A GOLDEN GOLDEN-RULE FOR WELFARE-
MAXIMIZATION IN AN ECONOMY WITH A
VARYING POPULATION GROWTH RATE

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The planner seeking to maximize welfare in an economy with a growing population is told that the most efficient golden-age path results from following the Swan-Phelps Golden Rule [4]. This Golden Rule states that, for constant population growth, the most efficient golden-age path equates the interest rate to the rate of population growth [4]. This is true for an economy with a constant rate of population growth. The question arises, is it still true for an economy in which the rate of population growth varies? The answer is no. The optimal golden-age state will be at a lower capital-labor ratio than the Swan-Phelps Golden-Rule State for the case where the rate of population growth is an increasing function of the capital-labor ratio, and it will be at a higher capital-labor ratio for the rate of population growth a decreasing function.

I. THE PLANNER’S ECONOMY

The planner is postulated to choose maximizing the utility of the representative man over the many years of his plan as the criterion for maximizing the welfare of the economy.

Consider a planner faced with an economy whose representative man’s fertility is affected only by his wealth or lack of it. The wealthier he becomes, the more children he has. The assumption of a representative man implies that all members of this society have the same tastes, education, and wealth. If the people in such a society enjoy having children, then it is reasonable to assume that the wealthier people become, the more children they have, and there is considerable empirical evidence for this when extraneous factors such as education and religion are allowed for by partial correlation.

It is possible that empirically the opposite could be true; namely, that fertility declines with per capita economic well-being (as e.g. when people can better afford to practice contraception effectively). Probably, a more realistic assumption would be a blend: for relatively low per capita wealth, increases in wealth would lead to increases in the rate of population growth until at some level of per capita wealth, the growth rate would reach a peak and then begin to decline with further increases. This decline would gradually slow down, and the growth rate would approach some constant level asymptotically. Actually, the derivation of the optimal golden-age path is the same whether the rate of population growth is an increasing or decreasing function of the capital-labor ratio. The main restriction on

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the function is that its second derivative have a particular lower-bound described in equation (8) later in the paper. However, except as otherwise noted, I suppose it to be strictly monotone-increasing.

II. THE PLANNER'S PROBLEM

Mathematically, the planner's problem is described as follows:

\[ \text{(1)} \quad \text{Maximize } \int_0^T U(c(t)) \, dt \text{ for } k(0) \leq k^0 \text{ ; } k(T) \geq k^T \geq 0 \]

where

\[ U(c) = \text{utility of the representative man} \]
\[ U \text{ is assumed monotone-increasing and strictly concave in } c, \text{ i.e. } U'(c) > 0 \text{ ; } U''(c) < 0 \]
\[ c(t) = \text{consumption per person = consumption per worker} \]

Constant returns to scale, diminishing marginal returns (to varying proportions), and capital-saturation at a finite capital-labor ratio are assumed.

\[ k(t) = \text{capital-labor ratio} \]
\[ \dot{k}(t) = \text{net change in capital per worker} \]

\[ \text{(2)} \quad f(k) = \text{annual per capita output,} \]

\[ f'(k) > 0 \text{ for } k < k_8 \]
\[ f'(k_8) = 0 \text{ where } k_8 = \text{Schumpeter capital-saturation point} \]
\[ f''(k) < 0 \]

Using the behavioral assumption previously described,

\[ n(k) = \text{the rate of population growth,} \]
\[ n(0) = 0 \text{ ; } n'(k) > 0 \]

The technological budget constraint is:

\[ c(t) = f(k) - n(k)k - \dot{k}. \]

Substituting the budget constraint for \( c(t) \), the problem becomes:

\[ \text{(3)} \quad \text{Maximize } \int_0^T V(k, \dot{k}) \, dt \text{ for } k(0) \leq k^0 \text{ ; } k(T) \geq k^T \geq 0 \]

where

\[ V(k, \dot{k}) = U(f(k) - n(k)k - \dot{k}) \]

1Because this derivation parallels Samuelson's derivation of the Per Capita Consumption Turnpike Theorem [61, I have tried to keep the same notation. As is usual for continuous models, new people are assumed to work from the time they are born (into the labor force). There are various other alternative expressions one might care to maximize instead of the utility of per capita consumption. One such variate, the Bentham-Lerner criterion, is explored later.
A necessary condition for a smooth interior maximum is the Euler equation:

$$E_t(k,\dot{k},k) = \left\{ \frac{d}{ds} [V_s] - V_s \right\} = 0$$

where

$$V_s = \frac{\partial V(k,\dot{k})}{\partial k} ; \quad \dot{V}_s = \frac{\partial V(k,\dot{k})}{\partial \dot{k}}$$

$$V_{s1} = \frac{\partial^2 V(k,\dot{k})}{\partial k \partial \dot{k}} = V_{s1}$$

$$V_{s2} = \frac{\partial^2 V(k,\dot{k})}{\partial \dot{k}^2} ; \quad \dot{V}_{s2} = \frac{\partial^2 V(k,\dot{k})}{\partial \dot{k}^2}$$

The Euler equation is also sufficient provided $V(k,\dot{k})$ is strictly concave because then the strong Legendre condition is satisfied, i.e.,

$$V_{s1} < 0; \quad V_{s2} < 0; \quad \Delta = \det \begin{bmatrix} V_{s1} & V_{s2} \\ V_{s1} & V_{s2} \end{bmatrix} > 0$$

where

$$V_{s1} = U''(c)(f'(k) - n'(k)k - n(k)) + U'(c)(f''(k) - 2n'(k) - n''(k)k)$$

$$V_{s2} = U''(c)$$

$$V_{s1} = V_{s1} = -U''(c)(f'(k) - n'(k)k - n(k))$$

$$\Delta = U''(c)U'(c)(f''(k) - 2n'(k) - n''(k)k)$$

Because it is assumed that $U'(c) > 0, U''(c) < 0,$ and $f''(k) < 0,$ the strict concavity conditions will be satisfied for:

$$n''(k) > 0, \quad U''(c) < 0, \quad f''(k) < 0$$

Therefore, assume $n''(k)$ satisfies this condition which also assures no conjugate points and uniqueness of solution for a given set of end points.

Let $k^*$ be the unique stationary solution of the Euler equation, $E_t(0,0,k^*) = 0.$ Because the turnpike is the optimal golden-age or stationary-state path, $k^*$ is the turnpike. Then,

$$E_t(0,0,k^*) = 0 - V_s(k^*,0)$$

$$= -U'(c^*)(f'(k^*) - n(k^*) - n'(k^*)k^*) = 0$$

where

$$c^* = f(k^*) - n(k^*)k^*.$$ 

Because it is assumed that $U'(c^*) > 0, k^*$ satisfies:

$$f'(k^*) = n(k^*) + n'(k^*)k^*.$$
This is our new Golden Golden-Rule.²

The Swan-Phelps Golden-Rule level for constant population growth, call it $\bar{k}$, satisfies $f'(\bar{k}) = n(\bar{k})$. Because $f'(k)$ is a decreasing function of $k$, $\bar{k}$ is greater (less) than $k^*$ for $n'(k)$ greater (less) than zero.

The essence of the Turnpike theorem is the local catenary behavior of optimal dynamic paths near the turnpike. The Turnpike theorem states that, for large enough $T$, the optimal path will spend most of the time arbitrarily near the turnpike. Samuelson has shown that, for large enough $T$, the optimal path will arch toward the turnpike like a catenary [6]. I suspect that the catenary behavior could be used to prove for consumption-turnpikes an analogous theorem to the one proved for production-turnpikes by Furuya and Inada [2]: namely, that any path, other than the one which spends most of the time near the turnpike, will move away from the turnpike and will eventually hit the axis, contradicting any claim to optimality.

Because of the importance of the catenary behavior, it is necessary to determine the local behavior of the dynamic solution to the Euler equation (5) near the $k^*$-turnpike (10). In the one-sector model with no time preferences, the catenary behavior is implied by the characteristic roots of the linear stability equation, associated with the Euler equation, being real, equal-in-magnitude, and opposite-signed.

To examine stability in a neighborhood of the $k^*$-turnpike, linearize the Euler equation as follows:

Let $y(t) = k(t) - k^*$ ; $\dot{y}(t) = \dot{k}(t)$ ; $\ddot{y}(t) = \ddot{k}(t)$.

Then, using equations (5) and (9), the linearized Euler equation becomes:

(11) $L_{11}\ddot{y}(t) - L_{12}y(t) = 0$

where

$L_{11} = U''(c^*) < 0$

$L_{12} = U'(c^*)(f'(k^*) - 2n'(k^*) - n''(k^*)k^*) < 0$

by the condition (8) on $n''(k)$.

The solution,

(12) $y(t) = A_1e^{\lambda t} + A_2e^{-\lambda t}$, $\lambda = \sqrt{L_{11}/L_{11}}$

is Samuelson’s catenary in the Per Capita Consumption Turnpike Theorem [6], but now applied to the case of endogenously variable population growth.

²Remark: The assumption that $n'(k) > 0$ is used implicitly in (10) to assure the existence of the $k^*$ solution. Nowhere else in the derivation of the turnpike solution was it needed. Therefore, $n'(k) < 0$ would give a similar result. However, from (2) $f'(k) \geq 0$ for $k \leq k_0$, and so the additional restriction that $n(k) \geq |n'(k)|k$ for $k \leq k_0$ would be required for existence of a stationary solution. Similarly, for the case where: $n'(k) > 0$ for $0 < k < k_1$; $n'(k) = 0$; $n'(k) < 0$ for $k_1 < k < k_2$; $n'(k) = 0$ for $k_2 < k < k_3$; $n'(k) > 0$ for $0 < k < k_4$, uniqueness and existence of a solution are assured provided the condition (8) on the second derivative is satisfied. Also it should be noted that (8) is satisfied when $n(k)$ is a constant.
III. THE PLANNER'S DILEMMA

Should the planner follow the Golden Rule for constant population growth and have society spend most of the time near the capital-labor ratio which equates the interest rate to the population growth rate, or should he follow the rule of the new Turnpike theorem and spend most of the time near the $k^*$-turnpike (10)?

To help solve the dilemma, consider the special case: for boundary conditions in (3), let $k^* = \bar{k}$ where:

$$(13) \quad f'(\bar{k}) = n(\bar{k}), \bar{k} \text{ is the Golden-Rule capital-labor ratio.}$$

In this case, a planner who follows the old Golden Rule would tell society to remain at the initial capital-labor ratio and to consume what is left after widening of capital is provided for. The planner who follows the rule of the new Turnpike theorem will tell society to consume more in the beginning, which will lower per capita wealth and hence the rate of population growth until the economy is operating near the new $k^*$ level. Then, society should continue to consume at this level until in the last moments of time period $(0,T)$, it would then move away from the turnpike back toward the pre-assigned old $k$ level—so that at time $T$, it would end-up satisfying the boundary condition, $k(T) = k^* = \bar{k}$ (see Figure 1).

Clearly, by the general Turnpike theorem, path ABCD gives a larger $\int_0^T U(c(t)) \, dt$ than the Swan-Phelps path AED. Further, because path AED is in the steady-state ($\dot{k} = 0$) for the whole period $(0,T)$ and path ABCD spends most of the period arbitrarily near the steady-state ($\dot{k}^* = 0$), it is of interest to compare per capita consumption between the two steady-states.

Mathematically, it is clear that per capita consumption among steady-states ($\dot{k} = 0$) is highest at the new $k = k^*$.

Figure 1

$\text{Figure 1}$

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*By a similar argument for $n'(k) < 0$ case, society should consume less in the beginning and raise per capita wealth which would lower the rate of population growth.*
Maximize $c = \text{Maximize } (f(k) - n(k)k)$

\[
dc/dk = f'(k) - n'(k)k - n(k) = 0 \text{ for } k = k^* \]

\[
d^2c/dk^2 = f''(k) - 2n'(k) - n''(k)k < 0 \text{ for } k = k^*, \text{ by condition (8)}
\]

So per capita consumption is maximized among golden ages at $k = k^*$.

The planner's problem is solved. With the rate of population growth variable, society should spend most of its time away from the Swan-Phelps Golden Rule for constant population growth and close to the $k^*$-turnpike, the Golden Golden-Rule.

The results of this section, which were formerly part of the main text, are essentially covered in a recent note by E. Davis [1].

**Generalizations for the Planner of an Economy with Different Behavior:**

1. **Population Growth Rate as a Function of Per Capita Consumption.** If the planner had faced an economy whose rate of population growth was a function of per capita consumption, $n = n(c)$, the resulting stationary solution to (9), the Euler equation, would be:

\[
(f(k^*) - n'(c^*))/(1 + n'(c^*)k^*) = 0,
\]

which implies $f'(k^*) = n'(c^*)$, the Swan-Phelps Golden Rule (i.e., $k^* = \tilde{k}$), if $1 + n'(c^*)k^* \neq 0$. Although $k = \tilde{k}$ clearly maximizes per capita consumption among stationary states ($\dot{k} = 0$), this case is qualitatively different from $n = n(k)$ because $c = c(k, \tilde{k})$ which implies $n = n(k, \tilde{k})$. Therefore, the dynamic behavior of the optimal path ($\dot{k} \neq 0$) is different. The required second-order conditions on $V(k, k)$, (7), which satisfy the strong Legendre conditions assuring uniqueness and a smooth interior maximum are complex to derive.

2. **Population Growth Rate as a Function of Per Capita Output.** If the planner had faced an economy whose rate of population growth was a function of per capita output, $n = n(f(k))$, the resulting stationary solution to the Euler equation would be:

\[
f'(k^*) = n(f(k^*))/(1 - n'(f(k^*))k^*)
\]

where $n'(f(k^*)) = dn/df|k^* = 1 - n'(f(k^*))k^* > 0$.

This is a particular case of the Golden Golden-Rule (10). $n = n(f(k))$ is a particular form of $n = n(k)$, and because $f(k)$ is a strictly monotone function, all order relations are preserved. The apparent new restriction that $n'(f(k))k^* < 1$ to assure a unique stationary-state solution is implicit in the original equation:

\[
f'(k^*) = (dn/dk)k^* + n(k^*)
\]

By assumption (2), $f'(k) > 0$ for $k < k_0$ and $n(k) > 0$. Therefore, to assure a solution to (10), $f'(k^*) > (dn/dk)k^*$. In this particular case, $dn/dk = n'(f(k^*))f'(k^*)k^*$ and because $f'(k^*) > 0$, this implies $1 > n'(f(k^*))k^*$

Notice for $n'(f(k)) > 0 (< 0)$, the $k^*$-turnpike level is lower (higher) than $\tilde{k}$, the Swan-Phelps Golden Rule (15), as was derived in the general case of the Golden Golden-Rule.

In the two-sector model where $f = f(k, h)$, a problem arises in deriving the required second-order conditions which satisfy the strong Legendre conditions, similar to the $n = n(c)$ case previously described.

3. **Time Preference**

If the planner has a systematic time preference, his problem would be to

\[
\text{Maximize } \int_0^\infty e^{-pt}U(c(t)) dt, \quad (p < 0)
\]

The new stationary turnpike solution, the Golden Golden-Rule with time preference, is

\[
f'(k^*) = n(k^*) + n'(k^*)k^* + \rho
\]

which is a lower (higher) turnpike for $\rho > 0 (< 0)$ than without time preference. The solution to the resulting linearized Euler equation corresponding to equations (11) and (12) is an unbalanced catenary [6].
IV. BENTHAM-LERNER CRITERION FOR SOCIAL WELFARE

Consider the plight of the planner who rejects the Samuelson-Diamond criterion for social welfare and instead prefers the Bentham-Lerner criterion of maximizing the total utility of all people who will ever live. For a constant rate of population growth, Samuelson has shown that the turnpike which optimizes Bentham-Lerner social welfare is the Schumpeter zero-interest level where capital is saturated [7]. Therefore, the planner already knows that society should spend most of its time away from the Golden-Rule state. However, is the Schumpeter point still the optimal capital-labor ratio for an economy in which the rate of population growth is a decision variable? The answer is no.

When the rate of population growth is treated as a rising function of the capital-labor ratio rather than as an exogenous constant, the Schumpeter zero-interest level cannot be an optimal solution, and the optimal path will always lie below it. In general, the optimal path for the Bentham-Lerner criterion will not be uniquely determined for the same sufficient conditions (8) specified for the Samuelson-Diamond criterion. However, for somewhat stronger sufficient conditions, a unique turnpike will exist.

Mathematically, the Bentham-Lerner criterion can be written as a constrained calculus of variations problem as follows:

\[ \text{Max} \int_0^T L(t)U(f(k) - n(k)k - \dot{k}) \, dt \quad k(0) \leq k^* ; \quad k(T) \geq k^* \]

subject to \( \dot{L}(t) = n(k)L(t) \)

For \( U(\cdot) \) and \( f(\cdot) \) concave functions, finding a solution to (18) is equivalent to finding the saddlepoint to the problem

\[ \text{Maximize Minimize} \int_0^T \mathcal{W}(k, \dot{k}, L, \dot{L}, \lambda) \, dt, \quad k(0) \leq k^* ; \quad k(T) \geq k^* \geq 0 \]

where

\[ \mathcal{W} = L(t)(V(k, \dot{k}) - M) + \lambda(t)(\dot{L}(t) - n(k)L(t)) \]

\( L(t) = \) labor force \( = \) population size

\( \lambda(t) = \) Lagrange multiplier

\( M = \) constant chosen by the planner (see the next paragraph)

\[ V(k, \dot{k}) = U(c) = U(f(k) - n(k)k - \dot{k}) \]

\( U, k, \dot{k}, f(k), n(k), c \) are as previously defined in (1) and (2).

*In addition to Lerner and Samuelson, the Bentham-Lerner criterion is discussed by Meade in his chapter on Optimum Population [3, Ch. 6].
Before deriving the optimality conditions, it is important to explain the meaning of the constant $M$. Because $U(c)$ is a measure of cardinal utility, it is uniquely determined only up to an affine transformation. Therefore, $U(c)$ must be written, in general, as $aU(c) - b$, where $a$ and $b$ are constants and $a > 0$. Clearly, only the origin of utility, $b$, could have any effect on the optimal solution (as e.g., maximize $\int_0^T L(t)(aU(c) - b)dt = a$ maximize $\int_0^T L(t)(U(c) - M)dt$ where $M = b/a$, $a > 0$). This shift of the utility origin did not effect the results of the Samuelson-Diamond criterion because the addition of the term $-\int_0^T Mdt$ is not a function of any decision variables and so does not effect the optimal path. Similarly, for the Bentham-Lerner criterion when $L(t)$ is not a decision variable, Ramsey [5] and Samuelson [7] could choose $M = U(CB)$, where $CB$ is bliss-point consumption, for convenience because the resulting optimal path was independent of the origin for $U(c)$. However, when the rate of population growth is endogenous and $L(t)$ is a decision variable, the value of $M$ chosen by the planner is crucial in determining the turnpike level of the capital-labor ratio.

One interpretation of the economic meaning of the constant $M$ is as follows: Define $M = U(cu)$, where $cu$ is that level of per capita consumption below which the planner would not want to bring any new people into the world. Thus, a planner, given a $cu$ so high that it is not possible for the economy to produce for a long period of time at a level where per capita consumption is greater than or equal to $cu$, would prefer, as an optimal solution, to "shut-down" the economy and the human race (i.e., set $L(t) = 0$).

If one maximizes $\int_0^T L(t)(V(k,\dot{k}) - M)dt$, $L(t) \geq 0$, and the optimal $k$ is such that $(V - M) < 0$ for most of the long time-period, $(0,T)$, then $\int_0^T L(t)(V(k,\dot{k}) - M)dt$ will be maximized by setting $L(t)$ identically equal to zero. Because $U(c) = V(k,\dot{k})$ is a strictly monotone-increasing function of $c$, the condition $(V - M)$ less than zero implies that $c$ is less than $cu$. For the rest of the paper, I shall refer to optimal paths $k$ for (18) as being admissible optimal paths if the associated optimal $L$ is positive rather than zero (e.g., if $k$ is a steady-state path and if $(V(k,0) - M)$ is non-negative, then $k$ is an admissible steady-state path).

Return now to finding the extremal paths to (18'). The necessary conditions for a saddlepoint are a set of Euler equations:

\[
\frac{d}{dt} \left[ \frac{\partial W}{\partial k} \right] - \frac{\partial W}{\partial k} = 0
\]

\[
\frac{d}{dt} \left[ \frac{\partial W}{\partial L} \right] - \frac{\partial W}{\partial L} = 0
\]

\[
\frac{d}{dt} \left[ \frac{\partial W}{\partial \lambda} \right] - \frac{\partial W}{\partial \lambda} = 0
\]
Equation (21) returns the constraint, \( \dot{L} = n(k)L \). Substituting this constraint and dividing out \( L(t) \) in equation (19), the resulting equations are:

\[
\begin{align*}
V_s \dot{k} + V_s k + n(k)V_s - V_s + \lambda n'(k) &= 0 \\
\dot{\lambda} + n(k)\lambda &= V(k, \dot{k}) - M
\end{align*}
\]

Let \( k^{**} \) and \( \lambda^{**} \) be the stationary turnpike solution to equations (19') and (20'). Then, \( \dot{k}^{**} = k^{**} = \dot{\lambda}^{**} = 0 \).

\[
\begin{align*}
(22) & \quad f'(k^{**}) = n'(k^{**})k^{**} + n'(k^{**})h^{**}/U'(c^{**}) \\
(23) & \quad \lambda^{**} = (U(c^{**}) - M)/n(k^{**})
\end{align*}
\]

where

\[
(24) \quad c^{**} = f(k^{**}) - n(k^{**})k^{**}
\]

By eliminating \( \lambda^{**} \) from (22), the \( k^{**} \)-turnpike solution for the Bentham-Lerner criterion is defined by

\[
(25) \quad f'(k^{**}) = n'(k^{**})k^{**} + n'(k^{**})(U(c^{**}) - M)/n(k^{**})U'(c^{**}).
\]

Query: is \( k^{**} \) unique for a given value of the parameter \( M \) (i.e., is \( k^{**}(M) \) single-valued)? A sufficient condition for the single-valuedness of \( k^{**} \) is that \( dM/dk^{**} \neq 0 \). Solve (25) for \( M \) and calculate \( dM/dk^{**} \):

\[
\begin{align*}
(26) & \quad M = U(c^{**}) - n(k^{**})U'(c^{**})b(k^{**})/n'(k^{**}) \\
(27) & \quad dM/dk^{**} = U'(c^{**})n(k^{**})(n''(k^{**})f'((k^{**}) - n'(k^{**})f''(k^{**}))/
\quad (n'(k^{**}))^2 - U''(c^{**})n(k^{**})(b(k^{**}) - n(k^{**}))b(k^{**})/n'(k^{**})
\end{align*}
\]

where

\[
h(k^{**}) = f'(k^{**}) - n'(k^{**})k^{**}.
\]

To examine the behavior of \( dM/dk^{**} \), divide the domain of \( k^{**} \) into two regions:

I. All \( k^{**} \) such that \( b(k^{**}) \geq n(k^{**}) \)

(28)

II. All \( k^{**} \) such that \( n(k^{**}) > b(k^{**}) > 0 \)

By inspection of (27), for \( k^{**} \) contained in I., a new sufficient condition,

\[
(29a) \quad n''(k) > n'(k)f''(k)/f'(k),
\]

Remark: for the special case, \( n \) constant (i.e., \( n'(k) = 0 \)), the \( \lambda^{**} \) disappears from (22) and, as expected, \( M \) has no effect on the solution, which is then the Schumpeter zero-interest level. In general, it is clear by inspection of (22) and (23) that \( k^{**} = k^{**}(M) \) and \( \lambda^{**} = \lambda^{**}(M) \).

Clearly, if \( b(k^{**}) < 0 \), from (25), \( U(c^{**}) < M \), which implies that \( L^{**}(t) = 0 \) is the optimal solution.
replacing (8), will assure \( dM/dk^\** \) is positive and \( k^\**(M) \) is a single-valued function monotone-increasing in region I. In region II, condition (29a) is not sufficient. A stronger condition on \( n''(k) \) which assures \( dM/dk^\** \) is positive in II is:

\[
(29b) \quad n''(k) > \left[ n'(k)/U'(c) \right] U''(c) b(k)(b(k) - n(k))/f'(k) \\
+ n'(k)f''(k)/f'(k)
\]

Therefore, for the remainder of the paper, assume \( n''(k) \) satisfies conditions (29), and \( k^\** = k^\**(M) \) is a single-valued, monotone-increasing function of \( M \).

Now consider the effect of various choices for the parameter \( M \) upon the optimal stationary-state solution. Because \( k^\** \) is an increasing function of \( M \), \( \bar{L}^\**/\bar{L}^\** = n(k^\**(M)) \) is an increasing function of \( M \) which implies that the optimal population size, \( \bar{L}^\**(t) \), will be larger for larger values of \( M \) at each instant of time. Therefore, a planner who favors a large population will be one with a large \( M \). Examination of three critical values of \( M \) will serve to illustrate the effect of \( M \) on the turnpike level.

1. Consider a planner with \( M = M_1 \) such that \( k^\**(M_1) = k^* \), the Golden Golden-Rule level of maximum per capita consumption (10). Then, \( f'(k^\**(M_1)) = n'(k^\**)(k^\**) + n(k^\**) \) and by substitution into (26),

\[
(30) \quad M_1 = U(c^*) - U'(c^*)n^*(k^*)/n'(k^*)
\]

where \( c^* = f(k^*) - n(k^*)k^* = c^*(M_1) \), as defined in (24). From (30), \( M_1 \) is less than \( U(c^\**) \) which implies that \( k^\**(M_1) \) is an admissible optimal solution and \( L^\**(M_1) \) is positive.

2. Consider a planner with \( M = M_2 \), where \( M_2 \) is the largest \( M \) such that \( k^\**(M) \) is an admissible optimal solution. By the definition of admissible optimal and because \( dM/dk^\** \) is positive, \( M_2 = U(c^\**(M_2)) \). Substituting into (25), the equation for the \( k^\**(M_2) \)-turnpike is:

\[
(31) \quad f'(k^\**(M_2)) = n'(k^\**)k^\**
\]

This turnpike level is the generalization of the Samuelson-Lerner-Ramsey turnpike when \( n \) is a constant (i.e., \( n'(k) = 0 \)). Essentially, when \( M = M_2 = U(c^\**) \), the original extremal integral (18) is minimizing the divergence from bliss (Ramsey, [5]). Also, to follow the technique used by Samuelson [7] to determine the general turnpike solution for the Benthan-Lerner criterion, one could apply a time preference, \( \rho = -n'(k^\**) \), to derive the above turnpike level.

*Because the population size in the turnpike state at a given time, \( L^\**(M) \), is a strictly increasing functional of \( M \), the planner with \( M = M_2 \) will favor the largest population size among all planners. A planner with \( M \) greater than \( M_2 \) will prefer to "shut-down" the economy, i.e., \( L^\**(M) = 0 \). One could consider \( c_w \) as a "Malthusian-type" lowest standard of living at which the society will continue to exist. Thus if that standard is higher than the economy can sustain (e.g., a society of aristocrats), then the optimal solution is for that society to cease to exist.
3. Consider a planner with $M = M_s$ such that $k^{**}(M_s) = k_s$, the Schumpeter capital-saturation level. Then, $f'(k^{**}(M_s)) = f'(k_s) = 0$, and by substitution into (26),

$$M_s = U(c_s) + n(k_s)k_sU'(c_s),$$

where $c_s = f(k_s) - n(k_s)k_s = c^{**}(M_s)$ as defined in (24). From (32), $M_s$ is larger than $U(c^{**})$ which implies that $k^{**}(M_s)$ is not an admissible optimal solution (i.e, $L^{**}(M_s) = 0$), and the planner would prefer to "shut-down" the economy. Therefore, the Schumpeter point can never be an optimal solution for $n'(k)$ positive.

To complete the description of the effect of $M$ on the turnpike solution, consider the two cases: (1) $n'(k) = 0$ and (2) $n'(k) < 0$. For $n'(k) = 0$, from (30), $M_s = -\infty$ which confirms earlier findings that the Golden Golden-Rule (or in this case, the Golden-Rule) level is never an optimal solution for $n$ positive under the Bentham-Lerner criterion. By taking limits on (26), $M_s = M_s$, and the solution from (31) is $k^{**}(M_s) = k^{**}(M_s) = k_s$, the Schumpeter point. These results were expected because, for $n$ fixed, the turnpike solution is unique, independent of $M$. For $n'(k) < 0$, there exists no stationary-state solution for the Bentham-Lerner criterion other than the trivial one, $L^{**}(M) = 0$. By substituting into (25) the condition for all admissible optimal solutions, $U(c^{**})$ greater or equal to $M$, the result is $f'(k^{**})$ less than zero which has no solution for $k$ less than $k_s$ by assumption (2). For $U(c^{**})$ less than $M$, $f'(k^{**})$ could possibly be non-negative, but the solution is not admissible and so, $L^{**}(M) = 0$.

In summary, for $n'(k)$ positive, the turnpike solution under the Bentham-Lerner criterion is a function of the origin of utility. There exists an upper-bound on the optimal capital-labor ratio which is always below the Schumpeter capital-saturation level. In general, when the optimal capital-labor ratio is an increasing function of the utility origin, the higher the utility origin specified for the planner, the larger is the population size favored by the planner at each moment of time. However, if the utility origin is specified too high, the planner will favor, as an optimal solution, a zero population size.

V. CONCLUSION

Throughout the paper, I have considered plans only for large finite time, and therefore, I have not been concerned with the convergence of the integrals of utility as the length of the period goes to infinity. For the Bentham-Lerner criterion, conditions for convergence of the integral may not exist because, for $n = n(k)$, it is no longer possible to pick the origin of utility arbitrarily as Ramsey and Samuelson did for $n$ constant.

Although some attempt was made to justify, on empirical grounds, the behavioral assumptions about the rate of population growth, clearly these are gross simplifications of the behavior of any actual society.
With these reservations, it was found that for an economy with a variable rate of population growth, the Samuelson-Diamond criterion for social welfare is not optimized at the Swan-Phelps Golden-Rule level for constant population growth, but instead at a lower level, the Golden Golden-Rule. The introduction of a subjective rate of time preference shifts the turnpike level in a straightforward manner.

If the Bentham-Lerner criterion is preferred, the turnpike solution depends upon the utility origin. However, there is an upper-bound on the optimal capital-labor ratio which is always below the Schumpeter capital-saturation level.

In general, for welfare maximization among golden ages, there is nothing sacred about the Golden Rule, the Schumpeter capital-saturation level, or even the Golden Golden-Rule. One would want now to consider the effect of changing the \( n(\dot{k}) \) function. Such a change would "invalidate" the Golden Golden-Rule.

\[ \text{I have analyzed, but not here, the cases Maximize} \int_0^T U(f(k) - k\ddot{L}/L - \dot{k}) dt \text{ and Maximize} \int_0^T U(f(k) - k\ddot{L}/L - \dot{k}L(t) e^{-\rho t}) dt. \]

REFERENCES