The optimality of a competitive stock market

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The findings of Jensen-Long and Stiglitz on the optimality of the stock market allocation have led to a controversy over whether the sources of the nonoptimality of value maximization are noncompetitive assumptions about the capital market or are inherent externalities associated with uncertainty which do not disappear even in a competitive market. At least, for the mean-variance model with constant returns to scale technologies, we claim that the answer is the former, and that the sources of the nonoptimality are nonprice-taking behavior by firms, restrictions on the availability of technologies to firms, and restrictions on the number of firms that can enter. Using both tatonnement and nontatonnement processes, it is shown that if firms value maximize, then the Jensen-Long and Stiglitz equilibria are unstable with respect to the number of firms. If the restrictions on the availability of technologies and on the number of firms that can enter are relaxed, then the equilibrium will be a Pareto optimum in most cases, and in no case will the aggregate amount of investment be less than the Pareto optimal amount. If firms act as price takers (with or without the other restrictions), then the equilibrium is a Pareto optimum.

In their important papers on the optimality of the stock market allocation, Jensen and Long and Stiglitz\textsuperscript{1} use a mean-variance model of the capital market to demonstrate that the investment allocation by value-maximizing firms will not in general be Pareto optimal. While both analyses are technically correct, the interpre-

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tation of their findings has led to a controversy over whether the sources of the nonoptimality of value maximization are non-competitive assumptions about the capital market or are inherent externalities associated with uncertainty which do not disappear even in a competitive market. If the answer is the former, then their findings are consistent with certainty theory analysis, where value or profit maximization by firms with some positive degree of monopoly power need not lead to Pareto optimal allocations. If the answer is the latter, then, indeed, the findings are more fundamental. In his comment on the Jensen-Long and Stiglitz papers, Fama\(^2\) argues that because both analyses use noncompetitive assumptions, they do not answer the question whether in the mean-variance model, a competitive equilibrium is a Pareto optimum. Fama does perform the analysis based on what he calls the appropriate competitive assumptions and concludes that, due to inherent externalities caused by uncertainty, a competitive equilibrium will not, in general, be a Pareto optimum.

Our paper addresses the issue of whether a competitive equilibrium in the mean-variance model is a Pareto optimum or not, and its principal conclusion is that it is. The reason that our conclusion differs from the previous analyses is not technical, but rather a difference in interpretation of what the competitive market assumptions are. Because of the apparent disagreement in the literature as to what the “correct” competitive assumptions are in a model of uncertainty with incomplete markets, we consider a number of alternative assumptions about firms’ beliefs and the market structure and examine the optimality of the resulting equilibria. We believe that these assumptions span the range of what most people would be willing to accept as descriptive of a “competitive” market. There are two distinct issues to be resolved: (1) if firms act as “price takers” and value-maximize (the usual competitive assumptions), is the resulting equilibrium a Pareto optimum? (2) if firms do not act as “price takers” and do value-maximize but the market structure is such that entry is free, is the resulting equilibrium a Pareto optimum?

We define a firm as a “price taker” if it perceives a horizontal supply curve for capital: i.e., it believes that the required expected return on a project (the “cost of capital”) is unaffected by actions taken by the firm. As will be shown for perfectly correlated outputs, the “price taker” assumption is equivalent to the firm’s taking the aggregate amount of investment in a project as “fixed” since the cost of capital depends only on the aggregate amount of investment taken by all firms in a given project. In all three of the aforementioned papers, firms do not act as “price takers” in this sense. While our definition of price-taking may be open to debate, it is difficult to imagine a definition for a competitive market

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1 In [4] and [13], respectively.
2 In [3].
where entry into that market is restricted. Since all three papers restrict entry, either explicitly in the case of Jensen and Long, and Stiglitz, or implicitly, in the case of Fama, their findings do not demonstrate that a competitive equilibrium is not a Pareto optimum.

If new firms are restricted from raising funds in the capital market on the same terms as already existing firms, then the capital market does not satisfy the pure competition assumptions. This restriction is implicitly built into a model which holds constant the number of (noncolluding) firms. To restrict entry into an industry by assuming that all technologies are not freely available to all firms (both existent and potential) is also a violation of the pure competition assumptions. This restriction can be built into a model by holding constant the number of firms, or more subtly, by assuming that the correlation between returns on the “same” project taken by different firms is not perfect. Since in a portfolio context, projects and technologies, or for that matter, assets and firms, are defined by their probability distributions, it is misleading and inaccurate to refer to a project being considered by two (or more) firms as the “same” project if for the same inputs, the cash flows from the project in each state of nature are not the same for each firm (i.e., if the cash flows from the same project taken by different firms are not perfectly correlated). For example, two projects or technologies whose cash flows or outputs are identically and independently distributed should not be considered the same project for the same reason that an investor does not treat two assets whose returns happen to be identically and independently distributed as the same asset or even as members of the same “risk class.”

Jensen and Long are careful to acknowledge that either the assumption of holding the number of firms fixed or the assumption that the correlation between cash flows on the same project taken by different firms is not perfect may not be consistent with free entry and pure competition. Stiglitz recognizes that by holding the number of firms fixed, he is restricting entry, but because he often assumes in his analysis that the distributions of returns for the “same” project across firms are independent, he is led to conclude that even if the number of firms allowed to enter is expanded, the Pareto optimal allocation is not achieved. Fama shows that when the returns from projects are less than perfectly correlated across firms, there are natural externalities in their production

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3 In the usual analysis of a competitive equilibrium under certainty (see, for example, [1] or [2]), it is assumed that there is a fixed number of firms, and, therefore, entry is restricted in the sense that we use it. However, because it is also assumed that firms are price takers, the entry restriction is never binding. Hence, the equilibrium is the same as if there were no such restriction. However, when firms do not act as price takers, restricting the number of firms to any finite number can affect the equilibrium allocation and allow for non-Pareto optimal results.

4 Risk class is defined as in Miller and Modigliani [8]: namely, two assets are in the same risk class if they are considered perfect substitutes for each other by investors.

5 In [13].

6 In [3].
decisions that violate the conditions of a perfectly competitive market. He then uses the linear market model which satisfies the conditions of perfect competition, to show that if the number of firms is finite and returns across firms are less than perfectly correlated, the resulting equilibrium is non-Pareto optimal. As shown later, for the technologies with imperfect correlation examined in our paper, the equilibrium solution to his equations (12)–(14) requires an infinite number of firms. For other technologies, there is some question as to the existence of a solution to his equations.

In Section 2, we reexamine the Jensen-Long model\(^7\) and show that if firms are price takers, then the equilibrium allocation of investment is Pareto optimal. It is also shown that if firms are not price takers but entry is free, then the equilibrium level of investment will be no smaller than the Pareto optimal amount. In Section 3, we modify the Jensen-Long model to allow for reversible investment and show that the competitive equilibrium is a Pareto optimum. In Section 4, we discuss the Jensen-Long alternative criterion of social wealth maximization. Because the strict assumption of technologies' being freely available to all firms (i.e., perfect correlation for the same project across firms) is not empirically descriptive, in Section 5, we examine the nonperfect competition case of imperfect (but positive) correlation and conclude that provided new firms can enter and raise capital on the same terms as preexisting firms, the resulting equilibrium will be a Pareto optimum.

2. Jensen-Long model

In this section, we use the same model as Jensen and Long: namely, all investors are risk-averse and characterize their decisions based on the mean and variance of end-of-period wealth; the market is perfect with all assets infinitely divisible; all investors have homogeneous expectations; transactions costs and taxes are zero. Under these conditions, they show\(^8\) that the equilibrium value of the firm is

\[
V_j = \frac{1}{r} \left[ \tilde{D}_j - \lambda \sigma_{jm} \right] \text{ for all } j, \tag{1}
\]

where

\[ \tilde{D}_j = \text{the expected value of the total end-of-period cash flow to the shareholders of firm } j, \]

\[ r = 1 + i, \text{ where } i \text{ is the one-period riskless rate of interest at which every investor can borrow or lend}, \]

\[ \sigma_{jm} = \sum_k \sigma_{jk} = \text{covariance of the total cash flows of the firm,} \]

\[ \tilde{D}_j \text{ with } \tilde{D}_m \text{ the total cash flows from all firms}, \]

\[ \sigma_{jk} = \text{Cov} (\tilde{D}_j, \tilde{D}_k) \text{ for all } j \text{ and } k, \]

\[ \sigma^2_m = \text{Var} (\tilde{D}_m), \]

\[ \lambda = (\tilde{D}_m - rV_m) / \sigma^2_m \text{ = market price per unit of risk, and} \]

\[ V_m = \sum_k V_k \text{ = total market value of all firms.} \]

\(^7\) In [4].

\(^8\) See [4], p. 153, (1).
We also assume that \( \lambda \) is a fixed number. This assumption is consistent with constant absolute risk aversion on the part of investors. Consider a new investment opportunity, or project with cash flow return per unit input, \( \bar{\rho} \), where

\[
E(\bar{\rho}) = \bar{\rho} \\
\text{Var}(\bar{\rho}) = \sigma_\rho^2 \\
\text{Cov}(\bar{\rho}, \bar{D}_j) = \sigma_{\rho j}.
\]

It is assumed that the distribution of \( \bar{\rho} \) is independent of which firm takes it and that there are stochastic constant returns to scale. I.e., if \( I_j \) denotes the amount of investment in the project by the \( j \)th firm, then \( I_j \bar{\rho} \) is the random variable cash flow to the \( j \)th firm as a result of taking the project and the random variable \( \bar{\rho} \) is independent of the choice of \( I_j \). As Jensen and Long do, we assume that the cash flows from all other assets remain fixed, independent of the amount of investment made in the new opportunity. I.e., it is assumed that firms cannot change the level of investment in other projects in response to investment in the new technology. Thus, not only is investment assumed to be irreversible, but, in addition, new investment cannot be made in the "old" technologies.\(^9\) While such an assumption is probably not very realistic, the qualitative results of the Jensen-Long model are preserved when this assumption is weakened to allow for reversible investment as is shown in Section 3.

As Jensen and Long have shown,\(^10\) the Pareto optimal amount of investment in the new project is\(^11\)

\[
I^* = \max \left\{ 0, \frac{1}{\lambda \sigma_\rho^2} \left[ \bar{\rho} - r - \lambda \sigma_{\rho p} \right] \right\}. \quad (2)
\]

Equation (2) is derived from the condition that, at the Pareto optimal level of investment,

\[
\bar{\rho} = r + \lambda \sigma_{\rho p} \quad (3)
\]

\(^9\)Thus, it is assumed that there is "putty" capital for the new technology only and "clay" for all other technologies, i.e., a type of "short-run" model. We believe this description to be consistent with that in Jensen and Long. The analysis used throughout the paper is that of comparing equilibria (comparative statics; see [1], Chapter 10). I.e., we compare the alternative equilibria resulting from the introduction of a new technology and changes in assumptions about firms' perceptions. Under the alternative interpretation of looking at a cross-section of firms for a given equilibrium, the distinction between "putty" and "clay" (long-run/short-run) is not necessary. Similarly, our notion of "new" versus "old" firms and our definition of "entry" suggests a kind of dynamics which is not necessary for examination of the conditions satisfied by firms investing in the \((N + 1)\)st technology for a given equilibrium. Nor is it necessary to assume that \( \lambda \) is a constant for this type analysis. However, since much of our discussion centers on the approaches to (stability of) the equilibria, this latter interpretation is of limited usefulness.

\(^10\)In [4], p. 156, (5).

\(^11\)As discussed in Section 1, a project is defined by its probability distribution, and hence, the correlation of returns on the same project across firms is perfect. Hence, the aggregate level of investment for a Pareto optimum is independent of the distribution of investment in the project across firms.
where $\tilde{D}_M = \tilde{D}_m + I\tilde{p}$ is the total cash flow for all firms when the amount of (aggregate) investment in the new project is $I$ and where $\sigma_{Mp}^2 = \sigma_{Mp}^2 + I\sigma_p^2$ is the covariance between $\tilde{p}$ and the return on the new market portfolio.

Jensen and Long\textsuperscript{12} also show that the value-maximizing level of investment in the project for the $j$th firm is

$$I^{*}_j = \max \left\{ 0, \frac{1}{2\lambda\sigma_p^2} [\tilde{p} - r - \lambda(\sigma_{Mp}^2 + \sigma_{jp}^2 + I\sigma_p^2)] \right\},$$

(4)

where $I'$ is the aggregate amount of investment in the project by all firms other than the $j$th firm.\textsuperscript{13}

Because the value-maximizing level of investment depends upon the other projects already taken by the firm, expression (4) seems to conflict with the notion of risk independence of projects.\textsuperscript{14} Suppose that a project is available which will not affect the distribution of cash flows of already existing projects and the distribution of the project's cash flow is the same, no matter which firm takes the project, then the doctrine of risk independence states that (1) the market values of the project and all other assets will be the same, independent of which firm takes it; (2) the firm can determine whether to take a project or not, independent of other assets held by the firm; i.e., it can act as if it were a new firm with no other preexisting assets.

Actually, part (1) of the statement does hold for the Jensen-Long model in the sense that for a given level of aggregate investment in the project, the distribution of investment in the project across firms does not affect the market value of the project or of other assets. The reason that part (2) does not obtain can be traced to the assumption by Jensen and Long that the firm believes that its investment decision will affect the aggregate amount of investment in the project. Further, through this belief, the firm perceives that its investment decision will not only affect the cost of capital for the new project, but in addition, the cost of capital on other projects previously taken by the firm. This is in contrast to Fama,\textsuperscript{15} whose "reaction principle" requires that a firm act as a price-taker in the sense of taking the aggregate level of investment in the project as fixed.

To see this, we use equation (1) to rewrite the formula for the value of the $j$th firm explicitly as a function of the amount of its own investment in the new technology, $I_j$, and the aggregate amount of investment by all firms in the new technology, $I$, as

$$V_j(I_j; I) = V_j(0; I) + I_{j0}(I),$$

(5)

where

\textsuperscript{12} In [4], p. 161, (18).

\textsuperscript{13} Equation (4) is valid only for perfect correlation of returns on the project across firms which has been assumed.


\textsuperscript{15} In [3], pp. 513–514, assumption (C6).
\[ V_j(0;I) = \frac{1}{r} [\bar{D}_j - \lambda (\sigma_{M_j} + I\sigma_{\rho^2})] \]  

(6a)

and

\[ g(I) = \frac{1}{r} [\bar{\rho} - \lambda (\sigma_{M_f} + I\sigma_{\rho^2})]. \]  

(6b)

\( V_j(0;I) \) is the market value of all other assets owned by the \( j \)th firm, given that the aggregate investment in the project is \( I \), and \( g(I) \) is the value per unit input of investment in the new technology (the same for all firms). \( I_j g(I) \) is the value of firm \( j \)'s ownership in the new technology and it is the amount that it would receive if it sold its part of the project to another firm or "spins it off" as a separate firm. The total market value of the project, \( \sum I_j g(I) = I g(I) \), is independent of the distribution of investment across firms. The required expected return per unit investment in the project equals \( \rho / g(I) \), the same for all firms. Hence the "cost of capital" for the new project depends only on the aggregate amount of investment. Thus, for a given cost of capital, the value of the project (as a specified level of investment) is the same, independent of which firm takes it.

It will throw light on the analysis to compute the value-maximizing solution using the representation for firm value in (5). Namely, to determine the optimal level of investment to take in a new project, the firm picks \( I_j = I^*_j \) so as to maximize \( V_j(I_j;I) - I \), which is the market value of the firm net of the cost of inputs. For \( I_j = I^*_j \) to be an interior solution, the first-order condition is

\[ \frac{d[V_j(I_j;I) - I]}{dI_j} = \frac{\partial V_j}{\partial I_j} - 1 + \frac{\partial V_j}{\partial I} \frac{dI}{dI_j} = 0 \quad \text{for} \quad I_j = I^*_j, \]

(7)

where

\[ \frac{\partial V_j}{\partial I_j} - 1 = \frac{1}{r} [\bar{\rho} - r - \lambda (\sigma_{M_f} + I\sigma_{\rho^2})], \quad \text{independent of} \quad I_j, \]

(8)

and

\[ \frac{\partial V_j}{\partial I} = -\frac{\lambda}{r} [\sigma_{\rho} + I\sigma_{\rho^2}]. \]  

(9)

If we assume, as Jensen and Long do, that the firm takes investment by other firms in the project as fixed (i.e., \( I' = I - I_j \) fixed) then, \( dI/dI_j = 1 \), and the value-maximizing solution \( I^*_j \) is as defined in (4). If, instead, Fama's "reaction principle" is used, then \( dI/dI_j = 0 \) and the optimal solution for each firm is the same. Namely, if \( [\bar{\rho} - r - \lambda (\sigma_{M_f} + I\sigma_{\rho^2})] < 0 \), the firm(s) will invest nothing; if \( [\bar{\rho} - r - \lambda (\sigma_{M_f} + I\sigma_{\rho^2})] > 0 \), each firm will be willing to invest an indefinite amount; if \( [\bar{\rho} - r - \lambda (\sigma_{M_f} + I\sigma_{\rho^2})] = 0 \), each firm is indifferent to the amount it invests. Obviously, the only equilibrium solution in this case is

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\[
I = \text{Max} \left\{ 0, \frac{1}{\lambda \sigma^2} \left[ \bar{\mu} - r - \lambda \sigma_{M_0} \right] \right\},
\]

which is the Pareto optimum amount of investment as defined in (2). As is usual for standard competitive models, while the aggregate amount of investment in the project is determinate, the scale and hence, the number of firms investing in the project is not. Note that from (5), each firm believes that its market value expands in proportion to the amount of investment it takes. Therefore, if firms believe they cannot affect the aggregate amount of investment in the project (or equivalently, its cost of capital), then the number of firms necessary to achieve a Pareto optimal allocation is finite and can be as few as one.\(^\text{16}\)

We now turn to the Jensen-Long case where firms act to maximize market value under the assumption that investment by all other firms, \(I'\), is fixed. Unlike the price-taker case, the resulting equilibrium allocation will depend on the number of firms that can invest in the new technology. In particular, Jensen and Long have shown that if there is a finite number of such firms, then the equilibrium allocation will not be an optimum, although it will be an optimum with an infinite number of firms.

However, as we shall argue, the assumption of a finite number of firms seems somewhat inconsistent with the notion of free entry and infinitely-divisible technologies with constant returns to scale. To allow for easier comparison with the Jensen-Long analysis and in keeping with the spirit of our analysis which is one of comparing equilibria, we will call those firms which have previous investments in the other technologies, “old” (or preexisting) firms and those firms which have no previous investments but can invest in the new technology, “new” firms. If, in equilibrium, a new firm takes a positive amount of investment, it is said to have “entered,” and if it has not, then it is called a “potential” new firm.

For the moment, let us make the Jensen-Long assumption that there are a finite number \((N + K)\) of firms where the first \(N(j = 1, 2, \ldots, N)\) firms are old firms and the last \(K(j = N + 1, \ldots, N + K)\) are new firms. From the value-maximization assumption, it must be that in equilibrium, there can be no firm which believes that it can raise its market value by a (feasible) change in its investment.\(^\text{17}\) Assume an equilibrium with respect to these assumptions with aggregate investment, \(I\). Now, consider what would happen if we allowed one more (potential) new firm—the \((K + 1)\)st. From equation (4), its value-maximizing investment demand would be

\[
I^*_N + K + 1 = \text{Max} \left\{ 0, \frac{1}{2 \lambda \sigma^2} \left[ \bar{\mu} - r - \lambda \sigma_{M_0} + I \sigma^2 \right] \right\}.
\]

\(^\text{16}\) Hence, as is the case for the usual competitive analysis (see note 3), if firms are price takers, then the restriction of entry by postulating a fixed number of firms has no effect on the investment allocation nor on the Pareto optimality of the allocation.

\(^\text{17}\) This equilibrium condition holds for all the models in the paper. However, what constitutes a “feasible” change in investment varies according to the particular model being discussed.
Now, if \( I^*_N + K + 1 > 0 \), then the firm would enter if it could. Why does it not? Formally, because it is assumed not to exist. However, if it is also assumed that the constant-returns-to-scale technology is freely available and that both old and new firms can raise funds in the capital market on the same terms, we see no apparent reason (other than implicit restrictions to entry hidden from the model's explicit assumptions) why such a firm would not be created by a profit-seeking entrepreneur.

If, on the other hand, \( I^*_N + K + 1 = 0 \) (as would be the case, for example, if other firms acted as price takers), then the (potential) new firm would have no incentive to enter and the equilibrium would be unaffected by expanding the number of firms which can enter. Thus, at least with respect to stability, the assumption of a fixed number of firms does not matter if the restriction is not binding in equilibrium as is the case for the standard competitive model.

Of course, our opposition to restricting the number of firms in a competitive model is an argument over an assumption, and hence, we cannot "prove" that it is inconsistent with free entry. The method of argument is stability of the equilibrium with respect to the number of firms, and we have shown that if in an equilibrium, profit opportunities exist for additional firms if the number of firms allowed were expanded, then such an equilibrium is not stable.

Since the assumption of a fixed number of firms seems to conflict with the notion of free entry, we replace it with the more compatible assumption that there is potentially an indefinitely large number of firms. From the value-maximization assumption, a necessary condition for equilibrium is that no potential new firm will have incentive to enter. From (11), this condition implies that in equilibrium, the aggregate amount of investment must satisfy \( \bar{\rho} - r - \lambda(\sigma_M + 1\sigma_y^2) \leq 0 \), or in other words, the condition for no entry by an additional new firm is that \( I \geq I^*_w \). Thus, the market cannot be in equilibrium unless aggregate investment is at least as large as the Pareto optimal amount. This result is in sharp contrast to the previous analyses where it was deduced that the equilibrium amount of investment was less than the Pareto optimal amount.

Using two different dynamic processes of adjustment toward equilibrium, we find the distribution of investment across firms which makes the equilibrium stable. Having done so, we then answer the question under what conditions will these stable equilibria be Pareto optimal?

To determine stability, we first consider a sequential-type analysis where firms arrive in the market place in a random fashion and make an investment decision according to (4). However, once they commit themselves they cannot reverse the decision (disinvest) although they can "come back" to the market place to make more investment. On completion of this analysis, we then consider the more conventional tâtonnement process for reaching equilibrium. For simplicity, we assume that old firms do not invest in
the project (i.e., \( I_j = 0, \ j = 1, \ldots, N \)) although the analysis would be virtually identical provided that \( \Sigma_{i} I_{i}^* \leq I_{\infty}^* \).  

The analysis for the case where both old and new firms invest in the new project (again, provided \( \Sigma_{i} I_{i}^* \leq I_{\infty}^* \)) is performed in the Appendix.

The process for getting to an equilibrium is as follows: a new firm (chosen at random) arrives at the market place and is given the following “macro” information (in addition to the “micro” information which it already has: namely, the distribution for \( \rho \): \( r, \lambda \), and the probability distribution for the market portfolio at the “time” of the firm’s arrival. “Time” is counted here as incrementing by one with each arrival of a new firm in the market place. Thus, if the current new firm making an investment decision is the \( k \)th such firm to do so, then the distribution for the market cash flow at time \( k \) is denoted by the random variable \( \tilde{D}_M(k) = \tilde{D}_M + \Sigma_{i}^{k-1} N_{i+1} \rho \) where \( N_{i+1} \) is the amount of investment in the project chosen by the \( t \)th (new) firm to arrive (\( \Sigma_{i}^{k-1} N_{i+1} \) is the aggregate amount of investment in the project by all previously arrived firms) and \( \tilde{D}_M(1) = \tilde{D}_M \). By choosing the information provided in this way, the new firm need not know what investments other firms have taken which is more in the spirit of decentralized decisions by firms and an impersonal market place for raising or issuing capital. Since it is assumed that each firm acts so as to maximize market value, the optimal investment by the \( k \)th (new) firm will be

\[
I_{n+k}^* = \max \left\{ 0, \frac{1}{2} \frac{\sigma_{M\rho}}{\lambda} \right\}, \quad \text{(12)}
\]

where \( \sigma_{M\rho}(k) = \text{Cov}(D_M(k), \tilde{p}) = \sigma_{M\rho} + (\Sigma_{i}^{k-1} N_{i+1}) \sigma_{\rho}^2 \). Formally, (12) and (4) are the same equations. However, (12) shows that direct knowledge of other firms’ investment decisions is not necessary. The process of arrivals by firms continues until no new firm would want to enter in which case the market will then be in equilibrium.

To determine the amount of investment taken by the \( k \)th new firm, we have that \( \Delta_k \sigma_{M\rho}(k) = \sigma_{M\rho}(k + 1) - \sigma_{M\rho}(k) = I_{n+k}^* \sigma_{\rho}^2 \) and therefore, \( \Delta_k I_{n+k}^* = -\frac{1}{2} I_{n+k}^* \), or

\[
I_{n+k+1}^* = \frac{1}{2} I_{n+k}^* = \left( \frac{1}{2} \right)^k I_{n+1}^* \quad \text{(13)}
\]

where \( I_{n+1}^* = \max \left\{ 0, \frac{1}{2} \frac{\sigma_{M\rho}}{\lambda} \right\} \). Further, the aggregate amount of investment taken by the first \( k \) firms, \( I(k) \), is

\[
I(k) = \max \left\{ 0, \frac{\left[ 1 - \left( \frac{1}{2} \right)^k \right]}{\lambda \sigma_{\rho}^2} \frac{\sigma_{M\rho}}{\lambda} \right\}
\]

\(^{18}\) See the Appendix.
By this process, it actually takes an infinite number of new firms 
\( k = \infty \) to reach equilibrium which also corresponds to the
Pareto optimal amount of investment. We did not assume that
many new firms enter, but deduced it as the result of free entry
and the assumption that as long as profit opportunities exist, firms
will enter. Note, however, that the convergence to equilibrium is
rapid: for example, after entry by ten firms, the aggregate amount
of investment taken is 99.9 percent of the Pareto optimal amount.
Because of the nontatonnement nature of the approach to equilib-
rium, there are wealth distributional effects if trades actually take
place at the "false" interim prices. However, given the assumption
of a constant price of risk, we have that the equilibrium will be
unaffected by these transfers of initial wealth, and it will be a
Pareto optimum.

An alternative interpretation of the process could be a se-
quence of (unstable) Jensen-Long-type equilibria where in each
subsequent equilibrium, investment taken in each previous equi-
librium is treated as "clay" in the same manner as investments in
other technologies are in their model. It should also be noted that
the dynamic process described corresponds to the static behavior
of a perfectly discriminating monopolist whose behavior, in the
standard theory under certainty, will produce a Pareto optimal
allocation.

We now look at the equilibrium solution achieved when the
approach is by the conventional tatonnement process, but where
we still allow entry by new firms as long as profit opportunities
exist. We can write the equilibrium value-maximizing level of in-
vestment for the \( j \)th firm from (4)

\[
I^*_j = \max \left\{ 0, \frac{1}{\lambda \sigma_p^2} \left[ \bar{p} - r - \lambda (\sigma_{Mj} + \sigma_{fj} + I^* \sigma_p^2) \right] \right\} 
\]

(15)

where \( I^* \) is the equilibrium aggregate amount of investment in
the project and \( I^*_w \) is the Pareto optimal amount of investment as
defined in (2).

To determine the equilibrium, we start by partitioning the firms
into three groups: group 1 contains all firms such that \( \sigma_{fj} > 0 \);
group 2 contains all firms such that \( \sigma_{fj} < 0 \); group 3 contains all
firms such that \( \sigma_{fj} = 0 \). Groups 1 and 2 contain only old firms
while group 3 contains all new firms and in addition, any old firms
whose \( \sigma_{fj} = 0 \). For notational simplicity and without loss of gen-
erality, we assume that there are no old firms in group 3. By
renumbering firms if necessary, let the firms in group 1 be num-
bered 1, 2, \ldots, \( N_1 \); the firms in group 2 be numbered \( N_1 + 1, \ldots, N \); the firms in group 3 to be numbered \( N + 1, \ldots, N + K \) (where \( K \) is the number of new firms which enter).

By inspection of (15), all new firms will choose the same
amount of investment, \( I_{new} = I^*_j = I^*_{j+1} = \ldots = I^*_{j+K} \).
Further, for \( I^*_j \) to be an equilibrium amount of aggregate invest-

\[
= \left[ 1 - \left( \frac{1}{2} \right)^h \right] I^*_w. 
\]
ment when free entry is allowed, there must be no incentive for any firm to change its chosen level of investment. In particular, consider the \((K + 1)\)st (potential) new firm: from (15), it will not enter only if \(I^*\) is such that
\[
\bar{p} - r - \lambda (\sigma_M + I^* \sigma_p^2) \leq 0. \tag{16}
\]
Hence, from (15) and (16), we have that the equilibrium amount of investment by all firms in groups 1 will be zero, i.e., \(I^*_{x_1} = I^*_{x_2} = \ldots = I^*_{x_N} = 0\). Further, strict inequality in (16) can only obtain if \(I^*_{\text{new}} = 0\) and no new firms enter.

Equation (16) is equivalent to the condition that \(I^* \geq I^*_{w}\) with \(I^* > I^*_{w}\) only if \(I^*_{\text{new}} = 0\). As in our analysis of the non-tatonnement approach to equilibrium, we have found that free entry ensures that the equilibrium amount of investment in the project will never be less than the Pareto optimal amount. A sufficient condition for \(I^* = I^*_{w}\) is that \(-\sum_{x_{N+1}}^{x_N} (\sigma_{f_{x_{x}}}/\sigma_p^2) \leq I^*_{w}\).

If the strict inequality holds, then new firms will enter and \(KI_{\text{new}} = I^*_{w} - \left[-\sum_{x_{N+1}}^{x_N} (\sigma_{f_{x_{x}}}/\sigma_p^2)\right]\).

Inspection of the Jensen-Long\(^{19}\) formula for the equilibrium amount of investment when project returns are perfectly correlated across firms shows that if any new firms enter, an infinite number will (i.e., \(K = \infty\)).

Hence, as in the previous non-tatonnement analysis, if at least one new firm enters, then the number of new firms entering will be infinite, and the stable equilibrium will be a Pareto optimum. Again, we emphasize that it was not assumed a priori that the number of firms entering was infinite, but deduced from the assumptions of value-maximizing by firms, free entry, and infinitely divisible assets. The convergence to the Pareto optimum is less rapid for the tatonnement process than in the previous case because entry by a new firm induces previously entered firms to contract their (contingent) investment, i.e., \(1 \geq I^*/I^*_{w} \geq K/(K + 1)\).

Actually, the convergence to the Pareto optimum does not even require that all firms are value-maximizers. Essentially, the analysis has shown that as long as the aggregate amount of investment is less than the Pareto optimal amount and as long as entry is not restricted, profit opportunities will exist and new firms will enter.

In the preceding analysis, it was shown that if, at least, one new firm enters, then the equilibrium is a Pareto optimum. However, it is possible that under our definition of free entry, the equilibrium allocation of investment can be larger than the Pareto optimum amount. Further, it is difficult to find an additional mechanism which will ensure that in all cases the equilibrium is a Pareto optimum. One mechanism which would substantially reduce the possibility for such “overinvestment” would be to assume that new firms, in addition to looking for profit opportunities for direct investment in the new technology, also consider the possibility of taking over old firms and rearranging their investment in the new technology so as to make a profit.

To see the effect of this assumption, we examine the case

\(^{19}\) In [4], p. 163, (27).
where the project’s returns are sufficiently negatively correlated with some of the currently existing assets so that the investment taken by firms in group 2 is greater than the Pareto optimum amount. For simplicity, consider the case where group 2 contains only one firm, the $N$th one, and the value maximizing $I^{*}_{N}(= I^{*}) > I^{*}_{w}$ which implies that $-\sigma_{Np} > I^{*}_{w}\sigma_{p}^2$.

The change in market value of the $k$th firm, $k = 1, 2, \ldots, N - 1$, going from an “equilibrium” with aggregate investment in the project of $I^{*}_{N}$ to one with aggregate investment equal to $I^{*}_{w}$ (with $I^{*}_{k} = 0$ in either case) would be

$$V_{k}(0; I^{*}_{w}) - V_{k}(0; I^{*}_{N}) = \frac{\lambda}{r} [I^{*}_{N} - I^{*}_{w}]\sigma_{Np} > 0, k = 1, \ldots, N - 1,$$

(17)

and the change in value of the $N$th firm if it divested itself of the investment in the project and (through the entry of new firms) the aggregate investment in the project were $I^{*}_{w}$, would be

$$V_{N}(0; I^{*}_{w}) + I^{*}_{N} - V_{N}(I^{*}_{N}; I^{*}_{N}) = \frac{\lambda}{r} [I^{*}_{N} - I^{*}_{w}]\sigma_{Np} + I^{*}_{w}\sigma_{p}^2]$$

(18)

$$= \frac{\lambda}{2r} [I^{*}_{N} - I^{*}_{w}]\sigma_{Np} + I^{*}_{w}\sigma_{p}^2] < 0.$$

Now, if the total change in the market value of all existing firms were positive by going from $I^{*}_{N}$ to $I^{*}_{w}$, then a profit opportunity would exist for a “new” firm to buy out all existing firms at the values associated with the $I^{*}_{N}$ “equilibrium” and then to divest the $N$th firm of its investment in the project and sell everything out at the new values associated with the $I^{*}_{w}$ equilibrium.

By summing (17) from $k = 1$ to $(N - 1)$ and adding it to (18), we have that the condition for the sum of the change in the market values of all firms to be positive is

$$\overline{\rho} - r + \frac{\lambda}{2} \sum_{1}^{N-1} \sigma_{kp} > 0.$$

(19)

Hence, provided that (19) is satisfied and that tender offers or mergers are allowed, the competitive equilibrium (where firms act so as to maximize their market value) will be a Pareto optimum.

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In the previous section, we assumed as did Jensen and Long that investment could only take place in the new technology and that investment in “old” technologies was held fixed. Both the investment demand by firms and the Pareto optimal amount of investment were computed on the basis of this assumption. While it may be realistic to assume that investment already taken is non-reversible (except possibly through some limited depreciation), there is no apparent reason to assume that investment in previous technologies might not be expanded.

In this section, we go to the other extreme and assume that all investment is reversible, i.e. there are no fixed costs, and cap-
ital can be freely transferred from one technology to another. An equivalent intertemporal assumption would be that capital depreciates completely in a single period. We choose the completely reversible assumption over the more realistic “mixed” reversible-irreversible assumption described above, not only for the obvious analytical simplifications, but also to preserve the one-period nature of the problem which would not be adequate if complete depreciation were not allowed.

Because we are only interested in the Pareto optimal allocation and because we maintain the assumption of a fixed price of risk $\lambda$, we can assume without further loss of generality a single representative investor with constant absolute risk aversion.

To determine the Pareto optimal allocation, we eliminate firms from the economy and derive the optimal investment allocation to each technology by the investor. One can view the problem as that faced by an economy with a single input, wheat (seed), which can be planted in various ways so as to produce a final output, wheat (grain) at the end of the period. The probability distributions for output from each technology are given and the assumption of stochastic constant returns to scale is maintained. The investor has some given initial endowment of wheat and must choose an optimal allocation across the various technologies.

First, consider the allocation when there are $N$ technologies with expected return per unit input, $\alpha_i$, and covariances of returns per unit input $\nu_{ij}, \ i, j = 1, 2, \ldots, N$. The optimal allocation of investment will be

$$I^*_j = \frac{1}{\lambda} \sum_{k=1}^{N} \mu_{jk} (\alpha_k - r), \ j = 1, \ldots, N,$$  \hspace{1cm} (20)

where $\mu_{ij}$ is the $ij$th entry of the inverse of the $N \times N$ variance-covariance matrix of returns.

Now consider how the allocation will change if a new $(N + 1)$st technology is introduced with expected return per unit input, $\alpha_{N+1} = \overline{\nu}$, and with covariances of returns $\nu_{jN+1} = \nu_{jN}, \ j = 1, \ldots, N$, $(\nu_{N+1} x_j = \sigma_\nu^2)$. The new optimal allocation of investment will be

$$I_{*j} = \frac{1}{\lambda} \sum_{k=1}^{N+1} \mu'_{jk} (\alpha_k - r), \ j = 1, \ldots, N + 1,$$  \hspace{1cm} (21)

where $\mu'_{ij}$ is the $ij$th entry of the inverse of the (new) $(N + 1) \times (N + 1)$ variance-covariance matrix of returns.

To investigate the changes in the optimal allocation of investment caused by the introduction of the new technology, we derive some relationships between the entries in the inverse of the “old”

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20 In contrast to the analysis in Section 2 (see note 9), the assumptions of this section correspond to a “long-run” model and the comparative statics is between a $N$-technologies “putty” equilibrium allocation and a $(N + 1)$-technologies “putty” one.

21 See Merton [7] for a discussion of an intertemporal capital asset pricing model where supplies are taken as variable.

22 See Merton [6], p. 1863. To avoid bothersome inequalities, it is assumed that all technologies are sufficiently productive so that $I^*_j \geq 0$, $j = 1, \ldots, N$. 

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$N \times N$ variance-covariance matrix, $\Omega^{-1} = [\mu_{ij}]$ and the inverse of the new $(N + 1) \times (N + 1)$ variance-covariance matrix, $\Gamma^{-1} = [\mu'_{ij}]$. Define the $N \times N$ matrix, $\gamma$; the $N$-vectors $B$ and $E$; and the scalar $D$ by

$$B' = [v_{1p}, \ldots, v_{Np}]$$

and

$$l_{N+1} = \Gamma^{-1} = \begin{bmatrix} \Omega & B' \\ B & \sigma_{\rho}^2 \end{bmatrix} \begin{bmatrix} \gamma & E' \\ E & D \end{bmatrix}$$

where $l_K$ is the $K \times K$ identity matrix. Then, $\gamma$, $E$, and $D$ must satisfy

$$\begin{align*}
\Omega \gamma + BE' &= l_y \\
B' \gamma + \sigma_{\rho}^2 E' &= l_1 \\
\Omega E + BD &= 0 \\
B' \gamma + \sigma_{\rho}^2 E' &= 0.
\end{align*}$$

By manipulating the conditions in (22), we have that

$$\begin{align*}
D &= [\sigma_{\rho}^2 - B'\Omega^{-1}B]^{-1} \\
\gamma &= \Omega^{-1} + D\Omega^{-1}BB'\Omega^{-1} \\
E &= -D\Omega^{-1}B.
\end{align*}$$

Noting that $[\mu'_{N+1,1}, \ldots, \mu'_{N+1,N+1}] = [E', D]$, we have from equation (21) that the optimal amount of investment in the new technology is

$$I^*_{N+1} = \frac{1}{\lambda} \left[ D(\alpha_{N+1} - r) + \sum_{j=1}^{N} \mu'_{ij}(\alpha_j - r) \right]$$

$$= \frac{D}{\lambda} \left[ \alpha_{N+1} - r - \sum_{j=1}^{N} \left( \sum_{i=1}^{N} v_{ip} \mu_{ij} \right) (\alpha_j - r) \right],$$

$$= \frac{D}{\lambda} \left[ \alpha_{N+1} - r - \sum_{i=1}^{N} v_{ip} \mu_{ij} (\alpha_j - r) \right],$$

$$= \frac{D}{\lambda} \left[ \alpha_{N+1} - r - \lambda \sum_{i=1}^{N} v_{ip} I^*_{i} \right],$$

from (23c) from (20). (24)

To compare (24) with the Pareto optimal level (2) in the previous section, we note that, as defined in Section 2, $\sigma_{\mu'_{\rho}} = \Sigma v_{ip} I^*_{i}$. Hence, rewriting (24) in terms of the notation of the previous section,

$$I^*_{N+1} = \frac{D}{\lambda} [\rho - r - \lambda \sigma_{\mu'_{\rho}}]$$

$$= (D\sigma_{\rho}^2) I^*_{\omega}$$

as defined in (2). Hence, unless $D\sigma_{\rho}^2 = 1$, the Pareto optimal amount of investment in the new technology when investment in the old technologies can be changed will not be the same as when investment in the old technologies is held fixed. Noting that $B'\Omega^{-1}B \geq 0$ because $\Omega^{-1}$ is positive definite, we have from (23a) that

$$D\sigma_{\rho}^2 = \frac{1 - (B'\Omega^{-1}B)/\sigma_{\rho}^2}{\sigma_{\rho}^2} = \frac{1 - \left( \sum_{i=1}^{N} v_{ip} v_{ip} \mu_{ij} \right)/\sigma_{\rho}^2}{\sigma_{\rho}^2} \geq 1,$$

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with strict equality holding if and only if \( v_{kp} = 0 \) for \( k = 1, 2, \ldots, N \). Thus, the Pareto optimal amount of investment in the new technology will be larger in the unconstrained case than in the constrained one unless the new technology is uncorrelated with all other technologies.\(^{23}\) (Note that while \( \sigma_{M,k}^2 = 0 \) is necessary, it is not sufficient to ensure that \( D\sigma_{j}^2 = 1 \).)

Although not directly related to the question of whether a competitive equilibrium is a Pareto optimum, an important difference between the Pareto optimum solution of this section and the one found by Jensen and Long is that to compute \( I^*_{N+1} \) requires knowledge not only of the distribution for the new technology and its relationship to the distribution of the (prenew technology) market portfolio, but also of the complete variance-covariance structure for the new technology and all the other technologies (i.e., unlike in equation (2), where one could compute the Pareto optimum with only information about the project and the aggregate market portfolio, to compute the Pareto optimum in equation (25), one would have to know the complete micro-structure for all other assets). The only statement that can be made without such information is that if \( [\bar{\rho} - r - \lambda \sigma_{M,I}] > 0 \), then some positive amount of investment at least as large as \( [\bar{\rho} - r - \lambda \sigma_{M,I}] / \lambda \sigma_{I}^2 \) should be taken in the new technology.

In a similar fashion, we can deduce the changes in investment in the other technologies to be

\[
I^*_{j} - I^*_j = - \left( \sum_{1}^{N} v_{kp} \mu_{kij} \right) I^*_{N+1} \quad i = 1, 2, \ldots, N. \tag{27}
\]

As in computing \( I^*_{N+1} \), a knowledge of the complete micro-structure for all assets is required to compute \( I^*_{j} \). Equation (27) also demonstrates that the assumption made by Jensen and Long that the riskless asset is available in unlimited amounts is not a sufficient condition to ensure that investment in the old technologies will not change in response to the introduction of a new technology.

The important questions to be answered are: (1) if we now introduce the value-maximizing firms and allow for free entry, can the resulting equilibrium be a Pareto optimum? (2) How much information is required by the firms so that the answer to (1) is yes?

Note that because capital is freely mobile among technologies, there is no need to distinguish between new and old firms. Hence, consider the \( k \)th firm which only has "micro" information about the \( j \)th technology. Suppose it acts so as to maximize its market value using the same assumption as in the previous section: namely, it takes investment by all other firms as given. Then, it will choose its level of investment \( I^*_j \) so as to

\(^{23}\) While, at first glance, it may not be clear why the Pareto optimal amount of investment in the new technology will always be larger in the "long-run" than in the "short-run" (independent of the covariance structure), this result is just a special case of the Le Chatelier Principle as applied to a constrained-maximum system. See Samuelson [10] for a discussion of the Le Chatelier Principle as applied in economics.
Max \( V^k - I^k \) 
\[ = \text{Max} \left\{ \frac{I^k}{r} \left[ (\alpha_j - r) - \lambda \left( \sum_{1}^{N+1} I_j v_{ij} + I^k v_j^2 \right) \right] \right\} \]
\[ = \text{Max} \left\{ \frac{I^k}{r} \left[ (\alpha_j - r) - \lambda (\sigma_M(k) + I^k v_j^2) \right] \right\} \]

(28)

where \( I_i \) is the aggregate investment by all other firms in the \( i \)th technology at the "time" of firm \( k \)'s decision and \( \sigma_M(k) \) is the covariance of the return (per dollar) invested in the \( j \)th technology with the cash flow of all investments by other firms at the "time" of firm \( k \)'s decision (\( \sigma_M(k) \) is defined in an analogous fashion to \( \sigma_M(p) \) in the previous section). The solution to (28) is

\[ I^k = \text{Max} \left[ 0, \frac{1}{2\lambda v_j^2} \left\{ \alpha_j - r - \lambda \left( \sum_{1}^{N+1} I_j v_{ij} \right) \right\} \right] \]  

(29)

Note that the firm uses precisely the same information available to firms in Section 2. (I.e., it did not need to know the micro characteristics of other firms or technologies.) Because we allow for free entry and exit, the market can only be in equilibrium if the \( I_i \) are such that any new firm considering entry into any technology will have no incentive to do so and similarly, no firm which has previously taken investment in any technology will have any incentive to exit. From (29), these conditions require that the equilibrium investments, \( I^*_i \), satisfy

\[ \alpha_j - r - \lambda \left( \sum_{1}^{N+1} I^*_i v_{ij} \right) = 0, j = 1, 2, \ldots, N+1 \]  

(30)

or that

\[ I^*_j = \frac{1}{\lambda} \sum_{1}^{N+1} \mu'_{ij}(\alpha_j - r), i = 1, 2, \ldots, N+1, \]  

(31)

which are the Pareto optimal amounts of investment from equation (21).

To complete the analysis, we go to the other extreme and assume that each firm is aware of all the available technologies, although it does not know what other firms are doing. Then, the \( k \)th firm will choose its investment levels, \( (I^k_1, \ldots, I^k_{N+1}) \), so as to

Max \( \left\{ V^k - \sum_{1}^{N+1} I^k_i \right\} = \text{Max} \left\{ \frac{1}{r} \left[ \sum_{1}^{N+1} I_j^k (\alpha_j - r) \right. \right. \right. 
\[ - \lambda \left( \sum_{1}^{N+1} I_j^k \sigma_M(k) + \sum_{1}^{N+1} \sum_{1}^{N+1} I_j^k I^k v_{ij} \right) \left. \right\} \}

(32)

which can be solved to yield
\[ I_i^k = \text{Max} \left[ 0, \frac{1}{2\lambda} \left\{ \sum_{j=1}^{N+1} \mu'_{ij} (\alpha_j - r - \lambda \sigma_{ij}(k)) \right\} \right], \]

\[ i = 1, \ldots, N + 1. \]

(33)

As before, equilibrium would require that a new firm would have no incentive to enter which from (33) requires that

\[ 0 = \sum_{j=1}^{N+1} \mu'_{ij} [\alpha_j - r - \lambda \left( \sum_{k=1}^{N+1} \sigma_{ij}^2 \right)], \]

where we have substituted for \( \sigma_{ij}(k) \). Noting that \( \sum_{j=1}^{N+1} \mu'_{ij} \sigma_{ij}^2 \) is 0 of \( i \neq k \), and \( = 1 \) if \( i = k \), we can rewrite (34) as

\[ 0 = \sum_{j=1}^{N+1} \mu'_{ij} (\alpha_j - r) - \lambda I^*_i, \quad i = 1, \ldots, N + 1, \]

(35)

which is the Pareto optimum solution as written in (21) or (31).

Thus, even if each firm has limited information with respect to other technologies and what other firms are doing, as long as entry is free and firms value-maximize, the resulting (competitive) equilibrium is a Pareto optimum.

4. On the social wealth maximizing criterion

In an analysis parallel to the ones deducing the amount of investment associated with Pareto optimality and value maximization of firm value, Jensen and Long examine the social wealth maximizing criterion and derive the amount of investment that maximizes the market value of all firms. The solution can be written as

\[ I^*_s = \text{Max} \left\{ 0, \frac{1}{2\lambda \sigma_p^2} [\bar{\rho} - r] - 2\lambda \sigma_{ij} \right\}. \]

(36)

Comparing (36) with (4), the social wealth maximizing solution is the same as for a firm which is the only one with access to the new project when in addition that firm owns all the other assets in the economy (i.e., when \( \sigma_{ij} = \sigma_{ij} \)). Thus, if (4) is the solution for a monopolist who has sole right to the new technology, but does not control all technologies, then one might interpret (36) as the solution for a “super” monopolist who not only has sole right to the new technology, but also owns all other technologies. If one interprets such an entity to be the government, then (4) could be termed the “private” monopoly solution and (36) termed the “public” monopoly solution.

Viewed in this light, it is not surprising that Jensen and Long found that the social wealth maximizing levels of investment differed from the Pareto optimal amount.24

A natural question to raise is, how “nonoptimal” is the social wealth maximizing solution? To answer this question, we consider a representative investor with a mean-variance utility function of

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24 See, for example, Long [5], where he shows that wealth maximization with monopoly does not lead to a Pareto optimum.
terminal wealth, \( H[E(W), \text{Var}(W)] \), with \( H_1 \equiv \partial H/\partial E(W) > 0 \) and \( H_2 \equiv \partial H/\partial \text{Var}(W) < 0 \). For analytical simplicity, suppose that there is a single (composite) risky asset with expected end-of-period cash flow, \( I\alpha \) and variance of the cash flow, \( I^2\sigma^2 \), where \( I \) is the amount of investment in the risky asset. Then, let

\[
\begin{align*}
\alpha_p & \equiv I\alpha/V \\
\sigma_p^2 & \equiv \sigma^2I^2/V^2 \\
W_0(I) & \equiv W_0 + V - I
\end{align*}
\]

where \( V \) is the market value of the risky asset; \( \alpha_p \) is the expected return per dollar invested in the risky asset; \( \sigma_p^2 \) is the variance of that return; \( W_0(I) \) is the initial wealth of the investor as a function of the amount of investment taken in the risky asset. For a given \( I \), the investor’s demand for the risky asset in his optimal portfolio can be written as

\[
D_p = (\alpha_p - r)/\sigma_p^2
\]

where \( \lambda \equiv - (2H_2/H_1) \) and is taken to be constant.

By definition, \( E(W) = D_p(\alpha_p - r) + rW_0(I) \) and \( \text{Var}(W) = D_p^2\sigma_p^2 \). Hence, substituting for \( D_p \) from (38), we have that for the optimal portfolio

\[
E(W) = (\alpha_p - r)^2/\lambda\sigma_p^2 + rW_0(I) \quad \text{(39a)}
\]

and

\[
\text{Var}(W) = (\alpha_p - r)^2/\lambda^2\sigma_p^2. \quad \text{(39b)}
\]

To determine the effect on the investor’s welfare of a change in \( I \), we first note that

\[
\frac{dH}{dI} = H_1 \left[ \frac{dE(W)}{dI} - \frac{\lambda}{2} \frac{d\text{Var}(W)}{dI} \right]. \quad \text{(40)}
\]

Second, the market value of the risky asset is \( V = I(\alpha - \lambda I\sigma^2)/r \) and hence,

\[
\frac{dV}{dI} = \frac{1}{r} [\alpha - 2\lambda I\sigma^2]. \quad \text{(41)}
\]

Combining (37), (39), and (41), we have that, along optimal portfolios,

\[
\frac{dE(W)}{dI} = (\alpha - r) \quad \text{(42a)}
\]

and

\[
\frac{d\text{Var}(W)}{dI} = 2I\sigma^2. \quad \text{(42b)}
\]

Substituting from (42) into (40) we have that

\[
\frac{dH}{dI} = H_1[\alpha - r - \lambda I\sigma^2], \quad \text{(43)}
\]

which is positive for \( I < (\alpha - r)/\lambda\sigma^2 \) reaching a maximum at the Pareto optimal level \( (\alpha - r)/\lambda\sigma^2 \). Therefore, investor wel-
fare increases monotonically as investment increases until the Pareto optimal level of investment is reached.

Under what condition will investors be worse off with a "public" monopoly than with a "private" one? Inspection of (4) and (36) shows that the condition is \( \sigma_{jp} < \sigma_{mp} \). How likely is it for this condition to obtain? If \( I_i \) is the aggregate amount of investment by firm \( i \), then \( \sigma_{Mp} = \sum_{j=1}^{N} I_j \nu_{ip} \). Hence, if \( \sum_{j=1}^{N} I_j \nu_{ip} > 0 \), then \( \sigma_{Mp} > \sigma_{jp} \). Further, if the \( \nu_{ip} \) are generally positive and of approximately the same size, then \( \sigma_{Mp} \gg \sigma_{jp} \) since \( \sum_{i=1}^{N} I_i \gg I_j \) for a large economy. Therefore, one might expect that the "public" monopolist will restrict investment in the new technology substantially more than would the "private" monopolist. Thus, we conclude that the social wealth maximizing rule has little value as a social welfare criterion.

5. Imperfect competition

In the preceding sections, it was established that given the assumptions required for perfect competition, the resulting equilibrium is a Pareto optimum when firms follow the value-maximization rule. One of the competitive assumptions is that the same techniques for production are available to all firms which implies that the returns across firms on the same project are perfectly correlated. Because of this strong assumption, one might reasonably question whether the derived optimality implications of the competitive model are descriptive of actual capital markets. To at least partially answer this question, we now examine the case of a non-competitive stock market where it is assumed that the same technology is not available to all firms. However, we retain the assumption of free entry to the capital market (i.e., that new firms can raise capital on the same terms as old firms). Further, we assume that all firms follow the value-maximization rule, and therefore, new firms will enter as long as profit opportunities exist.

The model we use is the same one used by Jensen and Long in their section three although it is also descriptive of the models used by Stiglitz and Fama. It is assumed that there exists a set of new projects whose joint distribution for per unit cash flows is symmetric,\(^{25}\) and that each firm has access to only one project. I.e., the expected cash flow per unit input for each firm's new project is \( \bar{\rho} \), the same for all firms, and similarly, the per unit variance \( \sigma_{\rho}^2 \) is the same for all firms. The correlation coefficient for the cash flows between any two new projects is \( \beta \), the same for all firms. (Clearly, the \( \beta = 1 \) case corresponds to all the new projects being the same project in which case the analysis reduces to that of the previous sections.)

While these assumptions about the joint probability distribution may seem somewhat specialized, they are descriptive of a (new) industry or product where the basic technology is freely available

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\(^{25}\) The random variables \( X_1, \ldots, X_n \) have a symmetric joint probability distribution \( P(x_1, \ldots, x_n) \) if \( P(\cdot) \) is a symmetric function in its arguments. See Samuelson [11] for further discussion.
but where the correlation between the outputs of different firms is not perfect due to (somewhat random) differences in each firm’s application of the technology or managerial talents, etc.\textsuperscript{26}

Jensen and Long\textsuperscript{27} deduce that the value-maximizing investment for the \( j \)th firm can be written as

\[
I^*_{j} = \text{Max} \left\{ \frac{1}{2\lambda \sigma_{\rho}^2} \left[ \tilde{\rho} - r - \lambda (\sigma_{\mu} + \sigma_{\rho} + \beta \sigma_{\rho}^2 I') \right] \right\}
\]

\[
= \text{Max} \left\{ \frac{1}{(2 - \beta)\lambda \sigma_{\rho}^2} \left[ \tilde{\rho} - r - \lambda (\sigma_{\mu} + \sigma_{\rho} + \beta \sigma_{\rho}^2 I) \right] \right\},
\]

(44)

where \( I \) is the aggregate investment taken by all firms in the set of new projects and \( I' = I - I^*_{j} \). Since we allow free access to the capital markets by new firms to raise capital to invest in their own versions of the new technology, a necessary condition for equilibrium is that \( I \) be such that no additional new, value-maximizing firms should have incentive to enter. Hence, from (44) for a new firm, the equilibrium aggregate investment must be\textsuperscript{28}

\[
I^* = \text{Max} \left\{ 0, \frac{1}{\beta \lambda \sigma_{\rho}^2} \left[ \tilde{\rho} - r - \lambda \sigma_{\mu} \right] \right\}.
\]

(45)

If \( \sigma_{\rho} > 0 \) for \( j = 1, \ldots, N \), then from (44) and (45), the equilibrium amount of investment by all old firms will be zero, and by symmetry, the equilibrium level of investment taken by each new firm will be the same and equal to \( I^* / K \) where \( K \) is the equilibrium number of new firms.\textsuperscript{29} By substituting \( I^*_{N+k} = I / K \), \( k = 1, \ldots, K \), into (44) and solving, we have that for \( K \) new firms, the aggregate amount of investment will be

\[
I = \text{Max} \left\{ 0, \frac{K}{[2 + \beta (K - 1)] \lambda \sigma_{\rho}^2} \left[ \tilde{\rho} - r - \lambda \sigma_{\mu} \right] \right\}.
\]

(46)

By comparing (46) with equilibrium condition (45), we have that the number of new firms that enter is infinite \((K = \infty)\) as was the case in the \( \beta = 1 \) analysis of previous sections.

When the number of new firms is restricted to be no larger than \( K \), Jensen and Long\textsuperscript{30} derived the (constrained) Pareto optimal solution for aggregate investment to be

\textsuperscript{26} Although they do not explicitly state it, we believe that Jensen and Long had this basic descriptive idea in mind when they formulated the model, particularly, since they refer to the set of projects as the “same” project although strictly for \( \beta < 1 \), it is not.

\textsuperscript{27} In [4], p. 161, (18).

\textsuperscript{28} For \([\tilde{\rho} - r - \lambda \sigma_{\mu}] > 0\), (45) is only valid for \( 0 < \beta < 1 \). Inspection of (44) shows that for \( \beta = 0 \), the only “equilibrium” is \( I^* = \infty \), although \( I^*_{j} \) is finite. While for \( \beta < 0 \), \( I^* \) and \( I^*_{j} \) are infinite. However, the \( \beta < 0 \) cases are empirically not relevant, particularly when the amount of investment becomes unbounded.

\textsuperscript{29} The case where \( \sigma_{\rho} < 0 \) for some old firms can be handled in a similar fashion to the analysis in Section 2.

\textsuperscript{30} In [4], p. 164, (31).
\[ I_w(K) = \text{Max} \left\{ 0, \frac{K + N}{[1 + \beta(K + N - 1)]\lambda \sigma_p^2} [\bar{p} - r - \lambda \sigma_{M_p}] \right\} \]

(47)

and the optimal amount of investment for each firm to be

\[ I_{wj} = \frac{I_w(K)}{(K + N)}, \quad j = 1, \ldots, N, \ldots, N + K. \]

(48)

Therefore, the unconstrained Pareto optimum will be (for \(0 < \beta \leq 1\))

\[
\lim_{K \to \infty} I_w(K) = \text{Max} \left\{ 0, \frac{1}{\beta \lambda \sigma_p^2} (\bar{p} - r - \lambda \sigma_{M_p}) \right\}
\]

\[ = I^* \text{ from (45).} \]

(49)

Hence, for \(0 < \beta \leq 1\), the equilibrium amount of aggregate investment will equal the Pareto optimal amount as long as entry to the capital market is unrestricted. Further, from (48) and the symmetry of the equilibrium, the equilibrium distribution of investment across firms will be Pareto optimal.\(^\text{31}\)

Since the model of this section is essentially the same as the ones used by Fama and Stiglitz, why did they deduce different results? Fama\(^\text{32}\) deduces from his model that unless returns are perfectly correlated across firms, the marginal rates of substitutions for investors will not equal the marginal rate of transformation for the economy as a whole, and therefore, the "competitive" equilibrium is not a Pareto optimum. In comparing his analysis with Jensen and Long's, he goes on to state,\(^\text{33}\) "They [Jensen and Long] interpret the result as due in part to the existence of monopolistic competition in the risky industry. But that explanation is not relevant here since the analysis in this section has assumed perfect competition. The result is that the competitive equilibrium is not Pareto optimal." Fama is mistaken in both his criticism of the Jensen-Long interpretation and his interpretation of his own model's results. Throughout our paper, it has been stressed that strict interpretation of the competitive assumptions imply that (1) returns on the same project across firms must be perfectly correlated and (2) the number of entering firms cannot be \(a \text{ priori}\) fixed without implying restricted entry to the capital market. While Fama's model does not violate the second condition, it does require that the number of firms be finite if the non-Pareto optimal result is to obtain. However, without restrictions on entry, we have shown that the equilibrium number of firms is infinite. If he had applied this additional condition to his equation (21), he would

\(^{31}\) Jensen and Long also derived this result as they let the number of firms tend to infinity. Further, they do state that as the number of firms become large, the resulting solution is a competitive equilibrium. However, they did not emphasize that value-maximization by firms plus unrestricted entry to the capital market make this the only stable equilibrium solution.

\(^{32}\) In [3], p. 524, (21).

\(^{33}\) Fama [3], p. 525.
have found that the resulting equilibrium was a Pareto optimum provided that the correlation of returns between firms is positive.

Stiglitz recognizes that he is implicitly restricting entry by holding the number of firms fixed, but he concludes,\textsuperscript{34} "Extensions of the model, allowing, for instance, for free entry of firms, reinforce the results reported here. In that case, not only will the total level of investments in the risky industries be too small, but also the number of firms will not be optimal. The economy will be operating below its mean variance frontier (allowing for a variable number of firms) even when the firms are independent."

The reasons that Stiglitz' conclusions differ from ours even when he allows for free entry is that he devotes most of his analysis to the case where returns across firms are independent (i.e. $\beta = 0$). Therefore, we reexamine our analysis for the $\beta = 0$ case. For simplicity, consider the case where there are no old firms (i.e. $N = 0$ and $\sigma_{M0} = 0$) and $p > r$. From (44), the equilibrium amount of investment by the $i$th new firm is\textsuperscript{35}

$$ I^* = \frac{1}{2\lambda \sigma_p^2} (\bar{p} - r), \quad (50) $$

which is half the Pareto optimal amount. Note, however, that as long as $p > r$, new firms will continue to enter, and therefore, the number of new firms that enter will become infinite. Further, unlike in the $\beta > 0$ case, each firm will invest a finite amount, and hence, the aggregate amount of investment demanded will become infinite which implies that no equilibrium exists under the hypothesized conditions.

One source of the nonexistence is the simplifying assumption that there exist unlimited amounts of riskless borrowing at the fixed rate $r$. While this assumption may be reasonable when the aggregate amount of investment is finite, it is absurd when the investment demand at that rate is infinite. Therefore, one would expect that $r$ would tend toward $\bar{p}$ in the limit. To see why, note that if $V_i = I_i$ (which is required in equilibrium for no further entry), then the return per dollar invested in the resulting market portfolio will have expected value $\bar{p}$ and variance $\sigma_p^2 / K$ where $K$ is the number of new firms. Clearly, as $K \to \infty$, the Law of Large Numbers takes over and the market portfolio dominates the riskless asset. On the other hand, if we require that $\bar{p}I_i / V_i = r$ to avoid this dominance, then $V_i > I_i$, and as $K \to \infty$, the market value of assets becomes infinite. Thus, for $\beta = 0$, and fixed $r$, no equilibrium exists under our definition of entry, and there is no stable Stiglitz-equilibrium with respect to the number of firms.

Alternatively, we could rule out the $\beta \leq 0$ cases as simply unrealistic when combined with entry. While there is no natural restriction on the sign of $\sigma_{M0}$ and $\beta$ could reasonably be less than one due to slightly different applications of the same technology, etc., it is difficult to imagine that the returns on firms entering the same industry could be distributed such that $\beta \leq 0$, particularly under the assumption that entrance by one firm does not affect the distribution of cash flow for another firm.

\textsuperscript{34} Stiglitz [13], p. 55.

\textsuperscript{35} See Jensen and Long [4], p. 168.
While it could also be argued that having an infinite number of firms enter is equally unrealistic, this result followed directly from the assumption of infinitely-divisible assets. Clearly, if investment were "lumpy," the number of firms entering would be finite. Further, if (as we believe) our nontatonnement approach to stability used in Section 2 is more descriptive of reality than the classical tatonnement process, then, even if a finite (and not very large) number of firms enter, the discrepancy between the aggregate amount of investment taken and the theoretical Pareto optimal amount will be quite small.

6. Conclusion

The cornerstones of our analysis are the assumptions that as long as firms (entrepreneurs) are value-maximizers and profit opportunities exist, then if they can enter, they will enter, and in a competitive market, entry is not restricted. We have shown that if firms act as price takers, the social welfare and value-maximization criteria are consistent. If firms do not act as price takers but the market structure is such that entry is free, then the resulting equilibrium investment allocation will be no less than the Pareto optimum amount. Further, this result still obtains for imperfect competition provided that access to the capital markets is free and that similar technologies are available to all firms entering an industry.

Our paper should not be interpreted as a criticism of the Jensen and Long, Stiglitz, and Fama articles, but rather as a clarification of whether their results apply to a competitive or "near" competitive market. By using stability analysis, we have shown that their equilibria are not stable with respect to the number of firms, and we have questioned the Stiglitz and Fama definitions of "price taker." Hence, without explicit additional assumptions of barriers to entry, we also question the assumption of a fixed number of firms by Jensen and Long and Stiglitz, and of a finite number of firms by Fama. However, their works do throw light on the important problems associated with noncompetitive markets with restricted entry.

Appendix

Consider the case of sequential investment decision making by firms (both old and new) that arrive in the market place in a random fashion. In general, the equilibrium distribution of investment across firms and the equilibrium total investment in the project will depend on the order in which firms arrive in the market place. However, independent of the ordering, the equilibrium total investment in the project will never be less than the Pareto optimal amount. Moreover, the equilibrium total investment in the project will equal the Pareto optimal amount for any sequence of arrivals with a partial sequence which includes all old firms with a negative covariance with the project (i.e., \( \sigma_{ij} > 0 \)) such that that at that point in the sequence, the total investment in the project is less than or equal to the Pareto optimum.
From (4), the optimal investment for the jth firm to arrive is

\[ I^*_j = \text{Max} \left\{ 0, \frac{1}{2\lambda \sigma_p^2} [\tilde{\rho} - r - \lambda (\sigma_{Mp}(j) + \sigma_{ip})] \right\} \] (A1)

\[ = \text{Max} \left\{ 0, 1/2 \left( I^*_w - \sum_{t=1}^{j-1} I^*_t \right) - \frac{\sigma_{ip}}{2\sigma_p^2} \right\}, \]

where \( \sigma_{Mp}(j) \equiv \text{Cov}(\tilde{D}_H(j), \tilde{\rho}) = \sigma_{Mp} + \left( \sum_{t=1}^{j-1} I^*_t \right) \sigma_p^2. \)

Since an equilibrium obtains only when no firm upon arrival would invest in the project, we have from (A1) that the equilibrium amount of total investment cannot be less than the Pareto optimal amount. Otherwise, a new firm would have incentive to enter.

Inspection of (A1) shows that the later a particular firm arrives in a given sequence, the smaller the amount of investment taken in the project by that firm. It is straightforward to show that the particular sequence of firm arrivals which produces the largest amount of equilibrium total investment in the project is the one where firms arrive in the order of nondecreasing covariance of the project with its own previous projects. I.e., if (by renumbering if necessary) firms are labeled by the order in which they arrive for the maximal-investment sequence, then \( \sigma_{1p} \leq \sigma_{2p} \leq \ldots \leq \sigma_{mp} \leq \ldots \) If there are no old firms with negative own covariances with the project, then in the maximal-investment sequence, the first (and only) firms to take positive investment in the project are new firms. But, this is the case examined in the text. Therefore, if there are no old firms with negative own covariances, then the equilibrium quantity of investment will be the Pareto optimal amount independent of the order of arrivals.

If there are \( m \) firms with negative own covariances and if \( I(m) \) denotes the aggregate amount of investment for the first \( m \) arrivals in the maximal investment sequence, then, from (A1), we have that

\[ I(m) = [1 - (1/2)^J]I^*_w - \frac{1}{2\sigma_p^2} \left[ \sum_{j=1}^{J} (1/2)^{J-j} \sigma_{jp} \right], \] (A2)

where \( J \) is the index of the last firm to undertake positive investment in the project (\( J \leq m \)). If \( I(m) \geq I^*_w \), then \( I(m) \) is a stable equilibrium amount of investment. If \( I(m) < I^*_w \), then new firms will enter, and the equilibrium amount of investment will be \( I^*_w \). Thus, from (A2), a sufficient condition for the equilibrium amount of investment to equal the Pareto optimal amount for every sequence of arrivals is that

\[ \frac{1}{2\sigma_p^2} \sum_{j=1}^{J} 2^j \sigma_{jp} \geq -I^*_w. \] (A3)

References


