OPTIMAL PORTFOLIO RULES IN CONTINUOUS-TIME WHEN THE NONNEGATIVITY CONSTRAINT ON CONSUMPTION IS BINDING

by

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1. Introduction

Merton (1971, pp. 388-394) derives optimal consumption and portfolio rules for the HARA (hyperbolic absolute risk aversion) family of utility functions in a continuous-trading environment with asset returns that are jointly log-normally distributed. As noted by Sethi and Taksar (1988), for some members of the HARA family, these derived policies are not feasible because they violate the nonnegativity constraints on either consumption or wealth. Using the same log-normality assumption, Karatzas, Lehoczky, Sethi, and Shreve (1986) apply dynamic programming to make an extensive study of the effects of the nonnegativity constraints on optimal policies and in particular, they solve for the optimal rules for HARA utility. Cox and Huang (forthcoming/a; forthcoming/b) use a martingale representation technology to derive an alternative method for solving the constrained optimal consumption/portfolio-selection problem. In contrast to dynamic programming, their approach does not require differentiability of the derived-utility function. Moreover, their method has the rather remarkable feature that the optimal consumption and portfolio policies can be determined by solving one algebraic transcendental equation and a linear partial-differential equation of the classic parabolic type. Unlike the nonlinear Bellman equation of dynamic programming, general existence and uniqueness conditions for a solution to this type of linear equation are well known.\(^1\) This paper applies both the dynamic programming and Cox-Huang
methodologies. Dynamic programming is used to analyze the portfolio behavior of long-lived investors for whom the nonnegativity constraint on consumption is currently binding. The Cox-Huang technique is used to determine the optimal consumption and portfolio rules of those HARA preference functions for which the unconstrained solutions in Merton (1971) are invalid.

As in Merton (1969; 1971), the assumption of an infinite-time-horizon investor with exponential time preference vastly simplifies the solutions for the optimal consumption and portfolio rules. Because the focus here is on the effects of the nonnegativity constraints on optimal policies, we concentrate our analysis on this simplifying case. Consider therefore an investor with an infinite time horizon \( T = \infty \) with preferences for consumption at time \( t \) given by:  

\[
U(C,t) = e^{-\rho t} V(C),
\]

for \( C = C(t) \), where: (i) \( V'(C) > 0 \) and \( V''(C) < 0 \) for \( C \geq 0 \); (ii) \( \rho \) is sufficiently large to satisfy the transversality conditions; and (iii) \( V'(C) \leq M < \infty \) for all \( C \geq 0 \) and some fixed number \( M \). Conditions (i) and (ii) ensure the existence of an optimal policy. Condition (iii) implies that for each \( t \), there exists a level of wealth, \( W^+(t) > 0 \), such that \( C^*(t) = 0 \) for \( W(t) \leq W^+(t) \). From (i), (ii) and (iii), we can posit with no loss in generality that \( V(0) = 0 \) and therefore, that \( V(C) > 0 \) for \( C > 0 \).

A particular optimal portfolio called the "growth-optimum" portfolio plays a central role in the Cox-Huang (forthcoming/b) analysis. As shown in Appendix A, the growth-optimum portfolio strategy maximizes the expected logarithm of wealth at any future date. If \( X(t) \) denotes the value
of the growth-optimum portfolio at time $t$, then we can express the dynamics for $X(t)$ as

$$dX = (\mu^2 + r)X \, dt + \mu X \, dz,$$

where $\mu^2 > 0$, $r$ is the riskless interest rate, and $dz$ is a standard Wiener process. As in Merton (1971, pp. 384-394), we assume throughout this paper that asset returns are jointly log-normally distributed. Hence, from Theorem II and its corollary in Merton (1971), all investors' optimal portfolios can be generated by combinations of any instantaneously mean-variance-efficient portfolio and the riskless security. From (A.4) in Appendix A, the continuous-time, growth-optimum portfolio strategy is instantaneously mean-variance efficient. Moreover, for the posited constant investment opportunity set, we have from (A.5) that $\mu^2$ and $r$ are constants and that $X(t+t)/X(t)$ is log-normally distributed for all $t$ and $t > 0$. Therefore, without loss of generality, we can assume that all investors select their optimal portfolios from just two assets: the growth-optimum portfolio and the riskless security. Thus, our model is identical to the one in Merton (1969, Section 6), except we now take account of the nonnegativity constraints on consumption and wealth.

Let $\hat{J}(W,t) \equiv \max E_t[\int_t^\infty U(C,s)ds]$ denote the indirect utility function for the investor at time $t$, given his wealth is $W$. To capture the nonnegativity constraint on wealth, we require that $W = 0$ is an "absorbing" state, and hence, that $\hat{J}(0,t) = \int_t^\infty U(0,s)ds$. From (34) and (35) in Merton (1969), we can transform the indirect utility function $\hat{J}$ so that Bellman's dynamic programming equation of optimality can be written as:

$$0 = \max \left\{ V(C) - pJ(W) + J'(W)[(\mu^2 + r)W - C] + \frac{1}{2}J''(W)\mu^2W^2 \right\},$$

for $\{C,W\}$.
subject to the constraint $C \geq 0$ and the boundary conditions:

(3a) \[ J(0) = 0 \]

and

(3b) \[ \lim_{t \to \infty} E[e^{-\rho t}J(W(t))] = 0, \]

where $J(W) = \exp[\rho t]J(W,t)$ and $J$ is solely a function of $W$, independent of explicit time or asset prices.\(^5\) Hence, the Bellman equation reduces to an ordinary differential equation.

2. **Optimal Portfolio Rules in the Region $C^*(t) = 0$**

Rather than solve for the optimal strategies using standard Kuhn-Tucker multipliers, we use a different method that exploits the special structure of (3). By inspection of (3), the optimal consumption and portfolio rules will depend only on current wealth (i.e., for $W = W(t)$, $C^*(t) = C^*(W)$ and $w^*(t)W = w^*(W)W$). Hence, from condition (iii) of (1), $W^+$, the critical level of wealth such that $C^*(W) = 0$ for $0 \leq W \leq W^+$, is a constant, independent of either time or asset prices. With this condition on $W^+$, we can express (3) as two linked-but-unconstrained optimization problems. Define $J_1(W) \equiv J(W)$ for $0 \leq W \leq W^+$ and $J_2(W) \equiv J(W)$ for $W > W^+$. The existence of a dynamic-programming solution in (3) requires that $J$ be twice-continuously differentiable for $0 < W < \infty$. From condition (iii) of (1), $0 < W^+ < \infty$. Thus, we require that $J_1$ and $J_2$ satisfy:

(4) \[
J_1(W^+) = J_2(W^+)
\]

\[ J_1'(W^+) = J_2'(W^+) \]

\[ J_1''(W^+) = J_2''(W^+) \]
Because $C^*(W) = 0$ for $W \leq W^+$ and $V(0) = 0$, we have from (3), for $0 \leq W \leq W^+$:

\begin{equation}
0 = \max \left\{ -\rho J_1'(W) + J_1''(W)(w\mu^2 + r)W + \frac{1}{2} J_1'''(W)\mu^2 w^2 W^2 \right\},
\end{equation}

subject to: $J_1(0) = 0$ and (4). Because $C^*(W) > 0$ for $W > W^+$, $J_2(W)$ will satisfy (3) with (3a) replaced by (4).

The first-order condition for (5) is given by:

\begin{equation}
0 = J_1'(W)\mu^2 W + J_1''(W)\mu^2 W^2 w^*.
\end{equation}

For $\mu^2 > 0$ and $W > 0$, we have from (6) that, for $0 \leq W \leq W^+$,

\begin{equation}
w^* = -J_1'(W)/[J_1''(W)W].
\end{equation}

In the usual dynamic-programming fashion, we substitute for $w^*$ in (5) from (7) to arrive at the Bellman equation:

\begin{equation}
0 = -\rho J_1(W) + rW J_1'(W) - \frac{1}{2} \mu^2 [J_1'(W)]^2 / J_1''(W).
\end{equation}

For $\rho > 0$, two solutions to (8) are given by:

\begin{equation}
J_1(W) = J_2(W^+)(W/W^+)^{\beta_1}, \ i = 1, 2,
\end{equation}

where $\beta_1 = [r + \rho + \mu^2/2 - \left( (r + \rho + \mu^2/2)^2 - 4r\rho/2 \right)^{1/2}] / 2r$ and $\beta_2 = [r + \rho + \mu^2/2 + \left( (r + \rho + \mu^2/2)^2 - 4r\rho/2 \right)^{1/2}] / 2r$. For $\rho > 0$ and $\mu^2 > 0$, $0 < \beta_1 < 1$ and $\beta_2 > 1$.

To determine which of the solutions in (9) is the optimum, we apply the boundary conditions from (5). Both solutions satisfy $J_1(0) = 0$ and $J_1(W^+) = J_2(W^+)$. By differentiation of (9),
\[ J_1'(W) = J_2(W^+)(W/W^+)^{\beta_1-1} \beta_1 - \beta_1 (W/W^+) \quad \text{and} \quad J_1''(W) = J_2(W^+)(W/W^+)^{\beta_1-2} \quad \left[ \beta_1 (\beta_1 - 1) \right]/(W^+)^2. \]

Evaluating these derivatives at \( W = W^+ \) and applying the conditions in (4), we have that, for \( i = 1,2 \):

\begin{align*}
(10a) \quad J_2'(W^+)/J_2(W^+) &= \beta_1/W^+ , \\
\text{and} \\
(10b) \quad J_2''(W^+)/J_2(W^+) &= \beta_1 (\beta_1 - 1)/(W^+) ^2 .
\end{align*}

From (1), \( V(C) \geq 0 \), \( V'(C) > 0 \) and \( V''(C) < 0 \) for \( C \geq 0 \). Therefore, for \( W \geq W^+ \), \( J_2(W) \) is a positive, strictly increasing and concave function. Because both \( \beta_1 \) and \( \beta_2 \) are positive, each solution in (9) is consistent with \( J_2'(W^+)/J_2(W^+) > 0 \) and (10a). However, only \( 0 < \beta_1 < 1 \) is consistent with \( J_2''(W^+)/J_2(W^+) < 0 \) and (10b). Hence, the optimal solution \( J \) for \( 0 \leq W \leq W^+ \) is given by:

\[ J_1(W) = J_2(W^+)(W/W^+)^\beta , \]

where \( \beta \equiv [r + \rho + \mu^2/2 - \left( (r + \rho + \mu^2/2)^2 - 4\rho_0 \right)^{1/2}/2r \) for \( r > 0 \) and \( \beta \equiv 2\rho/[2\rho + \mu^2] \) for \( r = 0 \).

To determine \( J \) for \( W > W^+ \), we have from (10a) and (10b) that \( J_2(W) \) satisfies (3) subject to (3b) and the conditions:

\[ J_2'(W^+)/J_2(W^+) = \beta/W^+ \quad \text{and} \quad J_2''(W^+)/J_2(W^+) = (\beta - 1)/W^+ . \]

The seemingly "extra" boundary condition is required to uniquely determine both \( J_2(W) \) and the optimal boundary point, \( W^+ \).

We summarize our findings for optimal portfolio behavior in the region where the nonnegativity constraint on consumption is binding in the following theorem and its corollaries:
THEOREM 1. If asset returns are jointly log-normally distributed and if an investor's preferences satisfy (1), then for every \( t \) such that \( C^*(t) = 0 \), the optimal-portfolio strategy is a constant-proportion, levered combination of the growth-optimum portfolio and the riskless security, with fraction \( w^*(t) = 1/(1 - \beta) > 1 \) allocated to the growth-optimum portfolio and fraction \([1 - w^*(t)] = -\beta/(1 - \beta) < 0\) in the riskless security.

Proof: \( C^*(t) = 0 \) if and only if \( 0 \leq W(t) \leq W^+ \). Hence, \( J = J_1(W(t)) \) as given in (11). By differentiation of (11) twice and substitution of \( J'_1 \) and \( J''_1 \) in (7), we have that for \( 0 \leq W(t) \leq W^+ \), \( w^*(t) = 1/(1 - \beta) \). From the definition of \( \beta \) in (11), \( \beta \) is a constant such that \( 0 < \beta < 1 \). Hence, \( w^*(t) \) is a constant greater than one. Thus, the fraction invested in the riskless security is \([1 - w^*(t)] < 0\).

Q.E.D.

To determine the investor's optimal allocations among individual risky assets under the hypothesized conditions of Theorem 1, we have that, for \( 0 \leq W(t) \leq W^+ \):

\[
(12) \quad w^*_k(t) = \frac{w^g_k(t)}{(1 - \beta)} = \frac{\sum_{i=1}^{m} v_{kj} (a_{ij} - r)/(1 - \beta)}{k = 1, \ldots, m},
\]

where \( w^g_k(t) \) is given in (A.1) of Appendix A. By inspection of (12) and the definition of \( \beta \) in (11), when the nonnegativity constraint on consumption is binding, the optimal-portfolio fractions depend only on the investment-opportunity set \( \{a_{ij}, r, \sigma_{ij}; i, j = 1, \ldots, m\} \) and the investor's rate of time preference, \( \rho \). The portfolio allocations do not depend on the functional
form of $V(C)$, the investor's current wealth, explicit time, or $W^*$. As an immediate consequence, we have that:

**Corollary 1.1.** If the hypothesized conditions of Theorem 1 obtain, then at each $t$, all investors with the same constant rate of time preference, and for whom $C^*(t) = 0$, will: (i) hold the identical optimal-portfolio fractions and (ii) choose their portfolios "as if" they had in common the iso-elastic utility function, $U(C,t) = 0$ and $B(W,T) = W^\beta/\beta$ for any $T > t$.

**Proof:** As noted, $\beta$ depends only on the investment-opportunity set and $\rho$. From Theorem 1, if $C^*(t) = 0$, then $w^*(t) = 1/(1 - \beta)$. Thus, all investors with the same rate of time preference $\rho$, and for whom $C^*(t) = 0$, will choose at time $t$ the same portfolio allocation, given by (12). From Merton (1969, equation 29), an investor with preferences such that $U(C,t) = 0$ and $B(W,T) = W^{\gamma}/\gamma$, $\gamma < 1$, for some $T > t$, will hold optimal-portfolio fraction in the risky asset, $w^*(t) = (\mu - r)/(\sigma^2(1 - \gamma))$, independently of $W(t)$, $t$ or $T$. Here, $\mu^2 = \alpha - r$ and $\sigma^2 = \mu^2$. Hence, for $\gamma = \beta$, such an investor's optimal-portfolio allocation is $w^*(t) = 1/(1 - \beta)$.

Q.E.D.

**Corollary 1.2.** If the hypothesized conditions of Theorem 1 obtain and if an investor's optimal consumption satisfies $C^*(s) = 0$ for $\tilde{t} \leq s \leq t^+$, then the investor's wealth at time $s$, $\tilde{t} \leq s \leq t^+$, can be expressed as:
\[ W(s) = W(t) \exp[-(a - 1)(r + \frac{3\mu^2}{2})(s - t)] \frac{X(s)}{X(t)}^a, \]

where \( a \equiv 1/(1 - \beta) > 1 \) and \( \beta \) is as defined in Theorem 1.

**Proof.** By hypothesis, for \( s \in [t, t^+] \), \( C^*(s) = 0 \) and therefore, from Theorem 1, \( W^*(s) = 1/(1 - \beta) \equiv a \), for all \( s \in [t, t^+] \). Hence, from (2), the dynamics of the investor's wealth can be written as
\[ \frac{dW}{W} = a \frac{dX}{X} + (1 - a) \mu \frac{dz}{\sqrt{t - s}} \text{ for } t < s < t^+. \]

Using Itô's Lemma, we can integrate this stochastic differential equation to obtain
\[ W(s)/W(t) = \exp[-(a - 1)(r + \frac{3\mu^2}{2})(s - \tilde{t})] \frac{X(s)}{X(\tilde{t})}^a \text{ for } \tilde{t} \leq s \leq t^+. \]

Q.E.D.

**COROLLARY 1.3.** If the hypothesized conditions of Theorem 1 obtain, if the investor's initial wealth, \( W(0) \), is positive, and if \( X(0) > 0 \), then, for all \( t \geq 0 \), (i) \( W(t) = 0 \) only if \( X(t) = 0 \) and (ii) \( \Prob\{W(t) > 0|W(0) > 0\} = 1 \).

**Proof.** By hypothesis, \( W(0) > 0 \), and from (1), \( W^+ > 0 \). Trivially, for all \( t \) such that \( W(t) \geq W^+ \), \( W(t) > 0 \). Because the investor's optimally-invested wealth has a continuous sample path and because \( W(0) > 0 \), if \( W(t) = 0 \) for some \( t > 0 \), then there exists a time \( \tilde{t} \) such that \( W(\tilde{t}) = W^+ \) and \( W(s) \leq W^+ \) for all \( s \in [\tilde{t}, t] \) where \( 0 < W^+ < W^+ \). Hence, the hypothesized conditions of Corollary 1.2 are satisfied for \( s \in [t, t] \). By inspection of the formula for \( W(t) \) in Corollary 1.2, \( W(t) = 0 \) only if \( X(t) = 0 \), which proves (i). From (A.3) of Appendix A,
\[ \text{Prob}[X(t) > 0 | X(0) > 0] = 1. \] Hence, from (i), \( \text{Prob}[W(t) > 0 | W(0) > 0] = 1 \), which proves (ii). \(^8\)

Q.E.D.

From Corollary 1.3, we have that all investors with preferences that satisfy (1) will choose optimal-portfolio strategies that do not risk ruin, and hence, the probability of bankruptcy (i.e., \( W(t) = 0 \)) is zero. Therefore, even when the nonnegativity constraint on consumption is binding, the nonnegativity constraint on wealth is not. This no-bankruptcy result was shown to obtain in the special case of infinite-lived investors and a constant investment-opportunity set. \(^9\) However, as proved in Merton (1990, Chapter 16, Theorem 2), it will also obtain for finite-lived investors with state-dependent utility in environments of relatively-general stochastic investment-opportunity sets. \(^10\)

3. **Intuition for the Optimal Rules: Clock Time Measured by Investor Impatience**

Having formally established the properties of optimal portfolios when the nonnegativity constraint on consumption is binding, we now try our hand at providing some intuition as to why Theorem 1 and its corollaries obtain. This to be followed by that promised analysis of the HARA utility functions.

Define the random-variable time interval \( \tilde{\tau} \), conditional on \( W(t) = W \), by:

\[
\tilde{\tau} = \inf\{s \in [0, \infty) : W(t + s) \geq W^+ \}. \tag{13}
\]
\( \tilde{\Delta t} \) is thus the first-passage time interval from time \( t \) such that
\[ C^\ast(t + s) = 0 \text{ for } 0 \leq s < \tilde{\Delta t} \text{ and } C^\ast(t + \tilde{\Delta t}) > 0. \] Obviously, for \( W > W^+ \), \( C^\ast(t) > 0 \) and \( \tilde{\Delta t} = 0 \). For \( W < W^+ \), \( \tilde{\Delta t} \) is a nondegenerate random time interval whose distribution depends on \( W \) and the investor's portfolio strategy between \( t \) and \( t + \tilde{\Delta t} \). For each investor with \( C^\ast(t) = 0 \) and for all \( s, 0 \leq s < \tilde{\Delta t}, C^\ast(t + s) = 0 \), and hence, \( V(C^\ast(t + s)) = 0 \). Therefore, at time \( t \), the indirect utility function \( \hat{J}(W,t) \) for such an investor can be expressed as:

\[
(14) \quad \exp[-\rho t]J(W) = \max_{\{w\}} E_t\{\exp[-\rho(t + \tilde{\Delta t})]J_2(W^+)\}
\]

\[
= \exp[-\rho t]J_2(W^+)\max_{\{w\}} E_t\{\exp[-\rho \tilde{\Delta t}]\}
\]

because \( J_2(W^+) > 0 \) and is nonstochastic. By inspection, the maximizing strategy in (14) depends at most on \( W \) and \( W^+ \).

Clearly, the strategy that maximizes \( E_t\{\exp[-\rho \tilde{\Delta t}]\} \) also minimizes \( E_t\{(1 - \exp[-\rho \tilde{\Delta t}])/\rho\} \) for \( \rho > 0 \). However, the latter representation has the rather-intuitive interpretation of minimizing the expected time until optimal consumption is positive, where time is measured by the appropriate clock. To see this, consider first the limiting case as \( \rho \to 0 \). In the limit, the criterion becomes \( \min E_t\{\tilde{\Delta t}\} \). For \( \rho > 0 \), imagine a clock that keeps time according to the time scale \( \tau(\rho) \), where:

\[
(15) \quad \tau(\rho) \equiv (1 - e^{-\rho t})/\rho .
\]

Note that \( \tau(\rho) \) has dimension of time and that \( d\tau(\rho) = \exp[-\rho t]dt = [1 - \rho \tau(\rho)]dt > 0 \). Initially, \( \tau(\rho) = t = 0 \) and \( d\tau(\rho) = dt \). However, as time passes, the \( \tau(\rho) \)-clock moves ever more slowly as measured against
ordinary-clock time. Indeed, for \( t = \infty \), \( \tau(\rho) = 1/\rho \). Moreover, the larger is \( \rho \), the slower is the \( \tau(\rho) \)-clock.

From (13) and (15), we can express the first-passage time interval until optimal consumption is positive in terms of \( \tau(\rho) \)-clock time as:

\[
\Delta \tau(\rho) = \frac{1 - e^{-\rho(t+\Delta t)}}{\rho} - \frac{1 - e^{-\rho t}}{\rho}
= e^{-\rho t} \frac{1 - e^{-\rho \Delta t}}{\rho}.
\]

### Theorem 2

If \( \{w^+(t)\} \) is the portfolio strategy at time \( t \) that minimizes \( E_c[\Delta \tau(\rho)] \) for \( \rho \geq 0 \) and \( 0 < W(t) < W^+ \), then (i) \( w^+(t) = 1/(1 - \beta) \) and (ii) \( \partial w^+(t)/\partial \rho > 0 \), where

\[
b = [r + \rho + \mu^2/2 - (r + \rho + \mu^2/2 - 4\rho)^{1/2}]/2r.
\]

**Proof.** Define \( I(W,t) = \min_{T_t} E_c[\Delta \tau(\rho)] \) for \( W = W(t) < W^+ \). It follows from (16) that \( I(W,t) = \min_{T_t} \{ \exp(-pt) \} \) where \( T = t + \Delta \).

By the principle of dynamic programming, \( I(W,t) \) will satisfy:

(a) \( 0 = \min\{\exp(-pt) + I_c(W,t) + \int_{W}^{W^+} [w^+ \mu^2 + r]W + \int_{W}^{W^+} I_{WW}(W,t)(w^+)^2 \mu^2W^2/2 \} \)

subject to the conditions: \( I(0,t) = \exp(-pt)/\rho \) because \( \Delta \tau = \infty \) if \( W(t) = 0 \)
and \( I(W^+,t) = 0 \) because \( \Delta \tau = 0 \) and \( t = T \) if \( W(t) = W^+ \).

Let \( I_1(W) = \exp(pt) I(W,t) \). By substitution into (a), we have that:

(b) \( 0 = \min\{1 - \rho I_1(W) + I_1'(W)[w^+ \mu^2 + r]W + I_1''(W)(w^+)^2 \mu^2W^2/2 \} \) subject to:

\( I_1(0) = 1/\rho \) and \( I_1(W^+) = 0 \). For \( \rho = 0 \), \( I(W,t) = I_1(W) = \log(W^+/W)/(r + \mu^2/2) \) and \( w^+(t) = 1.11 \). For \( \rho > 0 \), define \( I_2(W) = 1 - \rho I_1(W) \). Multiplying (b) by -1, which changes "min" to "max" and substituting for \( I_1(W) \), we have that:

(c) \( 0 = \max\{-\rho I_2(W) + I_2'(W)[w^+ \mu^2 + r]W + I_2''(W)(w^+)^2 \mu^2W^2/2 \} \) subject to: \( I_2(0) = 0 \) and \( I_2(W^+) = 1 \).

The first-order condition is \( 0 = \mu^2W[I_2'(W) + I_2''(W)W^+] \). The second-order
condition for a maximum is \( I''_2(W) < 0 \) for \( W > 0 \). By inspection, (c) here is identical to (5) with \( J_2(W^+) = 1 \). Hence, \( I_2(W) \) is given by (9) with \( J_2(W^+) = 1 \). Because \( I''_2(W) < 0 \), we have from (11) that \( I_2(W) = (W/W^+)^\beta \) and \( w^+(t) = 1/(1 - \beta) \). Note that for \( \rho = 0, \beta = 0 \), and (i) is proved. \( \partial w^+(t)/\partial \rho = (\partial \beta/\partial \rho)/(1 - \beta)^2 \). From the definition of \( \beta, \beta < 1 \) for \( \rho \geq 0 \), and therefore, \( \partial \beta/\partial \rho = [1 - \beta]/\left((x + \rho + \mu^2/2)^2 - 4\rho\right)^{1/2} > 0 \). Hence, \( \partial w^+(t)/\partial \rho > 0 \), which proves (ii).

Q.E.D.

As indicated by (14), we have from Theorems 1 and 2 that the optimal-portfolio strategy when the nonnegativity constraint on consumption is binding, \( \{w^*(t)\} \), is identical to the one that minimizes the expected time until optimal consumption becomes positive, with time measured by a \( \tau(\rho) \)-clock. Investors with the same rate of time preference have the same \( \tau(\rho) \)-clock. Because the minimizing strategy, \( \{w^+(t)\} \), is independent of either current wealth or the "target" level of wealth, \( W^+ \), all such investors will follow the same portfolio strategy.

Although the proof of Theorem 2 analyzes the optimal strategies in ordinary-clock time, one can formulate the investor's entire optimal consumption-investment problem using \( \tau(\rho) \)-clock time. Suppose each investor uses a \( \tau(\rho) \)-clock with his own rate of time preference to keep track of time. So, for example, the passage of one "hour" according to one investor's clock could correspond to the passage of a "day" according to a less-impatient (i.e., smaller \( \rho \)) investor's clock. From Cox and Miller (1968, pp. 228-229), a Wiener process measured in \( \tau(\rho) \)-clock and ordinary-clock times satisfies the relations:

\[
(17a) \quad dz'(\tau(\rho)) = e^{-\rho t/2}dz(t)
\]
and
\[ dz(t) = dz'(\tau(\rho))/\sqrt{1 - \rho \tau(\rho)}, \]

where \( z'(\tau(\rho)) \) is the Wiener process in \( \tau(\rho) \)-clock time. Hence, if asset-return dynamics can be described by diffusion processes in ordinary-clock time, then they are also described by diffusion processes in \( \tau(\rho) \)-clock time. Therefore, if a prime on a variable denotes its measurement in \( \tau(\rho) \)-clock time, then the dynamics of the investor's wealth can be written as:

\[ dW' = [(w'(\mu')^2 + r')W' - C'] dt + w'\mu'W'dz'(\tau), \]

where \( \mu' = \mu/\sqrt{1 - \rho \tau}, \ r' = r/(1 - \rho \tau) \) and \( C'(\tau) = C(\tau)/(1 - \rho \tau) \). The investor's program objective, \( \max E_0[\int_0^\infty \exp[-\rho t] V[C(t)] dt] \), is transformed into:

\[ \max E_0'[\int_0^T V[(T - \tau)C'(\tau)/T] d\tau], \]

where \( T \equiv 1/\rho \) and \( (1 - \rho \tau) = (T - \tau)/T \).

Although all investors have infinite-time horizons in ordinary-clock time, from (19), we see that in the appropriate clock time, these horizons become finite and have different durations depending on each investor's rate of time preference. In this sense, investors with different rates of impatience have different time horizons. Moreover, because \( dT/d\rho < 0 \), the greater the rate of time preference, the shorter is the effective horizon.

For an investor with \( 0 < W(0) < \bar{W}^+ \) (and hence, \( C^*(0) = 0 \)), the optimal-portfolio strategy formulated in \( \tau(\rho) \)-clock time can be expressed as the solution to:
(20) \[ \min E_0 \{ \min (\Delta \tau(\rho), T) \} . \]

By inspection of (20), the investor treats all \( \Delta \tau(\rho) \) values \( \geq T \) as if they were \( T \). This indifference follows because once \( \Delta \tau(\rho) \) exceeds his time horizon, the implications for his optimal consumption program are the same: he consumes nothing. The larger is \( \rho \), the smaller is \( T \) and the shorter is the time interval at which the investor becomes indifferent. The greater degree of concavity in (20) induced by this truncation of the distribution for \( \Delta \tau(\rho) \) reflects an urgency to get wealth up to the \( W^+ \) level before it doesn't matter. Perhaps this provides an intuitive explanation as to why investors with greater temporal impatience for consumption choose more aggressive investment strategies (i.e., \( \partial W(t)/\partial \rho > 0 \)).

In summary, explicit solutions have been derived for the optimal-portfolio behavior of investors with preferences given by (1) and for whom the nonnegativity constraint on consumption is currently binding. Hence, to determine the global optimal consumption and portfolio behavior for such investors, (3) need only be solved in the region \( W^+ \leq W < \infty \). To do so, the boundary condition (3a) is replaced with the conditions:

(21a) \[ J'(W^+)/J(W^+) = \beta/W^+, \]

and

(21b) \[ J''(W^+)/J'(W^+) = -(1 - \beta)/W^+, \]

with \( \beta = [r + \rho + \mu^2/2 - (r + \rho + \mu^2/2)^2 - 4rp/2r]^{1/2} \). These two conditions are sufficient to uniquely determine \( J(W) \) and \( W^+ \). Because \( W^+ \geq 0 \) and \( C^*(t) \geq 0 \) for \( W(t) \geq W^+ \), there is no need to explicitly impose the nonnegativity constraints on consumption and wealth. Therefore, (3),
subject to (21a) and (21b), can be solved as an unconstrained optimization problem as in Merton (1969; 1971).

4. Optimal Consumption and Portfolio Rules for HARA Utility

The HARA family of utility functions defined in Merton (1971, p. 389) can be expressed in terms of (1) here as:

\[ V(C) = \frac{(1 - \gamma)}{\delta} \left( \frac{\zeta C}{1 - \gamma} + \eta \right)^\gamma, \]

subject to the parameter constraints: \( \gamma \neq 1; \ \zeta > 0; \ \eta > 0 \) for \( \gamma > 1; \ \eta = 1 \) if \( \gamma = -\infty \); and the domain of \( C \) restricted so that \( \left[ \frac{\zeta C}{(1 - \gamma)} \right] + \eta > 0. \) To ensure that \( C = 0 \) falls within this domain and that \( V'(0) \) is finite, as required by condition (iii) of (1), we consider only those HARA functions with \( \eta > 0. \)

Let \( \delta \equiv 1/(1-\gamma) \) so that \( 0 \leq \delta < \infty \) for \( \gamma < 1 \) and \( -\infty < \delta < 0 \) for \( 1 < \gamma < \infty. \) Define \( q(\delta) \equiv r/(r - (r - \rho - \mu^2/2)\delta - \delta^2\mu^2/2). \) Provided that the parameter values satisfy the condition that:

\[ 0 < q(\delta) < \infty, \]

we have from (48) and (49) in Merton (1971, p. 390) that the unconstrained optimal consumption and portfolio policies for \( T = \infty \) can be written as:

\[ C^*_u(t) = \left[ \frac{r}{q(\delta)} W(t) + \frac{\eta}{\zeta q(\delta)} \left[ \frac{1 - q(\delta)}{\delta} \right] \right] \text{ for } W(t) < \bar{W} \]

\[ = r\bar{W} \text{ for } W(t) \geq \bar{W} \]

and

\[ W^*_u(t) = \delta W(t) + \frac{\eta}{\zeta r} \text{ for } W(t) < \bar{W} \]

\[ = 0 \text{ for } W(t) \geq \bar{W} \]
where $\bar{W}$ is the satiation level of wealth as defined in (B.5) of Appendix B. Hence, $\bar{W} = -\eta/(\delta \zeta r)$ for $\delta < 0$ and $\bar{W} = \infty$ for $\delta \geq 0$.

Using the Cox-Huang technique outlined in Appendix B, we now determine the optimal policies subject to the nonnegativity constraints on consumption and wealth. From (22),

$$U^*_C(C,t) = \xi \exp[\rho t] [[\xi C/(1 - \gamma)] + \eta]^{\gamma - 1}.$$  

From (B.8a) and (B.10a) in Appendix B, the investor's optimal consumption path can be written as:

$$C^*(t) = G(X(t),t) = \max\{0, [\xi y(t)/\lambda_1 y(0)]^{\delta} - \eta\}/\xi \delta,$$

where $y(t) = \exp[-\rho t] X(t)$, and by inspection, $G = G(y(t))$, does not otherwise depend on explicit time. By inspection of (24),

$$y^* = \lambda_1 y(0) n^{1/\delta}/\xi \text{ is the critical value such that } C^*(t) = 0 \text{ for } y(t) \leq y^* \text{ and } C^*(t) > 0 \text{ for } y(t) > y^*.$$

From (B.13) of Appendix B, the investor's optimally-invested wealth satisfies $W(t) = F(X(t),t)$ where $F$ is defined in (B.11). From Theorem B.3, $F$ satisfies the partial differential equation:

$$0 = \frac{1}{2} \mu X^2 F_{XX} + rX F_X + F_t - rF + G.$$

where subscripts on $F$ denote partial derivatives with respect to $X$ and $t$. The boundary conditions for (25) are determined as follows: From Corollary 1.3, $W(t) = 0$ only if $X(t) = 0$, and from Cox and Huang (forthcoming/b, Proposition 2.2), $\partial F/\partial X > 0$ for $0 < F < \bar{W}$ Hence, $F(0,t) = 0$.

From the condition, $\partial F/\partial X > 0$ for $0 < F < \bar{W}$, we have that as $X \to \infty$, $F \to \bar{W}$ But, as $W \to \bar{W}$, the optimal consumption policy approaches
from below, the unconstrained optimal policy given by (23b). Therefore, from (23b) with \( W = F \):

\[
\lim_{X \to \infty} \left\{ \frac{F(X,t)}{W} \right\} = 1 \quad \text{for } \delta < 0
\]

\[
\lim_{X \to \infty} \left\{ \frac{G(X,t)}{F(X,t)} \right\} = r/q(\delta) \quad \text{for } \delta \geq 0.
\]

Define \( f \equiv F(e^{\rho t},t) \). Noting that \( G \) depends only on \( y \), we have by inspection of (25) and its boundary conditions that \( f = f(y) \), independent of \( t \). By substitution of \( \exp[\rho t]y \) for \( X \) and \( f \) for \( F \), (25) can be rewritten as an ordinary differential equation given by:

\[
0 = \frac{1}{2}\mu^2 y^2 f''(y) + (r - \rho) y f'(y) - rf(y) + G(y)
\]

subject to:

\[
f(0) = 0
\]

and

\[
\lim_{y \to \infty} \left\{ \frac{f(y)}{W} \right\} = 1 \quad \text{for } \delta < 0
\]

\[
\lim_{y \to \infty} \left\{ \frac{G(y)}{f(y)} \right\} = r/q(\delta) \quad \text{for } \delta \geq 0.
\]

The general homogeneous (i.e., \( G = 0 \)) solution to (26) can be expressed as:

\[
f_h(y) = Ay^a + By^b,
\]

where \( a \equiv \frac{-(r - \rho - \mu^2/2) + \left( (r - \rho - \mu^2/2)^2 + 2r\mu^2 \right)^{1/2}}{\mu^2} > 0 \) and \( b \equiv -\left[ (r - \rho - \mu^2/2) + \left( (r - \rho - \mu^2/2)^2 + 2r\mu^2 \right)^{1/2} \right]/\mu^2 < 0 \), and \( A, B \) are arbitrary constants. If \( \psi(\lambda) \equiv \mu^2 \lambda^2/2 + (r - \rho - \mu^2/2)\lambda - r \), then \( \psi(a) = \psi(b) = 0 \), and \( \psi(\lambda) = (\mu^2/2)[(\lambda - a)(\lambda - b)] \). For \( \rho > 0 \), \( a > 1 \) and for \( \rho = 0 \), \( a = 1 \).
In the region $0 \leq y \leq y^+$, $G(y) = 0$, and $f(y) = f_h(y)$ for the appropriate selection of $A$ and $B$. From (26a), $f(0) = 0$, which implies that $B = 0$ because $b < 0$. Replacing $A$ by $f(\gamma^+)\gamma^+ - a$, where $f(\gamma^+)$ is a number yet to be determined, we have that:

\[(28) \quad f(y) = f(\gamma^+)\gamma^+ - a \quad \text{for} \quad 0 \leq y \leq y^+.\]

By the definitions of $q$ and $\psi$, we have that $q(\delta) = -r/\psi(\delta)$. For existence of a nontrivial optimal policy, the parameters $(\delta, r, \mu^2$ and $\rho)$ must satisfy (23a). It follows that for this admissible set, $\psi(\delta) < 0$. Therefore, $b < \delta < a$.

In the region $y^+ < y$ in which $G(y) > 0$, we have from (24) and (26) that for $b < \delta < a$, the solution for $f$ is given by:

\[(29) \quad f(y) = A'\gamma^a + B'\gamma^b + \eta[q(\delta)(\gamma^\delta - 1)]/\xi r \delta,\]

where $A'$, $B'$ are constants to be determined. Because $a > \delta > b$, the boundedness condition (26b) is satisfied if and only if $A' = 0$. From Theorem B.3 in Appendix B, $f(y)$ is twice-continuously differentiable, and therefore, from (28), $f'(\gamma^+)/f(\gamma^+) = a/\gamma^+$ and $f''(\gamma^+)/f'(\gamma^+) = (a - 1)/\gamma^+$.

Imposing these conditions on (29), we have that:

\[(30) \quad B' = \left(\frac{-\eta q(\delta)}{b \zeta r}\right)[(a - \delta)/(a - b)](-b)\gamma^b\]

\[= \frac{\eta a}{\xi r (a - b)(\delta - b)}(-b)\gamma^b > 0.\]

Substituting for $B'$ in (29) and evaluating $f$ at $y = \gamma^+$, we have that:

\[(31) \quad W^+ = f(\gamma^+)\]

\[= \frac{-\eta b}{\xi r (a - b)(\delta - b)} > 0.\]
From (28)-(31), the investor's optimally-invested wealth at time $t$ can be written as:

$$
(32) \quad f(y(t)) = W^+(y(t)/y^+)^a \quad \text{for} \quad 0 \leq y(t) \leq y^+
$$

$$
= W^+ + \frac{a(\delta)}{r} \{ C^*(t) - (\frac{\eta}{\zeta}(a-\delta)[(y(t)/y^+)^b - 1] \}
$$

for $y^+ < y(t)$,

where $C^*(t) = \eta[(y(t)/y^+)^\delta - 1]/\zeta \delta$ and $y(t) = \exp[-\rho t]X(t)$.

From (23b), (24), and (32), we can write the difference between the unconstrained optimal consumption rule and the constrained one as

$$
(33) \quad C^*_u(t) - C^*(t) = - \frac{a-\delta}{\zeta(b-a-b)} \frac{\eta(\delta-b)}{\zeta(a-b)} \left[ (y(t)/y^+)^a - 1 \right], 0 \leq W(t) \leq W^+
$$

$$
= - \frac{a-\delta}{\zeta(b-a-b)} (y(t)/y^+)^b, \quad W^+ < W(t) < \bar{W},
$$

where $W(t) = f(y(t))$. Noting that $\partial W/\partial y = f'(y) > 0$, we have from (33), that $\partial[C^*_u(t) - C^*(t)]/\partial W > 0$ for $W(t) < W^+$ and $\partial[C^*_u(t) - C^*(t)]/\partial W < 0$ for $W(t) > W^+$. $C^*_u(t) - C^*(t)$ is a continuous function for all $W$, but it is not differentiable at $W = W^+$ because $C^*(t)$ has a kink at that point. By inspection of (33), we confirm for large $W(t)$ that in the limit as $W(t) \to W$ (and $y(t) \to \infty$), the constrained solution approaches the unconstrained one.

From Theorem B.2 in Appendix B, the optimal demand for the growth-optimum portfolio is given by $f'(y(t))y(t)$. Hence, from (32), the optimal risky-asset holding can be written as

$$
(34) \quad w^*(t)W(t) = aW(t), \quad 0 \leq W(t) \leq W^+
$$

$$
= \frac{\eta g(\delta)}{\zeta r(a-b)} [(a-b)(y(t)/y^+)^\delta - (a-\delta)(y(t)/y^+)^b], \quad W^+ \leq W(t) \leq \bar{W}.
$$
From (34), \( \partial[w^*(t)W(t)]/\partial W > 0 \) for \( \delta \geq 0 \). For \( \delta < 0 \), \( \partial[w^*(t)W(t)]/\partial W > 0 \) for \( 0 \leq W(t) < f(\hat{y}) \) and \( \partial[w^*(t)W(t)]/\partial W < 0 \) for \( W(t) > f(\hat{y}) \) where

\[
\hat{y} = \frac{1}{y^+[(a-b)\delta/(a-\delta)]} > y^+ \text{ because } a > 0 > \delta > b.
\]

From (23c) and (34), we can write the difference between the unconstrained and constrained optimal demand functions for the growth-optimum risky asset as

\[
(35) \quad [w^*_u(t) - w^*(t)]W(t) = \frac{na}{\zeta r(a-b)} \left[ 1 - b[1 - (y(t)/y^+)^a]/a \right], \quad 0 \leq W(t) \leq W^+
\]

\[
= \frac{na}{\zeta r(a-b)} \frac{(y(t)/y^+)^b}{W}, \quad W^+ \leq W(t) \leq \bar{W}
\]

where \( W(t) = f(y(t)) \). By inspection of (35), \( [w^*_u(t) - w^*(t)]W(t) > 0 \) for all \( W(t) < \bar{W} \) However, \( \partial[w^*_u(t) - w^*(t)]W(t)]/\partial W < 0 \) for all \( W(t) \) and \( \delta \), and as with optimal consumption, the optimal constrained demand equals the optimal unconstrained one in the limit as \( W(t) \to \bar{W} \) and \( y(t) \to \infty \). At the critical wealth level, \( W^+ \), where optimal consumption becomes positive, \( w^*_u(t)/w^*(t) = -b/(a-\delta-b) \).

As noted at the outset, the constrained optimal solutions for HARA utility functions with finite time horizons are more complicated. The interested reader can find those solutions expressed as definite integrals in Cox and Huang (forthcoming/b).
Footnotes

* This paper is adapted from Merton (1990, Chapter 6, Section 3).

1. For further discussion and applications of the Cox-Huang method, see Merton (1990, Chapters 6, 14, 16).

2. As shown in Merton (1971, p. 400), ρ can also be interpreted as the "force of mortality" for an investor with no time preference and no bequest motive but who faces an uncertain date of death.

3. The general transversality condition is given in Merton (1969, equation 39). See also condition (2.6) in Karatzas, Lehoczky, Sethi, and Shreve (1986, p. 264) for the case of log-normally distributed asset returns.

4. From these continuity and boundedness conditions, V(0) is finite. For $\rho > 0$, $E_t\{\int_t^\infty \exp[-\rho s](V(C) - V(0))ds\} = E_t\{\int_t^\infty \exp[-\rho s]V(C)ds\} - \exp[-\rho t]V(0)/\rho$, and hence, the optimal rules for either criterion will be identical. For $\rho = 0$, technically, we cannot make this shift, because existence requires that $V(0) < 0$. However, as we shall see, if an optimal solution exists, it will be the set of optimal policies that obtain in the limit as $\rho \to 0$.

5. Condition (3a) ensures that $W(t) \geq 0$ and that $W(t) = 0$ is an absorbing state. From the boundary condition for the nonnegative wealth constraint, $\exp[-\rho t]J(0) = \int_t^\infty \exp[-\rho s]V(0)ds = 0$, because $V(0) = 0$. (3b) is the transversality condition.

6. As in footnote 4, for $\rho = 0$, V cannot be normalized so that $V(0) = 0$. Hence, in that case, replace "$-\rho J_1(W)$" with "$V(0)$" ($< 0$) in (5) and
$J_1(0) = -\infty$. The solution is $J_1(W) = [-2V(0)/(2r + \mu^2)] \log(W/W^+) + J_2(W^+)$ and from (7), $w^* = 1$.

7. For $r = 0$, $\beta_1 = 2r/[2r + \mu^2]$ and $\beta_2 = (\infty)$ does not exist.

8. An alternative proof follows from (ii) of Corollary 1.1. Because the iso-elastic utility function with the optimal-portfolio policy $w^*(t) = 1/(1-\beta)$ has infinite marginal utility at $W = 0$, this policy must have the feature that $W(t) > 0$, almost certainly, for all $t$.

9. Karatzas, Lehoczky, Sethi and Shreve (1986, p. 265, 3a) show this result. Cox and Huang (forthcoming/b, Proposition 3.1) derive the result for finite-lived investors.

10. The proof requires that $r(t) > -\infty$ and $0 < \mu^2(t)$ for all $t$. Although the purpose of the analysis is not to investigate general-equilibrium pricing, the no-bankruptcy result simplifies the proofs of existence and uniqueness of equilibrium prices. Because all indirect utility functions will exhibit infinite marginal utility at $W = 0$, "corner" solutions with $W = 0$ are ruled out.

11. We thus have that for log-normally-distributed asset returns, the growth-optimum portfolio policy is the optimal solution to the problem: given initial wealth $W(0)$ and any "target" level of wealth $W^+ > W(0)$, find the portfolio strategy that minimizes $E_0(\Delta t)$ where $\Delta t$ is the first time that wealth equals $W^+$. Although I do not have a reference, I believe that John Cox was the first to demonstrate this property of the growth-optimum portfolio.

12. This transformation can be applied more generally for investor preferences of the form: $U(C,t) = g(t)V(C)$, provided that $g(t) > 0$ and $\int_0^\infty g(s)ds \equiv T < \infty$. $\tau$-clock time is measured by $\tau \equiv \int_0^\tau g(s)ds$. Because $g > 0$, $\tau$ and $t$ are in one-to-one correspondence, and hence, we
can define $f(t) = g(t)$. The program objective can be expressed in $\tau$-clock time as $\max E_0^T \{ \int_0^T V[f(\tau)C'(\tau)]d\tau \}$, a finite-horizon program. Provided that $T < \infty$, the validity of this transformation does not require that $g(t)$ be monotonic in $t$. Although the formulation in $\tau$-clock time will not in general make the determination of explicit solutions any easier, it may be a helpful form to derive conditions for existence of an optimal solution, because $T$ is finite.

13. For $\delta > 0$ (i.e., $-\infty < \gamma < 1$), (23a) implies satisfaction of the transversality condition given in Merton (1969, equation 40), which is also required for a constrained optimal solution to exist. For $\delta < 0$ (i.e., $\gamma > 1$), violation of (23a) implies that the optimal solution is to consume at the satiation level of consumption, $\eta(\gamma - 1)/\xi$, until wealth is exhausted and consume nothing thereafter.

14. Unlike in the general case of Theorem B.3, $F$ does not depend on $P(t)$ here because the investment opportunity set is assumed to be constant over time.
Appendix A: The Growth-Optimum Portfolio Strategy

If \( W(t) \) denotes the value of a portfolio at time \( t \) that reinvests all earnings, then \( W(T)/W(t) \) is the cumulative total return per dollar from investing in the portfolio between \( t \) and \( T \). Let \( ACCR(t,T) \) denote the average continuously-compounded return on the portfolio over the period. Because the portfolio reinvests all earnings, \( ACCR(t,T) = \log[W(T)/W(t)]/(T - t) \), the average compound growth rate of the value of the portfolio. Consider a portfolio policy chosen so as to maximize the expected value of \( ACCR(t,T) \) at time \( t \), \( t < T \). The portfolio generated by this strategy is called the "growth-optimum" portfolio. By definition, \( E_t[ACCR(t,T)] = E_t[\log[W(T)] - \log[W(t)]]/(T - t) \). Because \( W(t) \) is known at \( t \) and \( (T - t) > 0 \), the objective of maximizing the expected growth rate of the portfolio can be restated as \( \max E_t[\log[W(T)]] \). Therefore, the growth-optimum portfolio will be the same as the optimal portfolio selected by an investor acting according to (B.1) of Appendix B with specific preferences \( U(C(t),t) = 0 \) and \( B(W(T),T) = \log[W(T)] \).

For the general asset-return dynamics posited in Merton (1971), we have from (104) of Section 9 there that the optimal-portfolio fractions for an investor with logarithmic preferences at time \( t \), \( \{w^g_k(t)\} \), can be written as:

\[
(A.1) \quad w^g_k(t) = \sum_{j=1}^{m} v_{kj}(t)[\alpha_j(t) - r(t)], \quad k = 1, \ldots, m,
\]

where \( \alpha_j(t) \) is the instantaneous expected return on risky asset \( j \) at time \( t \), \( r(t) \) is the interest rate, and \( v_{kj}(t) \) is the \( k-j \) element of the inverse
of the instantaneous variance-covariance matrix of returns, \( \sigma_{ij}(t) \). The fraction allocated to the riskless asset is given by
\[
1 - \sum_{k=1}^{m} w_{k}^r(t).
\]

By inspection of (A.1), the fractional allocations in the growth-optimum strategy at each point in time depend only on the current values of the investment-opportunity set \( \{\alpha_j(t), r(t), \sigma_{ij}(t)\} \), independently of whether or not these values will change in the future. The optimal fractions are also independent of both the level of the portfolio's value and the planning horizon \( T \) over which the expected average growth rate is maximized. Portfolio rules with these features are called "myopic" policies.

Let \( X(t) \) denote the value of the growth-optimum portfolio at time \( t \) and let \( P = (P_1(t), \ldots, P_m(t)) \) denote the \( m \)-vector of asset prices. From (A.1) and the posited asset-price dynamics in (5) of Merton (1971, p. 377), we can express the dynamics for \( X(t) \) in stochastic differential-equation form as:

\[\begin{align*}
\text{(A.2)} \quad dX &= \sum_{k=1}^{m} w_k^r \left( \frac{dP_k}{P_k} - r \ dt \right) + r \ dt X \\mu^2 + r \right) X \ dt + \mu X \ dz, \\
&= (\mu^2 + r) X \ dt + \mu X \ dz,
\end{align*}\]

where \( \mu^2 = \mu^2(P_1, \ldots, P_m, t) \equiv \sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij}^2 (\alpha_j - r)(\alpha_k - r) > 0 \) and \( dz = [\sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij}^2 (\alpha_j - r) \sigma_{ij} \ dz_k] / \mu \) is a standard Wiener process. Although in general, \( X(t) \) is not by itself a Markov process, it is jointly Markov in \( X \) and \( P \).

Because \( X = 0 \) is an absorbing state, it follows that if \( X(t) = 0 \), then \( X(t + \tau) = 0 \) for all \( \tau > 0 \). Thus, if \( X(t) = 0 \), then \( E_t[\partial B(X(T), t) / \partial X] \)

\[= E_t[1 / X(T)] = \infty. \]

But, (A.1) is the optimal rule for maximizing
\( E_t\{\text{ACCR}(t,T)\} \) and there is always a feasible strategy of holding just the riskless asset such that \( E_t\{\text{ACCR}(t,T)\} = r \). Therefore, it must be that:

\[
(A.3) \quad \text{Prob}\{X(t) > 0 | X(0) > 0\} = 1 \quad \text{for all } t \geq 0.
\]

That is, the growth-optimum policy never risks ruin and will never violate the nonnegativity condition on the value of the portfolio.

In Merton (1971, pp. 384-388), it is shown that if a portfolio is mean-variance efficient, then its portfolio fractions can be written as:

\[
(A.4) \quad \delta_k = \lambda \sum_{j=1}^{m} v_{kj} (\alpha_j - r), \quad k = 1, \ldots, m,
\]

where \( \lambda \) equals the ratio of the portfolio's variance to its expected excess return. By inspection of (A.2), the (instantaneous) expected excess return on the growth-optimum portfolio is \( (\mu^2 + r - r) = \mu^2 \) and its instantaneous variance rate is \( \mu^2 \). Hence, \( \lambda = 1 \). Setting \( \lambda = 1 \) in (A.4) and comparing it with (A.1), we have that \( w^g_k = \delta_k, \quad k = 1, \ldots, m \). Therefore, in the continuous-time model, the growth-optimum portfolio is instantaneously mean-variance efficient.

If, as in the text, the elements of the investment-opportunity set \( \{\alpha_j, r, \sigma_{ij}\} \) are constant over time, then, as shown in Merton (1971, p. 388), the risky-asset returns are jointly log-normally distributed. In that special case, the \( \{w^g_k\} \) are constant over time, and the growth-optimum strategy calls for a continuous rebalancing of the portfolio holdings to maintain these constant proportions. Because \( \mu^2 \) and \( r \) are now constants, we have from (A.2), that \( X(t) \) is, by itself, a Markov process. Moreover, (A.2) can be explicitly integrated so that, for \( \tau > 0 \) and all \( t \):
(A.5) \[ X(t + \tau) = X(t)\exp\{(r + \mu^2/2)\tau + \mu[z(t + \tau) - z(t)]\} \].

By inspection of (A.5), \( X(t + \tau) \) is log-normally distributed with 
\[ E_t[X(t + \tau)] = X(t)\exp[(\mu^2 + r)\tau] \] and \( \text{Var}_t[\log[X(t + \tau)]] = \mu^2\tau \).

An alternative expression for \( X(t + \tau) \), obtained from integration of (A.2) with \( \sum w_k^g \) constant, is given by:

(A.6) \[ X(t + \tau) = X(t)e^{at} \prod_{k=1}^{m} \frac{e^{-b_k\tau}w_k^g}{P_k(t)} \],

where \( a = r - \mu^2/2 \) and \( b_k = r - \sigma_k^2/2, k = 1, \ldots, m \). Therefore, conditional on knowing the value of the growth-optimum portfolio and the risky-asset prices for some time \( t \), we need only know the risky-asset prices, \( \{P_k(t + \tau)\} \), to determine the value of the growth-optimum portfolio at any future date, \( t + \tau \). That is, given initial conditions, the value of the growth-optimum portfolio is solely a function of the contemporaneous risky-asset prices and time.

As we have seen, \( X(t) \) is a mean-variance efficient portfolio. It follows from the Corollary to Theorem II in Merton (1971) that when the investment-opportunity set is constant, all investors' optimal portfolios can be generated by a simple combination of the growth-optimum portfolio and the riskless asset.
Appendix B: The Cox-Huang (forthcoming/b) Solution of the Intertemporal Consumption-Investment Problem

Consider the lifetime consumption and portfolio-selection problem as formulated in Merton (1971, Section 4), where the investor chooses a set of optimal consumption and portfolio rules so as to:

\[
\max_{\mathcal{C}(t), \mathcal{P}(t)} \mathbb{E}_0 \left[ \int_0^T U(C(t), t) dt + B(W(T), T) \right],
\]  

subject to the budget-constraint dynamics (14') of Merton (1971) and the appended feasibility restrictions that \( C(t) \geq 0 \) and \( W(t) \geq 0 \) for all \( t \leq T \). In the dynamic-programming approach to solving this problem (cf. Merton 1971, Theorem I), \( J(W(t), P(t), t) \) denotes the value of the optimized program in (B.1) at time \( t \) and the optimal consumption and portfolio rules are in the "feedback-control" form, \( \mathcal{C}^*(t) = C^*(W(t), P(t), t) \) and \( \mathcal{W}_k^*(t) = W_k^*(W(t), P(t), t) \), \( k = 1, \ldots, m \). The conditional-expectation operator \( \mathbb{E}_0 \) in (B.1) is computed over the joint probability distribution for the exogenously-specified asset-price dynamics and the endogenously-determined dynamics for the investor's wealth.

Consider now the seemingly different optimization problem in which the investor chooses a consumption path \( \hat{C}(t) \) and a bequest of wealth, \( \hat{W}(T) \), so as to

\[
\max_{\hat{C}(t)} \mathbb{E}_0 \left[ \int_0^T U(\hat{C}(t), t) dt + B(\hat{W}(T), T) \right],
\]

subject to the constraints that \( \hat{C}(t) \geq 0 \), \( \hat{W}(T) \geq 0 \), and

\[
X(0) \mathbb{E}_0 \left[ \int_0^T \frac{\hat{C}(t)}{X(t)} dt + \frac{\hat{W}(T)}{X(T)} \right] \leq W(0),
\]
where $X(t)$ is the value of the growth-optimum portfolio as defined in Appendix A and the text.

From (A.2), we have that the dynamics of $X(t)$ are jointly Markov in $X$ and $P$. Hence, the conditional expectation operator $E_0^t$ in (B.2) and (B.2a) is computed over the joint probability distribution for the individual asset prices and the value of the growth-optimum portfolio. Thus, we can denote the value at time $t$ of the optimized program in (B.2) by $\hat{J}(X(t), P(t), t)$. The optimal consumption-path and terminal-wealth functions will have the form: $\hat{C}^*(t) \equiv G(X(t), P(t), t)$ and $\hat{W}^*(T) \equiv H(X(T), P(T), T)$, where both $G$ and $H$ depend parametrically on $X(0)$, $P(0)$, and $W(0)$. Unlike the optimal rules for (B.1), $\hat{C}^*(t)$ and $\hat{W}^*(T)$ are not feedback controls, because the choices for $\hat{C}(t)$ and $\hat{W}(T)$ do not affect the time paths of either $X(t)$ or $P(t)$.

The connection between the optimization problems (B.1) and (B.2) is established in the following theorem proved by Cox and Huang (forthcoming/a, Sections 2-4; forthcoming/b, Assumption 2.3).

**THEOREM B.1.** Under quite mild regularity conditions, there exists a solution to (B.1) if and only if: (i) there exists a solution to (B.2) and (ii) $C^*(t) = \hat{C}^*(t)$ for $t \leq T$ and $W(T) = \hat{W}^*(T)$.

It follows as an immediate corollary that $J(W(t), P(t), t) = \hat{J}(X(t), P(t), t)$ for all $t \leq T$. The substantive economic intuition underlying the equivalence of these two optimization problems is discussed in Cox and Huang (forthcoming/b) and in Merton (1990, Chapter 16).

Because the joint probability distribution for $X(t)$ and $P(t)$ is not affected by the investor's choices for $\hat{C}(t)$ and $\hat{W}(T)$, (B.2) has the
structure of a static optimization problem that can be solved by the classical Lagrange-Kuhn-Tucker methods for constrained optimization. We can therefore rewrite (B.2) as:

\[
(B.3) \quad \max_{\hat{C}(t)} \int_0^T \left[ U(\hat{C}(t),t) dt + B(\hat{W}(T),T) + \lambda_1(\hat{W}(0) - X(0)\hat{W}(T)/X(T) \right. \\
- \int_0^T [X(0)\hat{C}(t)/X(t)] dt] + \int_0^T \lambda_2(t)\hat{C}(t) dt + \lambda_3\hat{W}(T) \\
\left. \right], \\
\]

where \(\lambda_1, \lambda_2(t),\) and \(\lambda_3\) are the usual Kuhn-Tucker multipliers reflecting the shadow costs of the constraints: (B.2a), \(\hat{C}(t) \geq 0,\) and \(\hat{W}(T) \geq 0.\)

For all \(t \leq T\) and all values of \(X(t)\) and \(P(t)\) that can occur with positive probability, the first-order conditions associated with (B.3) can be written as:

\[
(B.4a) \quad U_C(\hat{C}(t),t) = \lambda_1 X(0)/X(t) - \lambda_2(t), \\
\]

and

\[
(B.4b) \quad B_W(\hat{W}(t),T) = \lambda_1 X(0)/X(T) - \lambda_3, \\
\]

where subscripts on \(U\) and \(B\) denote partial derivatives.

If the investor's preferences are such that he becomes satiated at some finite level of consumption, \(\hat{C}(t),\) then \(U_C(\hat{C}(t),t) = 0.\) However, for \(C(t) < \hat{C}(t), U_C(C(t),t) > 0\) and \(U_{CC}(C(t),t) < 0.\) Similarly, if there exists a finite number \(\hat{W}(T)\) such that \(B_W(\hat{W}(T),T) = 0,\) then \(B_W(\hat{W}(T),T) > 0\) and \(B_{WW}(W(T),T) < 0\) for \(W(T) < \hat{W}(T).\) If for any \(t, W(t) \geq \hat{W}(t)\) where:

\[
(B.5) \quad \hat{W}(t) = \int_0^T C(\tau)e^{-r(\tau-t)} d\tau + e^{-r(T-t)} \hat{W}(T), \\
\]
then the investor will be satiated in wealth because he can achieve the absolutely maximal program of consumption and bequests by simply investing in the riskless asset. Therefore, we assume for the balance of the analysis that the investor's initial wealth is such that he is not satiated [i.e., \( W(0) < \bar{W}(0) \)]. This assumption assures us that strict equality applies in constraint (B.2a) for the optimal program and that the shadow price of wealth, \( \lambda_1 \), is strictly positive.

If for any \( t \), there exists a subsistence level of consumption, \( \bar{C}(t) \geq 0 \), such that \( U_C(\bar{C}(t),t) = \infty \), then a necessary condition for any optimal program is that \( \hat{C}(t) > \bar{C}(t) \geq 0 \) and the nonnegativity constraint on \( \hat{C}(t) \) will not be binding. In that case, because \( \lambda_2(t)\hat{C}(t) = 0 \), \( \lambda_2(t) = 0 \) for all \( t \). Similarly, if there exists a number, \( \bar{W}(T) \geq 0 \), such that \( B_\bar{W}(\bar{W}(T),T) = \infty \), then \( \hat{W}(T) > \bar{W}(T) \) and \( \lambda_3 = 0 \). Therefore, these marginal-utility conditions applying for all \( t \leq T \) are sufficient to ensure that the unconstrained solution to (B.2) and (B.2a) is a feasible solution.

In the general case, we have from the condition \( \lambda_2(t)\hat{C}(t) = 0 \) and (B.4a) that:

\[
\text{(B.6)} \quad \lambda_2(t) = \max[0,[\lambda_1 X(0)/X(t)] - U_C(0,t)],
\]

and from \( \lambda_3 \hat{W}(T) = 0 \) and (B.4b) that:

\[
\text{(B.7)} \quad \lambda_3 = \max[0,[\lambda_1 X(0)/X(T)] - B_\bar{W}(0,T)].
\]

Because \( U_{CC} < 0 \) and \( B_{WW} < 0 \), both \( U_C \) and \( B_\bar{W} \) are invertible. Hence, let \( Q(y,t) \equiv U_C^{-1}(y) \) and \( R(y,T) \equiv B_\bar{W}^{-1}(y) \). From (B.4), (B.6), and (B.7), we have that:
(B.8a) \[ \hat{C}^*(t) = \max\{0, Q(\lambda_1 X(0)/X(t), t)\}, \ t \leq T, \]
and
(B.8b) \[ \hat{W}^*(T) = \max\{0, R(\lambda_1 X(0)/X(T), T)\}. \]

To complete the solution for the optimal program, we need only determine \( \lambda_1 \). Under the assumption of no satiation, (B.2a) is a strict equality for the optimal program, and therefore, \( \lambda_1 \) can be determined as the solution to the transcendental algebraic equation given by:

(B.9) \[ 0 = -\overline{W}(0)/X(0) + E'_0\left[ \int_0^T \max\{0, Q(\lambda_1 X(0)/X(t), t)\}/X(t)dt \right. \]
\[ + \max\{0, R(\lambda_1 X(0)/X(T), T)/X(T)\} \] .

Because \( E'_0 \) is conditioned on the initial value of the growth-optimum portfolio and individual risky-asset prices, the solution to (B.9) will have the form \( \lambda_1 = \lambda_1[X(0), P(0), \overline{W}(0)] \). Substituting for \( \lambda_1 \) in (B.8a) and (B.8b), we have a complete solution for the time path of optimal consumption and the bequest of wealth. For compactness in notation, we express these optimal policies as:

(B.10a) \[ \hat{C}^*(t) \equiv G(X(t), t) \]
and
(B.10b) \[ \hat{W}^*(T) \equiv H(X(T), T) , \]

where the parametric dependence of \( G \) and \( H \) on the initial conditions \( X(0), P(0) \), and \( \overline{W}(0) \) is suppressed.

In sharp contrast to the dynamic-programming technique of Merton (1969; 1971), the Cox-Huang method requires only the solution of a single transcendental algebraic equation to derive a complete description of the
optimal intertemporal consumption-bequest allocation. However, from
inspection of (B.10) alone, we cannot determine the dynamic portfolio
strategy that the investor must follow to achieve this optimal allocation.
Nevertheless, given G and H in (B.10), we can derive the optimal portfolio
strategy.

For all \( t \leq T \), define the function \( F(X(t), P(t), t) \) by:

\[
F(X(t), P(t), t) = X(t)E_t^t \int \left[ G(X(\tau), \tau) / X(\tau) \right] d\tau + H(X(T), T) / X(T),
\]

(B.11)

where \( E_t^t \) is conditioned on \( X(t) \) and \( P(t) \). From (B.2a) and (B.11), we have
that \( F(X(0), P(0), 0) = W(0) \). Suppose that the investor who solved (B.2) for
his optimal intertemporal program at \( t = 0 \) reexamines his decisions at some
time \( t, 0 < t \leq T \). At that time, his optimal problem will be to:

\[
\max_{t} E_t^t \int \left[ U(C(\tau), \tau) d\tau + B(W(T), T) \right],
\]

subject to the constraint:

\[
X(t) E_t^t \int \left[ C(\tau) / X(\tau) \right] d\tau + W(T) / X(T) \leq W(t).
\]

(B.12a)

However, (B.10a) and (B.10b) are intertemporally-optimal solutions. Hence,
by the "time-consistency" condition for an intertemporal optimum, (i) \( W(t) \)
must be such that: \( \hat{C}(\tau) = G(X(\tau), \tau), t \leq \tau \leq T \), and \( \hat{W}(T) = H(X(T), T) \) are
feasible choices and (ii) \( G \) and \( H \) are the optimal rules that the investor
would select as the solutions to (B.12). Under the maintained assumption
of no initial satiation [i.e., \( W(0) < \hat{W}(0) \)], it follows that (B.12a) is
satisfied as a strict equality for \( \hat{C}(\tau) = G \) and \( \hat{W}(T) = H \). By inspection of
(B.11) and (B.12a), we have therefore that:
(B.13) \[ W(t) = F(X(t), P(t), t), \quad 0 \leq t \leq T, \]

where \( W(t) \) is the wealth of the investor at time \( t \), conditional on his having followed the optimal-allocation strategy in (B.10) from time 0 to time \( t \).

Provided that \( F \) is a twice-continuously differentiable function, we can use Itô's Lemma to derive the dynamics of the investor's optimally-allocated wealth. That is, given the dynamics of risky-asset prices in Merton (1971, equation 5) and the dynamics for \( X \) in (A.2) in Appendix A here, we have that:

\begin{equation}
\text{(B.14)} \quad dF = \ddot{\alpha}^t \, dt + F \mu X \, dz + \sum_{k=1}^{m} F_k \sigma_k P_k \, dz_k,
\end{equation}

with \( \ddot{\alpha} \) defined by:

\begin{equation}
\text{(B.14a)} \quad \ddot{\alpha}^t \equiv \frac{1}{2} \mu^2 X^{-2} F_{xx} + \sum_{k=1}^{m} F_{xk} \sigma_k X P_k + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} \sigma_{kj} \sigma_{jk} P_k P_j + (\mu^2 + r)XF_x
\end{equation}

\[ + \sum_{k=1}^{m} \alpha_k P_k F_k + F_t, \]

where subscripts on \( F \) denote partial derivatives and \( \sigma_{xk} \) denotes the instantaneous covariance between the returns on the growth-optimum portfolio and risky asset \( k, \ k = 1, \ldots, m \). From (B.13), \( dW = dF \), and therefore, by inspection of (B.14), \( \ddot{\alpha} \) is the instantaneous expected rate of growth of the investor's optimally-allocated wealth.

**THEOREM B.2.** If there exists an optimal solution to (B.2), \([C^*(t), W^*(T)]\), then for \( t \leq T \), the optimal portfolio strategy \( \{w_k^*(t)\} \) that achieves this allocation is given by:
\[ w^*_k(t)W(t) = F_x(X(t), P(t), t)w^g_{k}(t)X(t) + F_k(X(t), P(t), t)P_k(t), \ k = 1, \ldots, m, \]
with the balance of the investor's wealth in the riskless asset, where
\[ w^g_{k}(t) \]
is given in (A.1) of Appendix A.

**Proof.** From Theorem B.1, if there exists a solution to (B.2),
then there exists a solution to (B.1) such that \( C^*(t) = \hat{C}^*(t) \) for \( t \leq T \) and
\( W(T) = \hat{W}^*(T) \). Hence, the time path of optimally-allocated wealth in (B.1)
is identical to the one for optimally-allocated wealth in (B.2). Therefore, the portfolio strategies required to implement this common allocation
are also identical. Let \( \{w^*_k(t)\} \) denote the optimal portfolio fractions in
the risky assets. From (14') in Merton (1971), the dynamics of the
investor's optimally-allocated wealth can be written as:

\[
dW = \sum_{k=1}^{m} [\Sigma \ w^*_k(t)(\alpha_k - r) + r]W(t) - C^*(t) dt + \sum_{k=1}^{m} w^*_k(t)W(t)\sigma_k dz_k .
\]

From (B.13), \( W(t) = F(X(t), P(t), t), \) and therefore, \( dW - dF \equiv 0 \) for all \( t \leq T \). But, from (B.14), this condition is satisfied if and only if:

(i) \[ \tilde{\alpha}F = \sum_{k=1}^{m} \ w^*_k(t)(\alpha_k - r) + r]W(t) - C^*(t), \] and

(ii) \[ F_xX(t)dz + \sum_{k=1}^{m} F_k \sigma_k P_k dz_k = \sum_{k=1}^{m} w^*_k(t)W(t)\sigma_k dz_k . \] From (A.2),

\[ \mu dz = \sum_{k=1}^{m} w^g_{k}(t)\sigma_k dz_k . \] By rearranging terms, we can rewrite (ii) as

\[ \Sigma \ [F_xw^g_{k}(t)X(t) + F_kP_k - w^*_k(t)W(t)]\sigma_k dz_k \equiv 0 \] for all \( t \leq T \). Because \( dz_k \) is
not perfectly correlated with \( dz_j \), \( k \neq j \), this condition can only be satisfied if 
\[
F_k(X(t), P(t), t) w_k(t) X(t) + F_k(X(t), P(t), t) P_k(t) = w_k(t) W(t),
\]
k = 1, ..., m. From the portfolio-balance condition, the riskless-asset holding is given by 
\[
[1 - \sum_{k=1}^{m} w_k(t)] W(t).
\]
Q.E.D.

The Cox-Huang technique for solving the lifetime consumption-portfolio problem is summarized as follows: (a) determine the joint probability density function for \( X(t) \) and \( P(t) \), which involves at most the solution of a linear partial-differential equation. (b) determine the optimal intertemporal consumption-bequest allocations (B.10), which requires solution of the transcendental algebraic equation (B.9). (c) determine \( F(X(t), P(t), t) \) from (B.11), which requires mere quadrature. (d) determine the optimal portfolio strategy for each \( t \) from the formula in Theorem B.2.

In closing this appendix, we present Cox and Huang's alternative method to quadrature for determining \( F(X(t), P(t), t) \) that is used in Section 4 of the text.

**THEOREM B.3.** If \( F \), as defined in (B.11), is twice-continuously differentiable, then \( F \) is a solution to the linear partial-differential equation:

\[
0 = \frac{1}{2} \mu \sigma^2 X^2_{xx} + \sum_{k=1}^{m} \sigma_{xk} X_{xk} P_k + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} \sigma_{jk} P_j P_k + r X F_x + \sum_{k=1}^{m} \frac{r P_k F_k + F_t - r F + G_k}{1}
\]

subject to the boundary condition that \( F(X, P, T) = H(X, T) \), where \( G \) and \( H \) are as defined in (B.10).
Proof. In the proof of Theorem B.2, we showed that
\[
\tilde{\alpha}F = \left[ \sum_{1}^{m} w_k^* (\alpha_k - r) + r \right] W(t) - C^*(t) \text{ for all } t \leq T, \text{ where } \tilde{\alpha} \text{ is defined in (B.14a). From (B.13), } W(t) = F(X(t), P(t), t), \text{ and from Theorem B.1, } C^*(t) = G(X(t), t) \text{ for all } t \leq T. \text{ From inspection of (A.2), }
\]
\[
\mu^2 = \sum_{k}^{m} w_k^*(\alpha_k - r). \text{ Substituting for } \{w_k^*(t)\} \text{ from the formula in Theorem B.2, we have that } \left[ \sum_{1}^{m} w_k^*(\alpha_k - r) + r \right] W(t) - C^*(t) = F_x \mu^2 X(t) + \\
\sum_{1}^{m} F_k(\alpha_k - r) P_k + r F - C^*(t) = \tilde{\alpha} F. \text{ Substituting for } \tilde{\alpha} F \text{ from (B.14a) and rearranging terms, we have that}
\]
\[
0 = \frac{1}{2} \mu^2 X_F x_k x_k x_k x_k + \sum_{1}^{m} F_k \sigma_{j k} P_{j k} + \frac{1}{2} \sum_{1}^{m} F_k \sigma_{j k} P_{j k} + r X_F x_k + \sum_{1}^{m} r P_k F_k + F_t - r F + G.
\]
From the definition in (B.11), we have that \(F(X(T), P(T), T) = H(X(T), T)\).

Q.E.D.
BIBLIOGRAPHY


