extrinsic application is a prevailing theme of research in modern finance.

Sophisticated mathematical models and a strong influence on practice were not always hallmarks of finance theory. Indeed, the current wave of research in continuous-time mathematical finance began only about 25 years ago. However, the origins of much of the mathematics of finance can be traced to Louis Bachelier's magnificent dissertation on the theory of speculation. Completed at the Sorbonne in 1900, this work marks the twin births of both the continuous-time mathematics of stochastic processes and the continuous-time economics of option pricing. In analyzing the financial economics problem of option pricing, Bachelier provides two different derivations of the Fourier partial differential equation as the equation for the probability density of what is now known as a Wiener process or Brownian motion. In one of the derivations, he writes down what is now commonly called the Chapman-Kolmogorov convolution probability integral, which is surely among the earlier appearances of that integral in print. In the other derivation, he takes the limit of a

1991 Mathematics Subject Classification. Primary 90A09, 60H05, 60J40, 60J60.

1Much of this introductory section is taken from Merton (1994). Dedicated to the memory of Fischer Black, scholar, colleague, and friend.

2See Bernstein (1992) for carefully researched descriptions of several examples of this interplay between theory and practice that brought about some of the major financial innovations of the last few decades. See also Sanford (1993) for a practitioner prospective.
discrete-time binomial process to derive the continuous-time transition probabilities. Along the way, Bachelier also essentially developed the method of images (reflection) to solve for the probability function of a diffusion process with an absorbing barrier. This all took place five years before Einstein’s discovery of these same equations in his famous mathematical theory of Brownian motion. Not a bad performance for a thesis on which the first reader, Henri Poincaré, gave less than a top mark.

Nevertheless, as with Einstein’s 1905 mathematical theory, Bachelier’s earlier development of continuous-time stochastic processes was not entirely rigorous. Hence, we in finance join the countless others in the physical sciences as eternal beneficiaries of Norbert Wiener’s remarkable 1923 construction, which placed the subject on a firm and expanded foundation.

Completing the pre-1969 lineage of continuous-time finance, we have the work of Kiyoshi Itô (1951, 1987), whose formulation of the stochastic calculus is a fundamental tool of modern finance. Paul Samuelson’s 1965 rational theory of warrant pricing set the stage for later research in continuous-time finance and was the first economics/finance paper to apply the partial differential equations of diffusions since Bachelier’s work. He was also responsible for the rediscovery of Bachelier for the economics profession and much more. Henry McKean, Jr. (1965), in an appendix to Samuelson’s paper, set down the mathematics still used today for analyzing the free-boundary problem associated with early exercise of warrants.

A quantum jump in both the breadth and depth of the role of the Wiener process in finance theory came in 1969 with the formulation of the continuous-trading model of intertemporal portfolio selection. Although mathematically more complex, this formulation provides just enough additional specificity to produce both more precise theoretical solutions and more refined empirical hypotheses than can otherwise be derived from either its static or discrete-time dynamic counterparts. Perhaps the prime instance is the application of this model to refine the seminal Black-Scholes (1973) theory of option pricing, a theory that has had a profound impact on finance practice.

The basic insight underlying the Black-Scholes model is that a dynamic portfolio trading strategy in the stock can be found to hedge (or eliminate) the risk of holding an option on that stock. With its risk removed, the resulting hedged portfolio should earn an expected return equal to the riskless interest rate. Otherwise, trading the

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5D. Rich, Northeastern University, in private correspondence reports that Lord Kelvin discovered the method of images 20-30 years before Bachelier’s dissertation.

4See also McKean (1969). Merton (1992) contains a wide range of applications of Itô’s Lemma in finance as well as an extensive bibliography.


6See Merton (1992, Chs. 4, 5).
option would provide an excess-profit opportunity. Black and Scholes showed that the absence of such an opportunity is a sufficient condition to determine the price of the option.

In their initial formulation, Black and Scholes posited a trading strategy of short, discrete time intervals between revisions in the stock position that hedged the "systematic" or "beta" risk of the option. The resulting "zero-beta" portfolio should have an equilibrium expected return equal to the risk-free interest rate. In a subsequent refinement of their formulation, it was shown that in the limit of continuous trading, the hedged portfolio would have no uncertainty at all—a zero standard deviation in its return. It follows that the dynamic trading strategy in the stock must exactly replicate the returns on the option. Hence, to avoid arbitrage opportunities, the option price must always equal the value of this replicating portfolio. Since the absence of arbitrage opportunities is a necessary condition for any equilibrium model, the refined formulation is more robust with respect to the choice of equilibrium asset-pricing model. The "cost" of that robustness is the assumption of continuous trading. In the section to follow, we derive pricing formulas using this approach.

Black and Scholes, along with others, recognized that the replicating-portfolio methodology could be applied to a wide range of valuation problems, from the pricing of complex financial securities including corporate liabilities to the evaluation of loan guarantees and deposit insurance.

The subsequent scientific breakthroughs in financial modeling both shaped and were shaped by the extraordinary flow of financial innovation which coincided with revolutionary changes in the structure of world financial markets and institutions during the past two-and-a-half decades. And nowhere is this more apparent than in the area of contingent-claim or derivative securities.

Despite its name, the subject of derivative securities is not a likely topic to arise in a first calculus course, but as we will see, it could appear in a more advanced one in the stochastic calculus.

Contingent claims or derivatives are financial contracts or securities whose payoffs are contractly contingent on the prices of one or more other traded securities. Hence, the value of the derivative security depends on the price of the underlying traded securities. Examples are options and futures.

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For the detailed story of how Black and Scholes developed their model, see Bernstein (1992, Ch. 11).

Black and Scholes (1973) and Merton (1973).
2. Pricing of Contingent Claims: General Derivation\(^6\)

Let \( V = V(t) \) denote the price at time \( t \) of a limited-liability traded security, such as a share of stock, with posited dynamics given by

\[
dV = [\alpha V - D_1(V,t)] dt + \sigma V dZ
\]

where: \( \alpha \equiv \) instantaneous expected rate of return on the security; \( \sigma^2 \equiv \) instantaneous variance rate, which is assumed to depend, at most, on \( V(t) \) and \( t \) (i.e., \( \sigma^2 = \sigma^2(V, t) \)); \( dZ \) is a Wiener process; and \( D_1 \equiv \) dividend payment flow rate. With limited liability, to avoid arbitrage, \( V(t) = 0 \) for all \( t \geq t^* \) if \( V(t^*) = 0 \). Hence \( D_1 \) must satisfy \( D_1(0, t) = 0 \).

A contingent-claim \( (\text{or derivative}) \) security is a security with contractually determined payouts that can be described by functions of traded security prices and time.

**Terms of Contingent-Claim Security on \( V \)**

Let \( W(t) = \text{value (or price)} \) of a contingent-claim security at time \( t \):

\[
\begin{align*}
\text{If } V(t) & \geq \bar{V}(t) \text{ for } 0 \leq t < T, \text{ then } \ W(t) = f[V(t), t] \\
\text{If } V(t) & \leq \underline{V}(t) \text{ for } 0 \leq t < T, \text{ then } \ W(t) = g[V(t), t] \\
\text{If } t = T, \text{ then } \ W(T) = h[V(T)]
\end{align*}
\]

For \( 0 \leq t \leq T \), the security receives a payment flow rate specified by \( D_2(V,t) \). \( (2) \) is to be interpreted as follows: the first time that \( V(t) \geq \bar{V}(t) \) or \( V(t) \leq \underline{V}(t) \), the owner of the contingent claim must exchange the claim for cash according to schedule \( (2) \). If no such events occur for \( t < T \), then the security is redeemed at \( t = T \) for cash according to \( (2) \). \( T \) is called the *maturity date* (or expiration date, or redemption date) of the contingent claim. The contingent-claim security is essentially defined by specifying \( f, g, h, D_2, \) and \( T \). In some cases, the schedules or boundaries \( \bar{V}(t) \) and \( \underline{V}(t) \) are contractually specified; in others, they are determined endogenously as part of the valuation process.

By arbitrage restrictions, the contingent-claim security will have *limited liability* if and only if \( g \geq 0, h \geq 0, f \geq 0, \) and \( D_2(0, t) = 0 \).

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\(^6\)This derivation follows along the lines of Merton (1977, 1992).
Figure 1: Relevant Region of $V$: $V(t) \leq \bar{V}(t) \leq \overline{V}(t)$, $0 \leq t \leq T$

If (as we assume for the derivation) the boundaries $V(t)$, and $\overline{V}(t)$ are continuous functions, then because from (1), $V(t)$ has a continuous sample path through $t$, then because $V(t)$ has a continuous sample path in $t$ by (1), we have that (i) if $V(t) < V(\tilde{t})$ for some $\tilde{t}$, then there is a $\ell, t < \ell$, so that $V(\ell) = \overline{V}(\ell)$ and (ii) if $V(t) > \overline{V}(\tilde{t})$ for some $t$, then there is a $\ell, t < \ell$, so that $V(\ell) = \overline{V}(\ell)$. Hence, in this case, the inequalities for $V$ can be neglected in (2) and the only relevant region for analysis is $V(t) \leq V(t) \leq \overline{V}(t)$, $0 \leq t \leq T$. For expositional convenience, we also assume that the riskless interest rate, $r(t) = r$, is a constant over time.\textsuperscript{10}

2.1 Dynamic Hedging: The Production Technology for Creating Derivative Securities

To locate the derivation in a substantive framework, we posit a hypothetical financial intermediary that creates derivative securities for its customers by selling them contracts which are its obligation. It hedges the contractual liabilities so created by dynamically trading the underlying securities in a replicating strategy designed to replicate the cash flows of the contracts issued. Although surely a caricature, the description is not far removed from real-world practice.

Objective: find a feasible, continuous-trading portfolio strategy involving the traded security and the riskless security that has the following properties: if $P(t)$ denotes the value of the portfolio at time $t$, then for $0 \leq t \leq T$:

(i) at $t$, if $V(t) = \underline{V}(t)$, then $P(t) = g[V(t), t]$  

\textsuperscript{10}For the stochastic interest-rate case, see Merton (1973).
(ii) at \( t \), if \( V(t) = V(t) \), then \( P(t) = f(V(t), t) \)

(iii) for each \( t \), the payout rate on the portfolio is \( D_2(V, t) dt \)

(iv) at \( t = T \), \( P(T) = h[V(T)] \).

We call this replicating strategy 'portfolio (*)'.

A "cookbook" derivation of the production process for the financial intermediary to match the cash flows of the derivative security begins with a description of its quantitative ('quant') department activities. The first of those activities is to solve the following linear parabolic partial differential equation for \( F[V, t] \)

\[
0 = \frac{1}{2} \sigma^2(V, t) \frac{\partial^2 F}{\partial V^2} + \left[ rV - D_1(V, t) \right] \frac{\partial F}{\partial V} - rF[V, t] + F_t[V, t] + D_2(V, t)
\]

(3)

subject to the boundary conditions: for \( V(t) \leq V \leq V(t) \) and \( t < T \),

\[
F[V(t), t] = f(V(t), t) \geq 0
\]

(4)

\[
F[V(t), t] = g(V(t), t) \geq 0
\]

(5)

\[
F[V, T] = h[V] \geq 0
\]

(6)

where \( F_t = \frac{\partial F}{\partial t}, F_v = \frac{\partial F}{\partial V} \) and \( F_2 = \frac{\partial^2 F}{\partial V \partial t} \). Note that the non-negativity conditions in (4-6) together with \( D_2(0, t) = 0 \) implies that we are considering a contingent claim with limited liability. As a mathematical question, this is a "well-posed" problem, and a solution to (3-6) exists and is unique.

Having solved for the function \( F[V, t] \), the quant department has the prescribed tasks at each time \( t \) (0 \leq t \leq T) to:

(i) ask the trading desk for the current price of the traded security, \( V(t) \);

(ii) from the solution to (3-6) compute

\[
M(t) = F_t[V(t), t] V(t);
\]

(iii) tell the trading desk that portfolio strategy (*) requires that \$M(t)\) of the traded security be invested for the period \( t \) to \( t + dt \).
(iv) compute \( Y(t) = F[V(t), t] \) and store \( \tilde{Y}(t) \) in the intermediary's data files for (later) analysis of the time series (i.e., stochastic process) \( Y(t) \).

The prescription for the trading-desk activities of the intermediary is as follows: At time \( t = 0 \), give the trading desk \( S'P(0) \) as an initial funding (investment) for portfolio (*) which will contain the traded security and the riskless security. Let \( P(t) \) denote the value of portfolio (*) at \( t \), after having made any prescribed cash distribution (payment) from the portfolio. The trading desk has the job at each time \( t \) (\( 0 \leq t \leq T \)) to:

(a) determine the current price of the traded security, \( V(t) \), and send that price information to the quant department;

(b) by selling securities in the portfolio (if necessary), pay a cash distribution to "central headquarters" of \( D_2[V(t), t] \) which is paid out to the customer holding the derivative security;

(c) compute the value of the balance of the portfolio, \( P(t) \);

(d) receive instructions on \( M(t) \) from the quant department;

(e) readjust the portfolio allocation so that \( S'M(t) \) is now invested in the traded security and \( S[P(t) - M(t)] \) is invested in the riskless security.

It follows that the dynamics of the value of portfolio (*) are given by

\[
dP = M(t) \frac{dV}{V} + M(t) \frac{D_1(V, t)}{V} dt + [P - M(t)]r dt - D_2(V, t) dt
\]

where

\[
M(t) \frac{dV}{V} = \text{price appreciation}
\]

\[
M(t) \frac{D_1(V, t)}{V} dt = \text{dividend payments received into the portfolio}
\]

\[
[P - M(t)]r dt = \text{interest earned by the portfolio}
\]

\[
D_2(V, t) dt = \text{cash distribution to "central headquarters"}
\]

Noting that \( M(t) = F_4[V(t), t], V(t) \), we have by substitution from (1) to (7) that the dynamics of \( P \) satisfy

Return now to the quant department to derive the dynamics for \( Y(t) \) as follows:
\[ dP = F_1[V, t]dV + F_2[V, t]D_t(V, t)dt + (P - F_1[V, t]V)dt - D_2(V, t)dt \]

\[ = [F_1V(\alpha - r) + rP - D_2]dt + F_1VdZ. \]

From (iv), we have that \( Y(t) = F[V, t] \) for \( V(t) = V \). Because \( F \) is the solution to (3-6), \( F \) is a twice-continuously differentiable function of \( V \) and \( t \). Therefore, we can apply Itô's Lemma, so that for \( V(t) = V \),

\[ dY = F_1[V, t]dV + F_2[V, t]dt + \frac{1}{2}F_{11}(V, t)(dV)^2 \]

\[ = [\frac{1}{2}\sigma^2V^2F_{11} + F_1(\alpha V - D_2) + F_2]dt + F_1VdZ \]

because \((dV)^2 = \sigma^2V^2dt\). Because \( F[V, t] \) satisfies (3), we have that:

\[ \frac{1}{2}\sigma^2V^2F_{11} - D_1F_1 + F_2 = rF - rVF_1 + D_2 . \]

Substituting (10) into (9), we can rewrite (9) as:

\[ dY = [F_1(\alpha - r)V + rF - D_2]dt + F_1VdZ \]

Note that the calculation of \( Y(t) \) and its dynamics by the quant department in no way requires knowledge of the values of portfolio \((*)\), \( \{P(t)\} \), that are calculated by the trading desk. Putting these two time series together, we define \( Q(t) = P(t) - Y(t) \). It follows that \( dQ = dP - dY \). Substituting for \( dP \) from (8) and for \( dY \) from (11), we have that

\[ dQ = r(P - Y)dt = rQdt \]

Equation (12) is an ordinary differential equation \((dQ = r)\) with solution

\[ Q(t) = \frac{Q(0)e^{rt}}{r} \]

where \( Q(0) = P(0) - Y(0) = P(0) - F[V(0), 0] \). Therefore, if we choose the initial funding provided to the trading desk for portfolio \((*)\) so that \( P(0) = F[V(0), 0] \), then from (13), \( Q(t) = 0 \) for all \( t \) and
(14) \[ P(t) = F[V(t), t]. \]

Now suppose that we follow the (\(^*)\)-portfolio trading strategy with an initial investment \( P(0) = S F[V(0), 0] \). From trading-desk activity (b), this portfolio will generate cash payment flows of \( D_r[V(t), t] dt \) for \( 0 \leq t \leq T \). In addition, if \( V(t) = \bar{V}(t) \) at time \( t \), then from (4), \( F[V(t), t] = f[\bar{V}(t), t] \). Therefore, from (14),

(15) \[ P(t) = f[\bar{V}(t), t] \text{ if } V(t) = \bar{V}(t). \]

If \( V(t) = Z(t) \) at time \( t \), then from (5), \( F[V(t), t] = g[Z(t), t] \). Therefore, from (14),

(16) \[ P(t) = g[Z(t), t] \text{ if } V(t) = Z(t). \]

At \( t = T \), \( F[V(T), T] = h[V(T)] \) from (6). Therefore, from (14),

(17) \[ P(T) = h[V(T)]. \]

By comparison of (15-17) with (2), we have that the (\(^*)\)-portfolio strategy generates the identical payment obligations and terminal (and boundary) values as the contingent-claim security described at the outset of this section. That is, for a one-time, initial investment of \( S F[V(0), 0] \), we have found a feasible portfolio strategy that exactly replicates the payoffs to the "derivative" security. If the derivative security is traded, then to avoid ("conditional") arbitrage (conditional on \( \sigma, r, D_r \)), we must have that

(18) \[ W(t) = P(t) = F[V(t), t]. \]

Since the absence of arbitrage opportunities is a necessary condition for equilibrium, it follows that equilibrium prices for derivative securities must satisfy (18).

Note that the development of the (\(^*)\)-portfolio strategy did not require that the derivative security (defined by (2)) actually exists. Therefore, the (\(^*)\)-portfolio strategy provides the technology for "manufacturing" or synthetically creating the derivative security if it does not exist. I.e., if one describes a state-contingent schedule of outcomes for a portfolio (i.e. specifies \( f, g, h, D_r, T, Z(t), V(t) \)), then the (\(^*)\)-portfolio strategy provides the rules to create this pattern of payouts and specifies the cost of implementing these rules. The cost of creating the security at time \( t \) is \( F[V(t), t] \).
3. Impact of Portfolio Replication on Finance Practice

Dynamic replication of derivative securities' returns was applied primarily to equity derivative securities in the 1970s. The big, new applications of the 1980s were in the fixed-income area. The contingent-claim-analysis methodology of the preceding section is now used to price and hedge virtually every kind of derivative security, whether contingent on equities, interest rates, currencies or commodities. The huge U.S. national mortgage market could not operate effectively without mathematical models of this sort for pricing and hedging mortgages and mortgage-backed securities. Derivative securities have played an important role in the implementation of global capital markets by providing an efficient interface among the widely varying regulatory, tax, and institutional structures of individual national financial systems.

In the future, derivatives and the replicating-portfolio technology will widen their influence to include corporate and government activities. We close this section with a short example for each area.\(^{11}\)

3.1 A "Real" Option Example: The Synthetic Refinery

Consider a firm with extensive crude oil reserves and a chain of gasoline stations and a heating-oil distributor. Suppose that strategic analysis concludes that there are serious risk concerns about ensuring the firm's access to the production process which links those activities together. The need to eliminate that risk in the past would have been satisfied by perhaps acquiring a refinery. The alternative today, especially if the firm has no expertise in refining or managing a refinery, would be to enter into financial contracts in which the firm agrees to deliver at various dates a specified number of barrels of crude oil and, perhaps with some time delay, receives in return either so many gallons of high-grade gasoline or so many gallons of heating oil. The choice of which finished product is the firm's option at the delivery date. That contract functionally creates a synthetic refinery. It may not be appropriate for every such firm, but for many, entering into a simple contract is perhaps much safer and much more efficient than acquiring the refinery itself. The counterparty to the contract could be either a firm that specializes in refining or a financial intermediary which hedges its exposure with a replicating portfolio by dynamically trading oil, gasoline, and heating oil futures.

3.2 A Central-Bank Example: Automatic Stabilizers/Open-Market Operations

This example reverses the usual process of finding a replicating-portfolio strategy (*) that duplicates the payoffs to a derivative security. Take the prescribed dynamic trading strategy for open-market operations and issue a contingent-claim security which "replicates" that strategy. For example, let

\[
V(t) = \text{price of "standard" 10-year government bonds}
\]

\(^{11}\)These examples are developed more fully in Merton (1995).
\[ W(t) = \text{price of puttable 10-year government bonds which can be sold back to the government at a fixed price at the holder's option} \ (= F[V(t), t]). \]

The equivalent replicating portfolio is:

\[ SF_t[V(t), t]V(t) \] in standard 10-year bonds (F_t units)

\[ F[V(t), t] - F_t[V(t)]V(t) \] in short-term bills

\( F \) is convex and hence \( F_{11}[V(t), t] > 0 \). Thus, if interest rates rise and bond prices fall, then the equivalent number of units of standard bonds \( F_t[V(t), t] \) falls. If rates fall and bond prices rise, \( F_t \) increases. Therefore, issuing a puttable bond is economically equivalent to the stabilization policy of the government buying standard bonds when rates rise and selling bonds when rates fall, \textit{without} the central bank making any actual transactions.

\section*{4. Conclusion}

During the last two decades, replicating-portfolio models with continuous trading have become central tools for practitioners in financial institutions and markets around the globe. In the future, these models are likely to play an essential role in the functioning of the global financial system. As is evident from inspection of Section 2 here, the Wiener process is a fundamental part of those models' foundation.
References


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