An Asymptotic Theory of Growth Under Uncertainty [1, 2]

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1. INTRODUCTION

The neoclassical theory of capital accumulation and growth under certainty for both positive and optimal savings functions has received extensive study in the literature for almost two decades. However, the study of capital accumulation under uncertainty began much later and these analyses for the most part confined themselves to linear technologies. In his pioneering work, Phelps [19] and, later, Levhari and Srinivisan [10] ,Hahn [5] and Leland [23], examine the optimal consumption-saving decision under uncertainty with a given linear production technology. Hakansson [6], Leland [9] and Samuelson [21] in discrete time and Merton [12, 13] in continuous time, along with a host of other authors, have studied the combined consumption-saving-portfolio problem where the production functions are linear, but where there is a choice among alternative technologies.

There have been a few notable exceptions to this concentration on linear technologies. In a seminal paper, Mirrlees [17] tackled the stochastic Ramsey problem in a continuous-time neoclassical one-sector model subject to uncertainty about technical progress. Later, in [18], he expanded his analysis to other types of technologies. Mirman [16] for positive savings functions and Brock and Mirman [1] for optimal savings functions, using a discrete-time, neoclassical one-sector model, proved the existence, uniqueness and stability of a steady-state (or asymptotic) distribution for the capital-labour ratio. These steady-state distributions are the natural generalizations under uncertainty to the golden-age/golden-rule levels of the capital-labour ratio as deduced in the certainty case. While these papers are important contributions with respect to existence and uniqueness, they have little to say about the specific structure of these asymptotic distributions or about the biases (in an expected-value sense) induced by assuming a certainty model when, in fact, outcomes are uncertain.

The basic model used in this paper is a one-sector neoclassical growth model of the Solow-type where the dynamics of the capital-labour ratio can be described by a diffusion-type stochastic process. The particular source of uncertainty chosen is the population size although the analysis would be equally applicable to technological or other sources of uncertainties. The first part of the paper analyses the stochastic processes and asymptotic distributions for various economic variables, for an exogeneously given savings function, and deduces a number of first-moment relationships which will obtain in the steady-state. In addition, the special case of a Cobb-Douglas production function with a constant savings function is examined in detail and the steady-state distributions for

1 *First version received September* 1973; *final version accepted May* 1974 (*Eds.*).

2 The contents of this paper were presented in various forms at the NBER Conference on Growth Theory (Yale, December 1971); Mathematical Economics Seminar (Rochester, March 1973); Mathematical Economics Seminar (Columbia, April 1973). My appreciation to the participants for their helpful comments. My special thanks to R. M. Solow for many discussions. Aid from the National Science Foundation is gratefully acknowledged. F. Bourguignon has independently derived a number of the results in this paper. See " A Particular Class of Stochastic One Sector Growth Model in Continuous Time " (mimeo, University of Western Ontario, March 1973).

the capital-labour ratio, interest rate, etc., are derived. The second part investigates the stochastic Ramsey problem and a correspondence between this problem and an auxiliary problem involving the steady-state distribution only is derived which generalizes the notion of minimizing divergence from bliss to the stochastic case.

2. THE MODEL

We assume a one-sector neoclassical model with a constant returns to scale, strictly concave production function, $F[K, L]$, where $K(t)$ denotes the capital stock and $L(t)$ denotes the labour force which is assumed to be proportional to the population size. The capital accumulation equation can be written as

$$\dot{K}(t) = F[K(t), L(t)] - \lambda K(t) - C(t), \qquad \ldots(1)$$

where λ is the rate of depreciation (assumed to be non-negative and constant) and $C(t)$ is aggregate consumption.

The source of uncertainty in the model is the population size, $L(t)$. A reasonable stochastic process for the population dynamics can be deduced from a simple branching process for population growth.[1] Let h denote the length of time between " generations " and $X_i(t+h)$ denote the random variable number of offspring (net of deaths) for the ith person alive at time t. It is assumed that the expected number of offspring (net of deaths) per person per unit time, n, is a constant and the same for all individuals in every generation. It is also assumed that the random variable deviation from the mean can be written as the sum of two independent components: (1) a " systematic " component, $\sigma\eta(t;h)$, reflecting random effects common to all individuals at a given point in time t such as changes in social mores and tastes with respect to child-bearing, natural disaster, wide-spread disease, discovery of a " wonder " drug, national economic conditions, etc.[2] This component is assumed to be independently and identically distributed over time.[3] (2) a " non-systematic " component, $v_i\varepsilon_i(t;h)$, reflects random effects specific to the ith person alive at time t.[4]

This assumed process can be formally described by a conditional stochastic equation for $X_i(t+h)$, conditional on $L(t) = L$: namely,

$$X_i(t+h) = nh + \sigma\eta(t;\ h) + v_i\varepsilon_i(t;\ h), \quad i = 1, 2, \ldots, L, \qquad \ldots(2)$$

where n, σ, and v_i are constants; $E_t(\eta) = E_t(\varepsilon_i) = 0$; $E_t(\eta^2) = E_t(\varepsilon_i^2) = h$;

$$E_t(\eta\varepsilon_i) = 0 = E_t(\varepsilon_i\varepsilon_j),\ i \neq j;\ E_t[\eta(t;\ h)\eta(t+kh;\ h)] = 0, \quad k = 1, 2, \ldots$$

and " E_t " is the conditional expectation operator, conditional on knowledge of all (relevant) events which have occurred as of time t.

1 See Cox and Miller [2, p. 235] and Feller [4, p. 325]. However, as will be shown, we use a modified version of the processes presented there.

2 One might reasonably question the assumption that the distribution for η be exogenous and independent of L since *per capita* wealth, K/L, may affect both birth and death rates, and for finite amounts of land, L may also be influenced through " crowding ". However, since endogenously determined population growth is not central to the paper and its inclusion would follow along the same lines of analysis, we exclude it for brevity and simplicity. For a discussion of endogenous population growth in the certainty case, see Merton [15].

3 In an extension to the discussion in footnote 2, above, one might question the assumption of serial independence for η. The analysis could be modified to allow for serial dependence by introducing Ornstein-Uhlenbeck type processes [2, p. 225]. However, the cost of introducing these processes would be a more-complex, multi-dimensional dynamic structure, and it is not clear that the asymptotic distributions would be greatly affected by such serial dependencies.

4 The terms " individual ", " family ", and " group " are used interchangeably in much the same way as " population size " and " labour force " are in the standard analysis. Provided that the number of families is roughly proportional to the number of people and the number of people per family is not large, none of the analysis is materially effected by this interchange of interpretation.

To obtain a stochastic difference equation for the population size, note that

$$L(t+h)-L(t) = \Sigma X_i(t+h),$$

and hence, by summing equation (2) from $i = 1$ to L, we have that

$$L(t+h)-L(t) = nLh+\sigma L\eta(t;\ h)+\Sigma v_i\varepsilon_i(t;\ h) \qquad \ldots(3)$$

conditional on $L(t) = L$. From (3), the conditional expected change in population can be written as

$$E_t[L(t+h)-L(t)|\ L(t) = L] = nLh \qquad \ldots(4)$$

and the conditional variance as

$$\text{var}\ [L(t+h)-L(t)|\ L(t) = L] = \left[\sigma^2L^2+\left(\frac{1}{L}\Sigma v_i^2\right)L\right]h. \qquad \ldots(5)$$

If the v_i are bounded and approximately the same size, then $\Sigma v_i^2/L = 0(1)$ (e.g. if $v_i = v$, then $\Sigma v_i^2/L = v^2$). Hence, for large populations ($L \gg 1$) and $\sigma^2>0$, one can reasonably neglect the contribution of the " non-systematic " components to total population variance, and simplify the analysis by approximating (5) with

$$\text{var}\ [L(t+h)-L(t)|\ L(t) = L] \doteq \sigma^2L^2h. \qquad \ldots(6)$$

Because the major goal of the paper is to develop additional properties of the steady-state distribution beyond those of existence and uniqueness, we choose to work in continuous time and restructure the discrete time stochastic process for population size as a diffusion process.[1] The " surrogate "random variable for population size generated by the diffusion process approximation to (3) has a continuous density function on the non-negative real line, and its sample path over time will be continuous with probability one. Hence, for it to be a reasonable description of the population dynamics, the population size must be large enough to ignore the inherent discreteness of the birth-death process and large enough to justify the continuity assumption for changes over time. In addition, the approximation becomes more accurate for large values of the time variable, t, when compared with the interval between successive transitions, h. This is particularly important because we are primarily interested in the steady-state distribution where $t = \infty$.

The procedure of approximating discrete time processes by diffusion processes is useful because the mathematical methods associated with a continuum generally lend themselves more easily to analytical treatment than those associated with discrete processes. In addition, there is a large body of theory developed for the analysis of diffusion processes.

Apart from boundary conditions, the transition probabilities for a diffusion process are completely determined by a functional description of its instantaneous (infinitesmal), conditional mean and variance,[2] and hence, equations (4) and (6) are sufficient specifications to determine the appropriate " surrogate " diffusion process.

Although the diffusion sample path is continuous, it is not differentiable. Therefore, differential equations with standard time derivatives cannot be used to describe the dynamics. However, there is a generalized theory of stochastic differential equations developed by Itô and McKean [3] which is applicable to diffusion processes. In particular, the surrogate population dynamics corresponding to the discrete model described in (3) can be written as

$$dL = nLdt+\sigma Ldz, \qquad \ldots(7)$$

1 This combination provided enormous simplifications in the study of the consumption-portfolio problem. For examples, see Merton [12, 13, 14]. For further discussion of the diffusion approximation to the branching process, see Cox and Miller [2, p. 237] or Feller [4, 326]. Note: they analyse the case where birth rates across individuals are independent (i.e. $\sigma^2 = 0$), and hence, the variance of their process is proportional to L instead of L^2 as in our case.

2 See Feller [4, p. 321].

3 See Itô and McKean [7] and McKean [11].

where dz stands for a Wiener process and nL and $\sigma^2 L^2$ are the instantaneous mean and variance per unit time, respectively. Using Itô's Lemma,[1] equation (7) can be integrated, and by inspection, the random variable $L(t)/L(0)$ will have a log-normal distribution with

$$E_0\{\log [L(t)/L(0)]\} = (n - \tfrac{1}{2}\sigma^2)t$$

$$\equiv \mu t \qquad \text{...(8a)}$$

and

$$\text{var}\{\log [L(t)/L(0)]\} = \sigma^2 t. \qquad \text{...(8b)}$$

Having established a valid continuous time formulation for the population dynamics, we now determine the dynamics for capital accumulation. As in the certainty model, the dynamics can be reduced to a one-dimensional process by working in intensive (*per capita*) variables. Define

$k(t) \equiv K(t)/L(t)$, capital-labour ratio
$c(t) \equiv C(t)/L(t)$, *per capita* consumption
$f(k) \equiv F(K, L)/L = F(K/L, 1)$, *per capita* (gross) output
$s(k) \equiv 1 - c/f(k)$, (gross) savings per unit output.

Because smooth functional transformations of diffusion processes are diffusion processes, the dynamics for k will be a diffusion process whose stochastic differential equation representation can be written as [2]

$$dk = b(k)dt - \sqrt{a(k)}\,dz, \qquad \text{...(9)}$$

where $b(k) \equiv [s(k)f(k) - (n + \lambda - \sigma^2)k]$ is the instantaneous expected change in k per unit time and $a(k) \equiv \sigma^2 k^2$ is the instantaneous variance. Hence, the accumulation equation in *per capita* units follows a diffusion process and the transition probabilities for $k(t)$ are completely determined by the functions $b(k)$ and $a(k)$.

Before going on to analyse the distributional characteristics of k, it is important to distinguish between the stochastic process for k and the one for K. While the sample path for k is not differentiable, the sample path for K is. Since at a point in time, t, both $K(t)$ and $L(t)$ are known, output at that time, $F(K, L)$, is known, and from (1), K has a well-defined time derivative which is locally certain. Hence, competitive factor shares are well defined and the same as in the certainty model. Namely, the interest rate, r, and the wage rate, w, satisfy

$$r = f'(k) \qquad \text{...(10a)}$$

and

$$w = f(k) - kf'(k). \qquad \text{...(10b)}$$

Thus, unlike in the portfolio models, there is no " current " uncertainty, but only " future " uncertainty, and the returns to capital (and labour) over the next " period " (instant) are known with certainty. The returns to capital would be viewed by an investor as more like those obtained by continually reinvesting in (very) short-term bonds (i.e. " rolling-over shorts ") when the future interest rates are stochastic than those obtained by investing in common stocks with end-of " period " price uncertainty.[3]

3. THE STEADY-STATE DISTRIBUTION FOR k

Just as in the certainty model where the existence and quantitive properties of the steady state economy can be examined, so one can do so for the uncertainty model. However, instead of there being a unique point, k^*, in the steady state, there is a unique distribution

[1] See Appendix A. For the particular integration of equation (7), see McKean [11, p. 33].
[2] For a derivation of (9) using Itô's Lemma, see Appendix A.
[3] See Merton [14] for a discussion of the distinction between the two types of uncertainty with respect to interest rates and common stocks.

for k which is time and initial condition independent and toward which the stochastic process for k tends. As such it is the natural generalization of the certainty case which is included as a limiting case when dispersion tends to zero.

Since existence and uniqueness properties are not the major goals of the paper, we assume throughout the paper that the following set of sufficient conditions for existence are satisfied: (1) $f(k)$ is concave and satisfies the Inada conditions; (2) $s(k)>0$ for all $k<\bar{k}$, for some positive $\bar{k}$; (3) $n+\lambda-\sigma^2>0$.[1]

As discussed in the previous section, the stochastic process for k is completely determined by the functions $b(k)$ and $a(k)$ which in turn depend upon the particular production function and saving rule. However, it is possible to deduce a general functional representation for the steady-state probability distribution. Let $\pi_k(\cdot)$ be the steady-state density function for the capital-labour ratio. As is deduced in Appendix B, $\pi_k(\cdot)$ will satisfy.

$$\pi_k(k) = \frac{m}{a(k)} \exp\left[\int^k \frac{2b(x)}{a(x)}\, dx\right], \qquad \text{...(11)}$$

where m is a constant chosen so that $\int_0^\infty \pi_k(x)dx = 1$. Substituting for $b(k)$ and $a(k)$ from (9) into (11), we can rewrite (11) as

$$\pi_k(k) = mk^{\frac{-2(n+\lambda)}{\sigma^2}} \exp\left[\frac{2}{\sigma^2}\int^k \frac{s(x)f(x)}{x^2}\, dx\right]. \qquad \text{...(12)}$$

While equation (12) does show that the determination of the steady-state distribution reduces to one of " mere " quadrature, little more can be said about $\pi_k(\cdot)$ directly without further specifying the function $s(\cdot)f(\cdot)$. However, without further specification of this function, one can deduce certain moment relationships which must obtain in the steady state.

If $g(k)$ is a " well-behaved " function [2] of k and " E " is the expectation operator over the steady-state distribution for k, then

$$E\left\{g'(k)[s(k)f(k)-(n+\lambda-\sigma^2)k]+\frac{\sigma^2}{2}g''(k)k^2\right\} = 0. \qquad \text{...(13)}$$

The proof for (13) can be found in Appendix C.

Armed with (13), one can deduce a number of steady-state moment equalities among a variety of interesting economic relationships by simply choosing the appropriate function for $g(\cdot)$. For example, for $g(k) = k$, we have that

$$E[s(k)f(k)] = (n+\lambda-\sigma^2)E[k], \qquad \text{...(14)}$$

and if $s(k) = s$, a positive constant, then

$$E[f(k)] = \frac{(n+\lambda-\sigma^2)}{s} E[k]. \qquad \text{...(14')}$$

1 The sufficiency of these conditions for existence is shown in Appendix B. Actually, a weaker condition than (3) would be $n-1/2\sigma^2+\lambda>0 = \mu+\lambda>0$. However, in that case, certain first-moment relationships in the steady-state would not exist including $E[k]$ which would diverge. Since (3) is not much stronger than $\mu+\lambda>0$, we prefer it. Also, if $n>\sigma^2$, then $\mu>0$ which implies that with probability one, $L(t)\to\infty$ as $t\to\infty$. Hence, the $L\gg 1$ assumption of the approximation in (6) and the conditions under which the diffusion approximation is accurate will be satisfied for any positive initial population and sufficiently large t.

2 Sufficiently " well-behaved " would be that g is a C^2 function on the interval $(0, \infty)$ and that

$$\lim_{k\to 0} g'k^2\pi = \lim_{k\to\infty} g'k^2\pi = 0.$$

For $g(k) = \log(k)$, we have that

$$E\left[\frac{s(k)f(k)}{k}\right] = n+\lambda-\tfrac{1}{2}\sigma^2$$

$$= \mu+\lambda, \qquad \text{...(15)}$$

and if $s(k) = s$, then

$$E\left[\frac{f(k)}{k}\right] = \frac{\mu+\lambda}{s}. \qquad \text{...(15')}$$

The reader can try other forms for $g(\cdot)$ and deduce still more relationships.

4. THE COBB-DOUGLAS/CONSTANT SAVINGS FUNCTION ECONOMY

There is a specific functional form for $s(\cdot)f(\cdot)$ of no little interest where the steady-state distributions for all economic variables can be solved for in closed form. If it is assumed that the production function is Cobb-Douglas, $f(k) = k^{\alpha}$, $0<\alpha<1$, and that gross savings is a constant fraction of output [1] (s is a constant, $0<s \leq 1$), then by substituting the particular functional form in (12) and integrating, we have that π_k will satisfy

$$\pi_k(k) = mk^{\frac{-2(n+\lambda)}{\sigma^2}} \exp\left[\frac{-2s}{(1-\alpha)\sigma^2}\, k^{-(1-\alpha)}\right]. \qquad \text{...(16)}$$

While the constant, m, could be determined by direct integration, it will throw light on the whole analysis to compute it in an indirect way. If $R \equiv k^{\alpha-1}$, the output-to-capital ratio, and $\pi_R(R)$ is its steady-state density function, then, from (16),

$$\pi_R(R) = \pi_k(k)/\left|\frac{dR}{dk}\right|$$

$$= \frac{m}{(1-\alpha)} R^{\gamma-1}e^{-bR}, \qquad \text{...(17)}$$

where $\gamma \equiv [2(\mu+\lambda)/(1-\alpha)\sigma^2]>0$ and $b \equiv [2s/(1-\alpha)\sigma^2]>0$. By inspection, R has a gamma distribution,[2] and therefore, m must satisfy

$$m = (1-\alpha)b^{\gamma}/\Gamma(\gamma), \qquad \text{...(18)}$$

where $\Gamma(\cdot)$ is the gamma function. Because R has a gamma distribution, we have that the moment-generating function for R is

$$\phi[\theta] \equiv E\{e^{\theta R}\}$$

$$= \left[1-\frac{\theta}{b}\right]^{-\gamma} \qquad \text{...(19)}$$

and for non-integral or negative moments, we have that

$$\Phi[\theta] \equiv E\{R^{\theta}\}$$

$$= \frac{\Gamma(\theta+\gamma)}{\Gamma(\gamma)} b^{-\theta} \qquad \text{...(20)}$$

for $\theta> -\gamma$. The density functions and moments of the distributions for all the economic variables can be deduced from equations (17)-(20), and the more important ones are summarized in Table I.

[1] Therefore, $c = (1-s)f(k)$. The analysis of this section would be identical for a Modigliani-Pigou type consumption function where $c = (1-s)f(k)+\delta k$ with δ a positive constant. The formulas would be the same with " $(\lambda+\delta)$ " substituted wherever " λ " appears.

[2] For a description of the gamma distribution, see Feller [4, p. 46].

TABLE I

Steady-state probability distributions

$$b \equiv \frac{2s}{(1-\alpha)\sigma^2}$$

$$\gamma \equiv \frac{2(n+\lambda-\frac{1}{2}\sigma^2)}{(1-\alpha)\sigma^2} = \frac{2(\mu+\lambda)}{(1-\alpha)\sigma^2}$$

$$\eta \equiv \frac{(1-\alpha)}{\alpha}$$

Capital/labour ratio ($k \equiv K/L$)

Density function:

$$\pi_k(k) = 0, \qquad k \leqq 0$$

$$= \frac{(1-\alpha)b^\gamma}{\Gamma(\gamma)} k^{\frac{-2(n+\lambda)}{\sigma^2}} \exp[-bk^{\alpha-1}], \quad k>0$$

Moment-generating function:

$$\Phi_k[\theta] = E\{k^\theta\} = \frac{\Gamma[\gamma-\theta/(1-\alpha)]}{\Gamma[\gamma]}[b]^{\frac{\theta}{1-\alpha}}, \quad \theta<(1-\alpha)\gamma$$

"*Per capita*" *output* ($y \equiv f(k) = k^\alpha$)

Density function:

$$\pi_y(y) = 0, \qquad k \leqq 0$$

$$= \frac{\eta b^\gamma}{\Gamma(\gamma)} y^{-(\eta\gamma+1)} \exp[-by^{-\eta}], \qquad k>0$$

Moment-generating function:

$$\Phi_y[\theta] = E\{y^\theta\} = \frac{\Gamma[\gamma-\theta/\eta]}{\Gamma[\gamma]}[b]^{\theta/\eta}, \qquad \theta<\eta\gamma$$

Output/capital ratio ($R = f(k)/k = k^{\alpha-1}$)

Density function:

$$\pi_R(R) = 0, \qquad R \leqq 0$$

$$= \frac{b^\gamma}{\Gamma(\gamma)} R^{\gamma-1} \exp[-bR], \qquad R>0$$

Moment-generating function:

$$\Phi_R[\theta] = E\{R^\theta\} = \frac{\Gamma(\gamma+\theta)}{\Gamma(\gamma)}[b]^{-\theta}$$

$$\phi_R[\theta] = E\{e^{\theta R}\} = \left(1-\frac{\theta}{b}\right)^{-\gamma}$$

Interest rate ($r = f'(k) = \alpha k^{\alpha-1}$)

Density function:

$$\pi_r(r) = 0, \qquad r \leqq 0$$

$$= \frac{(b/\alpha)^\gamma}{\Gamma(\gamma)} r^{\gamma-1} \exp\left[-\frac{br}{\alpha}\right], \qquad r>0$$

Moment-generating function:

$$\Phi_r[\theta] = E\{r^\theta\} = \frac{\Gamma(\gamma+\theta)}{\Gamma(\gamma)}\left[\frac{b}{\alpha}\right]^{-\theta}$$

$$\phi_r[\theta] = E\{e^{r\theta}\} = \left(1-\frac{\alpha\theta}{b}\right)^{-\gamma}$$

Since most of the literature on growth models has neglected uncertainty, it is useful to know whether the steady-state solutions obtained in these analyses are unbiased estimates of the first-moments of the corresponding steady-state distributions. Unfortunately, the certainty estimates are biased as is illustrated in Table II using the closed-form solutions of this section. In particular, the certainty estimates for expected *per capita* consumption,

TABLE II

A comparison of steady-state expected values with steady-state certainty estimates

Variable	Expected value		Certainty estimate
Capital/labour ratio	$E\{k\} = \dfrac{\Gamma\left[\dfrac{2(n+\lambda-\sigma^2)}{(1-\alpha)\sigma^2}\right]}{\Gamma\left[\dfrac{2(n+\lambda-\frac{1}{2}\sigma^2)}{(1-\alpha)\sigma^2}\right]}\left[\dfrac{2s}{(1-\alpha)\sigma^2}\right]^{\frac{1}{1-\alpha}}$	$>$	$\left[\dfrac{s}{n+\lambda}\right]^{\frac{1}{1-\alpha}}$
Per capita output	$E\{f(k)\} = \dfrac{(n+\lambda-\sigma^2)}{s}E\{k\}$	$>$	$\left[\dfrac{s}{n+\lambda}\right]^{\frac{\alpha}{1-\alpha}}$
Per capita consumption	$E\{c\} = \dfrac{(1-s)(n+\lambda-\sigma^2)}{s}E\{k\}$	$>$	$(1-s)\left[\dfrac{s}{n+\lambda}\right]^{\frac{\alpha}{1-\alpha}}$
Capital/output ratio	$E\{k/f(k)\} = \dfrac{s}{n+\lambda-\frac{1}{2}(2-\alpha)\sigma^2}$	$>$	$\dfrac{s}{n+\lambda}$
Output/capital ratio	$E\{f(k)/k\} = \dfrac{n+\lambda-\frac{1}{2}\sigma^2}{s}$	$<$	$\dfrac{n+\lambda}{s}$
Interest rate	$E\{f'(k)\} = \dfrac{\alpha(n+\lambda-\frac{1}{2}\sigma^2)}{s}$	$<$	$\dfrac{\alpha(n+\lambda)}{s}$

output and capital are too small while the estimates for the output-capital ratio and the interest rate are too large. These results suggest that care must be taken in using the certainty analysis even as a first-moment approximation theory.[1]

In this and previous sections it has been shown that by working in continuous time and modelling the stochastic dynamics with diffusion processes, a number of important properties of the steady-state distributions in addition to existence and uniqueness can be determined. In the special case of this section, a complete analytical description was possible. Even in those cases where closed-form solutions are not deducible, powerful numerical integration techniques are available for solution of the parabolic partial differential equations satisfied by the transition probabilities and moment-generating functions. Hence, both simulation and estimation of the model are feasible. While the analysis presented assumed uncertain population size, the approach extends itself in a straightforward fashion to a variety of other specifications. For example, Mirrlees [17] has labour-augmenting technical progress as the source of uncertainty in his model where the (future) level of technical progress is log-normally distributed. The analysis presented here would be identical for his model where the intensive variables are in efficiency rather than *per capita* units.

There are partial differential equations for multi-dimensional diffusion processes corresponding to the ones for the one-dimensional process examined here. Hence, multi-sector models with more than one source of uncertainty can be studied with the same mode of analysis used here.

In addition, these analyses often provide " throw-offs " useful in other areas of research. For example, in developing a theory for the term structure of interest rates, it is usually necessary to postulate some process for the basic short-rate over time. Using the model of this section, we can derive an analytical description for the interest rate process. Since r ($\equiv \alpha k^{\alpha-1}$) is a smooth, monotone function of k, the interest rate dynamics will itself be generated by a diffusion process. From (9) and Itô's Lemma, we can deduce the form for the stochastic differential equation for r to be

$$dr = (Ar - Br^2)dt + vrdz, \qquad \ldots(21)$$

where

$$A \equiv (1-\alpha)\left(n+\lambda-\frac{\alpha}{2}\sigma^2\right) > 0 \qquad \ldots(22)$$

$$B \equiv (1-\alpha)s/\alpha$$

$$v^2 \equiv (1-\alpha)^2\sigma^2.$$

Using Itô's Lemma again, we can stochastically integrate (21) to obtain an expression for the random variable $r(t)$, conditional on $r(0) = r_0$, in terms of random variables with known distributions: namely,

$$r(t) = \frac{(r_0 \exp[(A-\frac{1}{2}v^2)t + vZ(t)])}{\left(1 + r_0 B \int_0^t \exp[(A-\frac{1}{2}v^2)s + vZ(s)]ds\right)}, \qquad \ldots(23)$$

[1] It should be pointed out that although the first-moment of the steady-state distribution does not equal the certainty estimate, the mode of the steady-state distribution for k is the same as the certainty steady-state value for k. By differentiating $\pi_k(k)$ in (12) and setting it equal to zero, we have that

$$s(M)f(M) = (\eta+\lambda)M$$

where M is the mode of the steady-state distribution for k, independent of σ^2. Hence, k and all other variables in Tables I and II converge in distribution to a spike at the certainty value. I am indebted to an editor for pointing this out.

where $Z(t) \equiv \int^{t} dz$ is a Gaussian distributed random variable with a zero mean and $E\{Z(s)Z(s')\} = \min(s, s')$. By inspection of (23), $1/r(t)$ is equal to a weighted integral of log-normally-distributed random variables. Since it has already been shown that $r(t)$ has a gamma distribution as $t \to \infty$, we have as a curious side-result that the distribution of an infinite integral of log-normal variates is inverse gamma.

5. THE STOCHASTIC RAMSEY PROBLEM

In the previous sections an expression for the steady-state distribution of the capital-labour ratio was determined for an arbitrary savings function. We now turn to the problem of determining the optimal savings policy under uncertainty. Formally, the finite-horizon problem is to find a savings policy, $s^*(k, T-t)$, so as to

$$\max E_0 \left\{ \int_0^T U[(1-s)f(k)]dt \right\} \qquad \text{...(24)}$$

subject to $k(T) \geqq 0$ with probability 1 and where $U[\cdot]$ is a strictly concave, von Neumann-Morgensten utility function of *per capita* consumption for the representative man. The technique used to solve the problem is stochastic dynamic programming. Let

$$J[k(t), t;\ T] \equiv \max E_t \left\{ \int_t^T U[(1-s)f(k)]d\tau \right\} \qquad \text{...(25)}$$

$J[\cdot]$ is called the Bellman function and by the principle of optimality,[1] J must satisfy

$$0 = \max_{\{s\}} \left\{ U[(1-s)f(k)] + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial k}[sf - \beta k] + \tfrac{1}{2}\frac{\partial^2 J}{\partial k^2}\sigma^2 k^2 \right\}, \qquad \text{...(26)}$$

where the stochastic process for k satisfies (9) and $\beta \equiv (n+\lambda-\sigma^2) > 0$. The first-order condition to be satisfied by the optimal policy s^* is

$$U'[(1-s^*)f] = \frac{\partial J}{\partial k}, \qquad \text{...(27)}$$

where $U'[c] \equiv dU/dc$. To solve for s^* (in principle), one solves (27) for s^* as a function of k, $T-t$ and $\partial J/\partial k$, and substitutes into (26) which becomes a partial differential equation for J. Having solved this equation, one substitutes back into (27) to determine s^* as a function of k and $T-t$.[2]

Because of the non-linearity of the Bellman partial differential equation, closed-form solutions are rare. However, in the limiting infinite-horizon ($T \to \infty$) case of Ramsey, the analysis is substantially simplified because this partial differential equation reduces to an ordinary differential equation. Since the stochastic process for k is time-homogeneous and $U[\cdot]$ is not a function of t, we have from (25) that

$$\frac{\partial J}{\partial t} = -E_t\{U[(1-s^*[k, T-t])f[k(T-t)]]\}. \qquad \text{...(28)}$$

[1] Sufficient differentiability of the Bellman function, J, is assumed in the dynamic programming formulation. However, provided that an optimal solution with bounded controls exists, the strict concavity of U and the smoothness of the dynamics for k are sufficient to ensure differentiability for $k > 0$. For a rigorous development of the optimality equation (26), see Kushner [8], and for a less-formal discussion, see Merton [12, 13].

[2] This is the standard procedure for solving continuous-time, dynamic programming problems. See Merton [12, 13] for explicit examples of solution.

If an optimal policy exists,[1] $f(\cdot)$ satisfies the Inada conditions, and $\beta>0$, then

$$\lim_{T\to\infty} s^*(k, T-t) = s^*(k, \infty) = s^*(k),$$

and from the analysis in the previous sections, there will exist a steady-state distribution for k, π_k^*, associated with the optimal policy $s^*(k)$. Taking the limit in (28) we have that

$$\lim_{T\to\infty}\left\{\frac{\partial J}{\partial t}\right\} = -E^*\{U[(1-s^*)f(k)]\}$$

$$\equiv -B, \qquad \ldots(29)$$

where E^* is the expectation operator over the steady-state distribution π_k^* and B is the level of expected utility of *per capita* consumption in the Ramsey-optimal steady-state which is independent of the initial condition, $k(t)$.

From the Bellman equation, (26), and (29), we have that, as $T\to\infty$, J must satisfy the ordinary differential equation

$$0 = U[(1-s^*)f]-B+J'[s^*f-\beta k]+\tfrac{1}{2}J''\sigma^2k^2, \qquad \ldots(30)$$

where primes denote derivatives with respect to k.[2] By differentiating the first-order condition (27) with respect to k, we have that

$$J'' = U''[(1-s^*)f(k)]\left[(1-s^*)f'(k)-\frac{ds^*}{dk}f(k)\right] \qquad \ldots(31)$$

substituting for J'' and J' from (27) and (31) into (30) and rearranging terms, we can rewrite (30) as

$$0=(-\tfrac{1}{2}\sigma^2k^2fU'')\frac{ds^*}{dk}+(fU'-\tfrac{1}{2}\sigma^2k^2U''f')s^*+\tfrac{1}{2}\sigma^2k^2U''f'-U'\beta k+U-B \quad \ldots(32)$$

which is a first-order differential equation for s^*. Note that for the (degenerate) case of certainty ($\sigma^2 = 0$), (32) reduces to

$$(s^*f-\beta k) = \frac{B-U}{U'} \qquad \ldots(32')$$

which is " Ramsey's Rule ", where B is the " bliss level " of utility associated with maximum steady-state consumption and $\dot{k} = (s^*f-\beta k)$ along the optimal certainty path.

In the certainty case and without regard to the time-optimal path associated with $\max \int_0^\infty U[c]dt$, the optimal steady-state capital-labour ratio can be determined by the static maximization of $U[c(k)]$ in the steady-state (i.e. with $\dot{k} = 0$ and $c(k) = f(k)-\beta k$). The solution for all strictly concave utility functions is the well-known Golden Rule, $f'(k^*) = \beta$. Hence, it is natural to ask whether there exists a corresponding method using only the steady-state distribution for determining the optimal savings policy under uncertainty.

To answer this question, we consider the problem of finding the savings policy, $s^{**}(k)$, that maximizes the expected utility of *per capita* consumption over the steady-state distribution. I.e.

$$\max_{\{s\}} E\{U[(1-s)f(k)]\} = \max_{\{s\}}\int_0^\infty U[(1-s)f(k)]\pi_k(k)dk, \qquad \ldots(33)$$

[1] There is an extensive literature on the existence of an optimal policy for the Ramsey problem under certainty. For a discussion of existence under uncertainty, see Mirrlees [17, 18].

[2] The boundary condition for (30) is a transversality-type condition that as $t\to\infty$, limit $E_0\{J[k(t), t]\} = 0$. Note: because (30) does not contain J explicitly, any candidate solution, $\hat{J}$, which satisfies $E_0\{\hat{J}[k(t), t]\} = H$, a constant, can be made to satisfy the transversality condition by setting $J[k(t), t] = \hat{J}[k(t), t]-H$.

which is the natural generalization to uncertainty of the static maximization under certainty.[1]

From equation (12), we can rewrite the steady-state density function for k as

$$\pi_k(k) = mk^{-\delta} \exp\left[\frac{2}{\sigma^2} h(k)\right], \qquad \text{...(34)}$$

where

$$\delta \equiv 2 + \frac{2\beta}{\sigma^2}$$

$$h(k) \equiv \int^k s(x)f(x)x^{-2}dx$$

$$\dot{h}(k) \equiv \frac{dh}{dk} = s(k)f(k)k^{-2} \qquad \text{...(35)}$$

$$\ddot{h}(k) \equiv \frac{d^2h}{dk^2} = \frac{ds}{dk} fk^{-2} + sf'k^{-2} - 2sfk^{-3}$$

and m is a constant chosen such that $m \int_0^\infty k^{-\delta} e^{2h(k)/\sigma^2} dk = 1$. Substituting from (34) for π_k and noting that from (35), $(1-s)f = f - k^2\dot{h}$, we can rewrite (33) as the constrained maximization problem

$$\max \left\{\left[m \int_0^\infty U[f - k^2\dot{h}]k^{-\delta} e^{2h/\sigma^2} dk\right] + \lambda\left[1 - m \int_0^\infty k^{-\delta} e^{2h/\sigma^2} dk\right]\right\}, \qquad \text{...(36)}$$

where λ is the usual multiplier for the constraint. Inspection of (36) shows that, formally, it is identical to a standard intertemporal maximization problem under certainty where the independent variable is k instead of time. Hence, either the classical calculus of variations or the maximum principle can be employed to solve it. The Euler equations for (36) can be written as

$$0 = \frac{d}{dk}[U'k^{2-\delta}e^{2h/\sigma^2}] + \frac{2}{\sigma^2} k^{-\delta} e^{2h/\sigma^2}(U - \lambda) \qquad \text{...(37a)}$$

$$0 = \int_0^\infty U\pi_k(k)dk - \lambda \int_0^\infty \pi_k(k)dk \qquad \text{...(37b)}$$

$$0 = 1 - m \int_0^\infty k^{-\delta} e^{2h/\sigma^2} dk. \qquad \text{...(37c)}$$

Carrying out the differentiation in (37a), substituting for $\ddot{h}$, $\dot{h}$, and δ from (35), and rearranging terms, we can rewrite (37a) as

$$0 = -(U''k^2f)\frac{ds^{**}}{dk} + \left(-U''k^2f' + \frac{2}{\sigma^2} fU'\right)s^{**} + k^2f'U'' - \frac{2\beta}{\sigma^2} kU' + \frac{2}{\sigma^2}(U - \lambda), \qquad \text{...(38)}$$

where $s^{**}(k)$ is the optimal policy associated with (36) and (37).

[1] Unlike in the certainty case where a single k^* is chosen, we must choose a steady-state distribution, π^*, which is completely determined by the policy variable $s(k)$. In the certainty case, $\pi^* = \delta(k - k^*)$ where $\delta(\cdot)$ is the Dirac delta function and k^* depends on s^* through the steady-state constraint that $s^* = \beta k^*/f(k^*)$. Hence, from the monotonicity of $\beta k/f(k)$, (33) reduces to $\max\{U[f(k) - \beta k]\}$, where the choice variable is k.

A comparison of (38) and (32) shows that the two differential equations are identical except for the constant terms λ and B. However, from (37b), we see that

$$\begin{aligned}\lambda &= \int_0^\infty U[(1-s^{**})f]\pi^{**}(k)dk \\ &= \max E\{U[c]\} \\ &= B, \text{ by its definition in (29).}\end{aligned} \qquad \ldots(39)$$

Hence, the optimal policy associated with (36) and the one associated with (24) for $T = \infty$ are identical. I.e. $s^{**}(k) = s^*(k)$. Just as in the certainty case, the criterion

$$\max E_0\left\{\int_0^\infty [U-\lambda]dt\right\}$$

has the interpretation of minimizing the (expected) divergence from bliss and clearly in the certainty case, λ is the utility of maximum substainable consumption. One major difference in the uncertainty case is that the steady-state maximization gives the optimal savings policy for all time and not just the asymptotically optimal savings policy. Further, while we have demonstrated the correspondence between the two problems only for the special case of continuous-time diffusion processes, it is probably not a difficult task to prove it for general time-homogenous Markov processes and time-independent utility functions.

Unfortunately, inspection of (38) shows that there is no unique optimal steady-state distribution for k for all concave utilities corresponding to the Golden Rule under certainty. However, there is a special case where unanimity obtains.

Suppose $f(k)$ is Cobb-Douglas and we ask the question what *constant* savings function is optimal? From the correspondence between (24) with $T = \infty$ and (36), the problem can be formulated as choose the constant s^* so as to

$$\max_{\{s\}} \int_0^\infty U[(1-s)k^\alpha]\pi(k;\ s)dk, \qquad \ldots(40)$$

where from (16) and (18)

$$\pi(k;\ s) = \frac{(1-\alpha)}{\Gamma(\gamma)} b^\alpha k^{-\delta} \exp[-bk^{\alpha-1}]$$

and δ is as defined in (35); $\gamma \equiv (\delta-1)/(1-\alpha)$ and $b \equiv 2s/(1-\alpha)\sigma^2$. The first-order condition for a maximum in (40) is

$$0 = \int_0^\infty \left[\frac{\partial\pi}{\partial s} U - k^\alpha U'\pi\right] dk. \qquad \ldots(41)$$

Define $V(k;\ s) \equiv U[(1-s)k^\alpha]$. Noting that $V' \equiv dV/dk = \alpha(1-s)k^{\alpha-1}U'$ and

$$\partial\pi/\partial s = [(\gamma/s)-(2k^{\alpha-1}/(1-\alpha)\sigma^2)]\pi,$$

we can rewrite (41) as

$$0 = \left[\int_0^\infty \left\{\alpha(1-s^*)\left(\frac{\gamma}{s^*} - \frac{2k^{\alpha-1}}{(1-\alpha)\sigma^2}\right) V\pi - k\pi V'\right\} dk\right] / \alpha(1-s^*). \qquad \ldots(41')$$

Using integration by parts, we have that

$$\begin{aligned}\int_0^\infty (k\pi)V'dk &= Vk\pi\Big|_0^\infty - \int_0^\infty V\frac{d}{dk}(k\pi)dk \\ &= 0 - \int_0^\infty V\frac{d}{dk}(k\pi)dk\end{aligned} \qquad \ldots(42)$$

by the definition of π and the concavity of V. Using $d(k\pi)/dk = [1+b^*(1-\alpha)\,k^{\alpha-1}-\delta]\pi$ in (42) and substituting (42) into (41′), we can rewrite (41′) as

$$0 = \int_0^\infty V\pi\left\{\left[b^*(1-\alpha)-\frac{2\alpha(1-s^*)}{(1-\alpha)\sigma^2}\right]k^{\alpha-1}+\left[\frac{\alpha(1-s^*)\gamma}{s^*}+1-\delta\right]\right\}dk. \qquad \ldots(43)$$

By inspection, the integrand of (43) will be identically zero for all V, π and k if $s^* = \alpha$. Hence, in the class of constant savings rules with a Cobb-Douglas production function, the optimal rule is $s^* = \alpha$ for all concave utility maximizers.

APPENDIX A

Itô's Lemma

While the sample paths of diffusion-type stochastic processes are continuous with probability one, they are not differentiable. Hence, standard differential equation representations cannot be used to describe the dynamics of such processes. However, a complete theory of stochastic differential equations for processes of this type has been developed (cf. [7] and [11]) which allows for (stochastic) integration and differentiation in a manner similar to that of the ordinary calculus. The stochastic analog to the Fundamental Theorem of the Calculus is called Itô's Lemma which for one-dimensional, time-dependent diffusion processes can be stated as [11, p. 32]:

Itô's Lemma. *Let $F(X, t)$ be a C^2 function defined on $R^2 \times [0, \infty)$ and take the stochastic integral*

$$X(t) = X(0)+\int_0^t b(x, s)ds+\int_0^t \sqrt{a(x, s)}dz,$$

then the time-dependent random variable $y \equiv F$ is a stochastic integral and its stochastic differential is

$$dy = \frac{\partial F}{\partial X}dx+\frac{\partial F}{\partial t}dt+\tfrac{1}{2}\frac{\partial^2 F}{\partial X^2}(dx)^2,$$

where

$$dx = b(x, t)dt+\sqrt{a(x, t)}\,dz$$

and the product of the differentials $(dx)^2$ is defined by the multiplication rule

$$dzdz = 1\ dt$$

$$dzdt = 0.$$

Itô's Lemma is a particularly powerful practical tool for the analysis of stochastic dynamics. For examples of its application to some economic problems see [12], [13] and [14]. The lemma shows exactly how to differentiate and hence integrate functions of Wiener processes. Since diffusion processes can be written as functional transformations of Wiener processes, the lemma allows one to immediately deduce the dynamics for any well-behaved function of a diffusion-process random variable. Thus, by inspection of the resulting Itô equation, one can determine the instantaneous mean and variance for the transformed process and hence all the information necessary to determine the transition probabilities and moments of the transformed process. Further, as is illustrated in the text by deducing the distribution for future interest rates in the Cobb-Douglas example, it is sometimes possible to use Itô's Lemma to integrate the differential equation directly to obtain a representation for the random variable as a function of the initial value, time and a random variable whose distribution is well known (e.g. Gaussian), even when no closed-form solution exists for the transition probabilities.

To determine the stochastic differential for the capital-labour ratio, $k \equiv K/L$, we apply Itô's Lemma as follows:

$$\begin{aligned} k &= K/L \equiv G(L,t) \\ \frac{\partial G}{\partial L} &= -\frac{K}{L^2} = -\frac{k}{L} \\ \frac{\partial^2 G}{\partial L^2} &= \frac{2K}{L^3} = \frac{2k}{L^2} \\ \frac{\partial G}{\partial t} &= \frac{\dot{K}}{L} = (sf(k)-\lambda k) \quad \text{from (2).} \end{aligned} \qquad \ldots(A.1)$$

From Itô's Lemma,

$$dk = \frac{\partial G}{\partial L}dL + \frac{\partial G}{\partial t}dt + \tfrac{1}{2}\frac{\partial^2 G}{\partial L^2}(dL)^2. \qquad \ldots(A.2)$$

From (7) and Itô's Lemma, we have that

$$\begin{aligned} dL &= nLdt + \sigma L dz \\ (dL)^2 &= \sigma^2 L^2 dt. \end{aligned} \qquad \ldots(A.3)$$

Substituting from (A.1) and (A.3) into (A.2), we have that

$$\begin{aligned} dk &= \left(-\frac{k}{L}\right)(nLdt+\sigma Ldz)+(sf(k)-\lambda k)dt+\tfrac{1}{2}\left(\frac{2k}{L^2}\right)\sigma^2 L^2 dt \\ &= [sf(k)-(\lambda+n-\sigma^2)k]dt-\sigma k dz, \end{aligned} \qquad \ldots(A.4)$$

which is equation (9) of the text.

Finally, there is a multi-dimensional version of Itô's Lemma [11, p. 32] for vector-valued diffusion processes.

APPENDIX B

The Steady-state Distribution for a Diffusion Process

Let $X(t)$ be the solution to the Itô equation

$$dx = b(x)dt + \sqrt{a(x)}\,dz, \qquad \ldots(B.1)$$

where $a(\cdot)$ and $b(\cdot)$ are C^2 functions on $[0, \infty)$ and independent of t with $a(x)>0$ on $(0, \infty]$ and $a(0) = b(0) = 0$. Then $X(t)$ describes a diffusion process taking on values in the interval $[0, \infty]$ with $X = 0$ and $X = \infty$ natural absorbing states. I.e. if $X(t) = 0$, then $X(\tau) = 0$ for $\tau > t$ and similarly, for $X(t) = \infty$.

Let $p(X, t;\ X_0)$ be the conditional probability density for X at time t, given $X(0) = X_0$. Because $X(t)$ is a diffusion process, its transition density function will satisfy the Kolmogorov-Fokker-Planck "forward" equation (Feller [4, p. 326] and Cox and Miller [2, p. 215]).

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[a(x)p(x,t;\ X_0)] - \frac{\partial}{\partial x}[b(x)p(x,t;\ X_0)] = \frac{\partial p(x,t;\ X_0)}{\partial t}. \qquad \ldots(B.2)$$

Suppose that X has a steady-state distribution, independent of X_0. I.e.

$$\lim_{t\to\infty} p(X,t;\ X_0) = \pi(x).$$

Then, $\lim_{t\to\infty} (\partial p/\partial t) = 0$, and π will satisfy

$$\frac{1}{2}\frac{d^2}{dx^2}[a(x)\pi(x)] - \frac{d}{dx}[b(x)\pi(x)] = 0. \qquad \ldots(B.3)$$

By standard methods, one can integrate (B.3) twice to obtain a formal solution for $\pi(x)$: namely,

$$\pi(x) = m_1 I_1(x) + m_2 I_2(x), \qquad \ldots(\text{B.4})$$

where

$$I_1(x) \equiv \frac{1}{a(x)} \exp\left[2\int^x \frac{b(y)}{a(y)}dy\right]$$

and

$$I_2(x) \equiv \frac{1}{a(x)} \int^x \exp\left[2\int_y^x \frac{b(s)}{a(s)}ds\right]dy$$

and m_1 and m_2 are constants to the be chosen such that

$$\int_0^\infty \pi(x)dx = 1.$$

While the formal solution was straightforward, the proof of existence and the determination of the constants is more difficult. Formally, a steady-state distribution will always exist in the sense that x will either (1) be absorbed at one of the natural boundaries (i.e. a degenerate distribution with a dirac function for a density) or (2) it will have a finite density function on the interval $(0, \infty)$ or (3) it will have a discrete probability mix of (1) and (2). However, we are interested in the conditions under which a strictly non-trivial steady-state distribution exists (possibility (2)). Under such conditions, the boundaries are said to be *inaccessible*. I.e. prob $\{X(t) \leqq \varepsilon\} \to 0$ and prob $\{X(t) \geqq 1/\varepsilon\} \to 0$ as $\varepsilon \to 0$. Further, it can be shown that the boundaries are inaccessible if and only if $\int_0^x I_2(y)dy$ and $\int_x^\infty I_2(y)dy$ both diverge and $\int_0^\infty I_1(y)dy$ is bounded. Hence, under these conditions, we can conclude that $m_2 = 0$.

We now prove that the boundaries are inaccessible for the stochastic process (9) described in the text. From (9) and the assumptions of Section 3, we have that

$$b(k) \equiv [s(k)f(k) - (n+\lambda-\sigma^2)k] \qquad \ldots(\text{B.5})$$

and

$$a(k) \equiv \sigma^2 k^2, \qquad \ldots(\text{B.6})$$

where $f(k)$ is a concave function satisfying $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$;

$$0 < \varepsilon \leqq s(k) \leqq 1; \; n+\lambda-\sigma^2 > 0.$$

The method of proof is to compare the stochastic process generated by (9) with another stochastic process which is known to have inaccessible boundaries and then to show that the probability that k reaches its boundaries is no larger than the probability that the comparison process reaches its boundaries.

Using Itô's Lemma (Appendix A), we can write the stochastic differential equation for $x \equiv \log(k)$ as

$$dx = h(x)dt - \sigma dz, \qquad \ldots(\text{B.7})$$

where

$$h(x) \equiv e^{-x}s(e^x)f(e^x) - (n+\lambda-\tfrac{1}{2}\sigma^2). \qquad \ldots(\text{B.8})$$

Using the assumptions that $0 < \varepsilon \leqq s(e^x)$ and $f'(0) = \infty$ along with L' Hospital's Rule, we have that

$$\lim_{x\to-\infty} h(x) = \infty$$

and similarly, using the assumptions that $s(e^x) \leqq 1$ and $f'(\infty) = 0$, we have that

$$\lim_{x\to\infty} h(x) = -(n+\lambda-\tfrac{1}{2}\sigma^2) < 0. \qquad \ldots(\text{B.9})$$

By continuity, there exists an $\underline{x} > -\infty$ such that for all $x \in [-\infty, \underline{x}]$, there exists a $\delta_1 > 0$ such that

$$h(x) \geqq h(\underline{x}) \geqq \delta_1 > 0. \qquad \text{...(B.10)}$$

Similarly, there exists an $\bar{x} < \infty$ such that for all $x \in [\bar{x}, \infty]$, there exists a $\delta_2 < 0$ such that

$$h(x) \leqq h(\bar{x}) \leqq \delta_2 < 0. \qquad \text{...(B.11)}$$

Consider a Wiener process $W_1(t)$ with drift δ_1 and variance σ^2 defined on the interval $[-\infty, \underline{x}]$ where $\underline{x}$ is a reflecting barrier. I.e.

$$dW_1 = \delta_1 dt - \sigma dz \qquad \text{...(B.12)}$$

for $W_1 \in [-\infty, \underline{x}]$. Cox and Miller [2, p. 223-225] have shown that such a process with $\delta_1 > 0$ has a non-degenerate steady-state, and hence $-\infty$ is an inaccessible boundary. Comparing (B.12) and (B.7), we see that the two processes differ only by the drift term. Further, from (B.10), the drift on x is always at least as large as the drift on W_1 in the interval $[-\infty, \underline{x}]$. Therefore, the probability that x will be absorbed at $-\infty$ is no greater than for W_1, and hence, $-\infty$ is an inaccessible boundary for x. But, $x \equiv \log k$. Thus, zero is an inaccessible boundary for the k process.

Consider a Wiener process $W_2(t)$ with drift δ_2 and variance σ^2 defined on the interval $[\bar{x}, \infty]$ where $\bar{x}$ is a reflecting barrier. Again, using the Cox and Miller analysis, W_2 will have a non-degenerate steady-state provided that $\delta_2 < 0$. But from (B.11), the drift on x will be at least as negative as δ_2 on the interval $[\bar{x}, \infty]$, and hence ∞ is an inaccessible boundary for x. Therefore, ∞ is an inaccessible boundary for k. Hence, we have proved that under the assumptions of the text, both boundaries of the k process are inaccessible and that a non-trivial steady-state distribution for k exists. Note: as mentioned in footnote 1, p. 379, we only required the weaker assumption that $(n + \lambda - \frac{1}{2}\sigma^2) > 0$ used in (B.9) to prove existence.

Because the boundaries are inaccessible, we also have that $m_2 = 0$ in (B.4). A first integral of (B.3) gives

$$\frac{1}{2}\frac{d}{dx}[a(x)\pi(x)] - b(x)\pi(x) = \tfrac{1}{2}m_2 = 0 \qquad \text{...(B.13)}$$

for a non-degenerate steady-state. We use this result in Appendix C.

Finally, the solution for the non-degenerate steady-state distribution can be written as

$$\pi(x) = \frac{m}{a(x)} \exp\left[2\int^x \frac{b(y)}{a(y)} dy\right], \qquad \text{...(B.14)}$$

where m is chosen so that $\int_0^\infty \pi(x)dx = 1$.

APPENDIX C

More Steady-state Properties

Let $X(t)$ be a random variable whose dynamics can be written as the Itô stochastic differential equation

$$dx = b(x)dt + \sqrt{a(x)}\, dz, \qquad \text{...(C.1)}$$

where $a(x)$ and $b(x)$ are such that x has a steady-state distribution $\pi(x)$ which satisfies (B.14) in Appendix B.

Let $g = g(x)$ be a time-independent function of x. Provided that g is a sufficiently well-behaved function, the stochastic process generating g will also be a diffusion process with a stochastic differential equation representation

$$dg(x) = b_g(x)dt + \sqrt{a_g(x)}\, dz, \qquad \text{...(C.2)}$$

where by Itô's Lemma (Appendix A),

$$b_g(x) \equiv g'(x)b(x)+\tfrac{1}{2}g''(x)a(x)$$
$$a_g(x) \equiv [g'(x)]^2 a(x). \qquad \ldots(C.3)$$

If $g(\cdot)$ is twice continuously differentiable and $g'(\cdot)$ satisfies the conditions:

$$\lim_{x\to 0}[g'(x)a(x)\pi(x)] = \lim_{x\to\infty}[g'(x)a(x)\pi(x)] = 0,$$

then

$$E\{b_g(x)\} \equiv E\{g'(x)b(x)+\tfrac{1}{2}g''(x)a(x)\} = 0, \qquad \ldots(C.4)$$

where " E " is the expectation operator over the $\pi(\cdot)$ distribution.

Proof of (C.4) follows directly from integration by parts:

$$E\{b_g(x)\} = \int_0^\infty [g'(x)b(x)+\tfrac{1}{2}g''(x)a(x)]\pi(x)dx. \qquad \ldots(C.5)$$

Integrating by parts,

$$\int_0^\infty g''(x)a(x)\pi(x)dx = [g'(x)a(x)\pi(x)]\Big|_0^\infty - \int_0^\infty g'(x)\frac{d}{dx}[a(x)\pi(x)]dx$$
$$= -\int_0^\infty g'(x)\frac{d}{dx}[a(x)\pi(x)]dx \qquad \ldots(C.6)$$

from the limit conditions imposed on $g'(x)$. Substituting from (C.6) into (C.5) and rearranging terms, we have that

$$E\{bg(x)\} = \int_0^\infty g'(x)\left\{b(x)\pi(x)-\tfrac{1}{2}\frac{d}{dx}[a(x)\pi(x)]\right\}dx \qquad \ldots(C.7)$$
$$= 0$$

because $\pi(\cdot)$ satisfies (B.13) in Appendix B, and hence the term in curly brackets, { }, is identically zero. QED

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