

Asset Allocation with Endogenous Labor Income: The Case of Incomplete Markets

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Preliminary

Comments Are Welcome

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Abstract

This paper investigates optimal consumption and portfolio decisions with nontradable labor income and flexible labor supply. This paper considers risky labor income in a setting where wage income and stock returns are not perfectly correlated. The paper provides approximate closed-form solutions to the problem, allowing for a thorough characterization of the optimal consumption and portfolio policies. These solutions show that, when labor income risk is idiosyncratic, the presence of labor/leisure choice can have dramatic positive effects on portfolio allocations relative to the benchmark in which labor income is exogenously given to the investor. The main mechanism delivering this result is that consumption becomes less sensitive to financial downfalls, thus raising the incentive to participate in the stock market.

I Introduction

This paper studies optimal asset allocation in the presence of endogenous, nontradable labor income. For many investors, human capital constitutes a major fraction of their total wealth. Human capital is in fact a major component of most developed economies. For example, labor income (the yield on human capital), measured as total wage compensation, accounted for almost three quarters of GNP in the US in 1997.² Labor income is uncertain for most individuals, and it is also endogenous through the labor supply decision. This creates moral hazard problems that prevent individuals from efficiently sharing their labor income risk by selling contracts contingent on the future value of their labor income.

This uninsurability aspect of labor income has been the focus of attention of recent literature on portfolio choice. This literature analyzes how the nontradability nature of labor income affects the optimal allocation of financial wealth between risky and riskless assets. Much of this literature also tries to empirically characterize individual labor income processes and how they correlate with the returns on tradable assets. For example, Heaton and Lucas (2000) use the PSID data set and find that income risk is weakly positively correlated with stock returns. Cocco, Gomes and Maenhout (1998) also estimate the labor income process from the PSID data, with the emphasis on controlling the population characteristics. They find that the evolution of labor income over the life cycle is humped-shaped. In the presence of uninsurable labor income and borrowing constraints, this hump-shaped profile also creates a hump-shaped portfolio allocation to stocks over the life cycle. Viceira (2001) studies the effect of labor income and retirement on consumption and portfolio choice. He finds that when labor income risk is purely idiosyncratic, the allocation to risky asset is unambiguously higher for an employed investor than for a retired investor. Merton (1977), Weil (1994), Svensson and Werner (1993), and others have also addressed this problem.

However, this literature assumes away the labor supply is endogenous, mostly for tech-

²*Economic Report of the President*, February 1999.

nical reasons. An important exception is the work of Bodie, Merton and Samuelson (1992). They consider a setting in which an investor with isoelastic preferences over consumption and leisure must choose every period her labor supply as well as how to allocate her financial wealth between a risky and a riskless asset. However, to find a tractable analytical solution they must restrict their attention to either non-stochastic wages or stochastic wages that are perfectly correlated with the return on the risky asset. Either of these restrictive assumptions make markets complete, effectively allowing investors to perfectly hedge their labor income risk.

This paper generalizes the work of Bodie, Merton and Samuelson (1992) to an incomplete markets setting, as in Viceira (2001). It analyzes optimal portfolio choice in a setting where the investor is allowed to freely choose her labor supply and where wages are stochastic and not necessarily perfectly correlated with the return on risky assets. Hence markets are not effectively complete, and the investor cannot perfectly hedge his labor income risk. When labor supply is flexible, investors can react to negative shocks to their financial wealth by increasing their labor supply. Thus labor supply provides a buffer to financial downfalls. However, we do not know exactly how it affects both consumption and portfolio policies. This paper explores this question.

This paper also reexamines the theoretical result presented in Viceira (2001) on labor income risk hedging demands. Viceira shows that when shocks to stock return are positively correlated with those to labor income, hedging demand is unambiguously negative. We prove that, in the presence of labor supply flexibility, hedging demand consists of two counteracting components, but it remains true that hedging demand is negative. Interestingly, the hedging components can account for a *larger* proportion of total portfolio demand than when labor income is exogenous. Furthermore, despite these larger negative hedging demands, total demand for stocks can be much higher. This result has important implications for the life-cycle pattern of asset allocation.

An additional contribution of this paper is to provide some insights on the relation between labor income risk and the equity premium puzzle. Weil (1994) suggests that a large positive correlation between labor income shock and stock return can help to reconcile the equity premium puzzle. Our model examines this logic from a partial equilibrium perspective in the presence of labor supply flexibility.³ It shows that the equity premium puzzle can indeed become worse.

The lack of theoretical work on this problem is mainly due its analytical complexity, which makes very difficult to find closed-form analytical solutions. Recently, there is an increasing interest in the field of portfolio choice,⁴ mainly spurred by advances in computational methods, new approximation techniques and identifying conditions under which closed-form solutions are possible. These methods provide many new insights to the consumption-savings, portfolio choice problem. This paper finds analytical solutions based on the approximation techniques developed in Campbell (1993), Campbell and Viceira (1999,2001), Chacko and Viceira (1999) and Viceira (2001). These techniques work by considering log-linear approximations to the intertemporal budget constraint and the first-order conditions of the intertemporal optimization problem faced by the investor and finding analytical solutions that verify the approximate problem.

The paper is organized as follows. Section II describes the basic elements of the model and the solution method; Section III analyzes optimal consumption, labor supply, and portfolio policies; Section IV presents some calibration results, and section V concludes.

³Basak (1999) recently considers a general equilibrium model with flexible labor supply. However, his analysis relies on the assumption of complete markets, which in the case of human capital is again a strong assumption.

⁴Balduzzi and Lynch (1999), Barberis (1999), Brandt (1998), Brennan, Schwartz and Lagnado (1997), Campbell and Viceira (1998, 1999), Campbell, Chan and Viceira (1999), Chacko and Viceira (1999), Cocco, Gomes and Maenhout (1999), Heaton and Lucas (1996, 1997), Koo (1995a,b), Liu (1998), Wachter (1998,1999)

II The Model

The focus of the paper is on optimal consumption and portfolio choice in the presence of endogenous labor supply, given exogenous preferences and exogenous processes for asset returns and the wage rate. We study this problem using a non-trivial extension of the analytical framework in Viceira (2001) to allow for flexible labor supply. Viceira (2001) shows that it is possible to build an analytically tractable infinite-horizon model that captures finite-horizon effects by introducing a positive probability of retirement as in Blanchard (1985). We use this framework and introduce labor/leisure choice to endogenize the labor income process. In the following subsections, we provide a detailed description of the model.⁵

II.1 Assumptions

II.1.1 Preferences

We assume that the investor in the model is infinitely lived and derives utility from both a consumption good and leisure. In each period, she has some fixed time endowment, which we normalize to 1. When she is employed, she can choose to forgo part of her leisure time and devote it to work. When she retires, she simply consumes all her time endowment and earns no labor income. Specifically, her preferences take the following time- and state-separable form:

$$u(C_t, N_t) = \frac{1}{1-\gamma} [C_t^\theta (1-N_t)^{1-\theta}]^{1-\gamma} \quad (1)$$

⁵We could explicitly consider a life-cycle, finite-horizon model. This is the approach taken by Cocco, Gomes and Maenhout (1998), Campbell, Cocco, Gomes and Maenhout (1999), and others. However, under this alternative approach, analytical solutions are rarely available under realistic assumptions about wages and asset returns. Therefore, most papers in this approach proceed by using numerical techniques to solve the model. The disadvantage is that comparative statics have to be carried out on a case-by-case basis and robustness of the results is not guaranteed.

where C_t is consumption good and $N_t \leq 1$ is the amount of time devoted to work at time t . θ captures the relative importance of leisure ($1 - N_t$) to the investor, and γ is a curvature parameter. We assume that $0 < \theta \leq 1$ and $\gamma > 0$. This specification is a special case of the constant elasticity of substitution (CES) family of utility functions, in which the elasticity of substitution between consumption and leisure is set to one.⁶ Note that (1) nests the standard power utility model with no leisure as a special case ($\theta = 1$). It also allows consumption and leisure to enter separately when $\gamma = 1$.

Parameter γ represents the coefficient of relative risk aversion over the composite good $C_t^\theta(1 - N_t)^{1-\theta}$. It also represents the reciprocal of the elasticity of intertemporal substitution of the composite good. Note also that the coefficient of relative risk aversion over consumption C_t depends on both preference parameters θ and γ . It is given by

$$-\frac{C \cdot u_{CC}}{u_C} = 1 + \theta(\gamma - 1) \equiv \hat{\gamma}.$$

Parameter $\hat{\gamma}$ also represents the reciprocal of the elasticity of intertemporal substitution over consumption C_t .

II.1.2 Employment and Wage Process

Following Viceira (2001), we postulate that the investor faces an exogenous uncertainty over her employment status in each period. In the beginning of each period, the investor stays

⁶Our specification is the only member in the CES family consistent with the empirical observation that, for the US postwar data, there is no apparent trend in the hours worked per household, despite the upward growth in the average real wage. See, for example, Prescott (1986) and Kydland (1995).

Eichenbaum, Hansen and Singleton (1988) consider more elaborate forms of preference specifications which allow for habit formation in both consumption good and leisure. However, their focus is on the estimation of the preference parameters that are consistent with the data, while our focus is to analyze optimal consumption and labor supply policies and asset allocation for an individual investor. Thus we adopt a simpler form as more general preferences with habit formation may obscure our understanding of the basic mechanism.

employed with probability π^e , while she enters retirement (i.e., permanent unemployment) with probability $(1 - \pi^e)$. To keep things simple, we assume that π^e is constant over time and does not depend on the demographic characteristics of the investor. This way of modelling retirement effectively introduces an exogenous, finite expected retirement horizon in the model. The number of periods the investor expects to stay employed is $1/(1 - \pi^e)$. This finite horizon framework is very flexible as we can vary the employment probability π^e to investigate horizon effects on optimal policies. This exercise is undertaken in section IV.

In the employment state, the investor decides how much to work at the market-prevailing wage rate, in addition to making consumption and investment choices. The wage rate (denoted by H_t) is uncertain. It is subject to both permanent and temporary shocks. We model the log wage as a composite process of a random-walk with drift plus a transitory shock:

$$h_{t+1} = h_{t+1}^p + (\varepsilon_{t+1} - \frac{1}{2}\sigma_\varepsilon^2), \quad (2)$$

$$h_{t+1}^p = g + h_t^p + \xi_{t+1}, \quad (3)$$

where $h_{t+1} \equiv \log(H_{t+1})$, $h_{t+1}^p \equiv \log(H_{t+1}^p)$, and ε_{t+1} and ξ_{t+1} are serially uncorrelated, normally distributed disturbances with mean zero and constant variances σ_ε^2 and σ_ξ^2 respectively. These disturbances can be cross-sectionally correlated with $\text{Cov}_t(\varepsilon_{t+1}, \xi_{t+1}) = \sigma_{\varepsilon\xi}$. H_{t+1}^p is the permanent component of the wage rate, in the sense that shocks to this component of the wage rate have a permanent, multiplicative effect on the level of the wage rate. The second component of the wage rate $(\varepsilon_t - \sigma_\varepsilon^2/2)$ captures mean-one, multiplicative transitory shocks to the wage process.⁷ Expected growth in the log wage rate is given by $g \geq 0$. Apart from including a demographic component, the wage process (2)-(3) is consistent with the empirical evidence available from micro data on wages (Altonji 1986)⁸, and it is standard

⁷Note that

$$E\left(\frac{H_t}{H_t^p}\right) = \exp\left(E(\varepsilon_t) + \frac{1}{2}\text{Var}(\varepsilon_t) - \frac{1}{2}\sigma_\varepsilon^2\right) = 1.$$

⁸See equation (19) and (20) in Altonji (1986), where he argues that this specification is “an empirically

in the literature (see Carroll 1996, 1997, Cocco, Gomes and Maenhout 1998, Koo 1995a, b, Quah 1990 and Viceira 2001).

II.1.3 Securities

There are two financial assets available to the investor for trading: a risky asset (stocks) and a riskless instrument (bonds). Stocks offer a stochastic return whereas the return on bonds is constant. We denote these log returns as $r_{1,t+1}$ and r_f , respectively. Investment opportunities are constant.⁹ Accordingly, the excess log return on stocks is assumed to be composed of a constant risk premium μ plus idiosyncratic noise u :

$$r_{1,t+1} - r_f = \mu + u_{t+1}, \quad (4)$$

where u_{t+1} has mean zero and variance σ_u^2 and is serially uncorrelated and conditionally homoskedastic. For stocks to be a potential hedging device against wage fluctuations, we allow shocks to unexpected excess return to be correlated with those in the wage process. Denote these covariances as

$$\begin{aligned} \text{Cov}_t(u_{t+1}, \varepsilon_{t+1}) &= \sigma_{u\varepsilon}, \\ \text{Cov}_t(u_{t+1}, \xi_{t+1}) &= \sigma_{u\xi}. \end{aligned}$$

As in Viceira (2001), we further assume that u_{t+1} is independent of the retirement shock.

relevant model of the wage process" (pp.S198).

⁹Although there is abundant evidence that expected returns in many financial markets are time varying, we ignore this in order to isolate the effect of flexible labor supply on optimal consumption and portfolio choice. For evidence of time-variation in expected returns, see, for example, Campbell (1987, 1991), Campbell and Shiller (1988, 1991), Fama and French (1988, 1989), Bekaert and Hodrick (1992), and Hodrick (1992), and Bailey and Chan (1993).

II.2 Optimality Conditions

The investor faces the classical consumption-investment problem, augmented with a labor supply decision. At time t , she inherits wealth from last period and chooses optimal consumption, labor supply and portfolio allocation in order to maximize her life-time utility, given the prevailing prices. Formally, conditional on being in the employment state, she solves

$$\max_{\{C_t, N_t, \alpha_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t, N_t) \right]$$

subject to the period-by-period budget constraint:

$$W_{t+1} = (W_t + N_t H_t - C_t) R_{p,t+1}, \quad (5)$$

where β is the time discount factor, H_t is the real wage rate, W_t denotes the investor's wealth, and $R_{p,t+1}$ is the return on the portfolio. The expectation $\mathbb{E}_0(\cdot)$ is taken with respect to all the exogenous shocks in the model (u, ε, ξ) and the retirement shock. α_t represents the fraction of the investor's savings invested in stocks, and $(1 - \alpha_t)$ is the fraction of savings invested in bonds. Thus the gross return on the investor's portfolio is given by

$$R_{p,t+1} = \alpha_t (R_{1,t+1} - R_f) + R_f. \quad (6)$$

The intertemporal budget constraint differs from most of the literature such as Heaton and Lucas (1997) and Viceira (2001), in that nontraded labor income is endogenous here—the investor can optimally adjust her labor income by varying working time, whereas labor income is taken as exogenous in other studies.¹⁰

Using a standard perturbation argument, it is easy to show that optimal consumption

¹⁰We will use “labor supply” and “work effort” interchangeably. Barro and Sala-i-Martin (1995) argue that when N_t is broadly interpreted to include variations in work effort, there is no obvious upper bound, whereas Becker (1985) and Bils and Chang (1999) impose such a “energy bound”.

in the employment state must verify the following pair of Euler equations:

$$E_t \left\{ \beta \left[\pi^e \frac{u_C (C_{t+1}^e, N_{t+1})}{u_C (C_t^e, N_t)} + (1 - \pi^e) \frac{u_C (C_{t+1}^r, 0)}{u_C (C_t^e, N_t)} \right] R_{i,t+1} \right\} = 1, \quad (7)$$

where $i = 1, f$, and we use the superscripts “ e ” and “ r ” to denote the employment and retirement states, respectively.

In the retirement state, the Euler equation is standard. It is given by

$$E_t \left\{ \beta \left[\frac{u_C (C_{t+1}^r, 0)}{u_C (C_t^r, 0)} \right] R_{i,t+1} \right\} = 1. \quad (8)$$

Note that N_t enters both the Euler equation (7) and the budget constraint (5) in the employment state. N_t is an endogenous choice variable determined by the following first order condition for the optimal trade-off between leisure and consumption:

$$u_C (C_t, N_t) = \frac{1}{H_t} u_L (C_t, N_t). \quad (9)$$

Equation (9) says that at the optimum, marginal utility of consumption good per unit price is equal to the marginal utility of leisure deflated by the real wage. Since consumption is the numeraire, its price is 1 by definition.

II.3 An Approximation Framework

The model we have just described, while retaining a relatively simple structure, is rich enough to allow the study of optimal consumption and portfolio decisions in the presence of labor supply flexibility. However, despite its simplicity, there is no known exact analytical solution for the model. This is largely due to the nonlinearity inherited in the Euler equation and the budget constraint in the employment state. To solve for the optimal policies in the employment state we can use numerical methods along the lines of Koo (1996) or Heaton and Lucas (2000) or we can find an approximate analytical solution along the lines of Viceira

(2001). Viceira (2001) extends the solution approach in Campbell (1993, 1996) and Campbell and Viceira (1999, 2001) to the analysis of consumption and portfolio choice problems with nontradable risky labor income.¹¹

Deaton (1991) and Carroll (1996, 1997) show that, in the case of exogenous labor income, the wealth-income ratio becomes the state variable and optimal policies can be characterized as functions of this state variable. This insight is used in Viceira (2001) to find analytical solutions to a loglinearized version of the model around the unconditional mean of the log wealth-income ratio. In our model, labor income is endogenous, so that the log wealth-income ratio is no longer the state variable. Instead, Appendix A shows that the state variable becomes the log wealth-permanent wage ratio ($w_t - h_t^p$). We can loglinearize the model around the unconditional mean of this variable as well as the means of the log consumption-permanent wage ratio ($c_t - h_t^p$) and the log labor supply (n_t). To simplify exposition, we simply present here the loglinearized intertemporal budget constraint, the Euler equations and the optimality condition for consumption and leisure, and direct the reader to Appendix A for details.

The log-linear form of the intertemporal budget constraint (5) is the following:

$$w_{t+1}^e - h_{t+1}^p \approx k^e + \rho_w^e (w_t^e - h_t^p) - \rho_c^e (c_t^e - h_t^p) + \rho_n^e n_t + \rho_n^e \varepsilon_t - \Delta h_{t+1}^p + r_{p,t+1}^e, \quad (10)$$

where k^e , ρ_w^e , ρ_c^e , ρ_n^e are loglinearization parameters that depend on the unconditional means $E(w_t^e - h_t^p)$, $E(c_t^e - h_t^p)$, and $E(n_t)$, and $\Delta h_{t+1}^p = h_{t+1}^p - h_t^p$. Similarly, Campbell (1993) shows that the log-linear budget constraint in the retirement state takes the form

$$w_{t+1}^r - w_t^r = k^r + \left(1 - \frac{1}{\rho_c^r}\right) (c_t^r - w_t^r) + r_{p,t+1}^r \quad (11)$$

where ρ_c^r and k^r are loglinearization parameters that depend on the unconditional mean

¹¹The accuracy of the approximation has been examined by Campbell and Koo (1996), Campbell, Cocco, Gomes, Maenhout, and Viceira (1998) and Viceira (2001). These papers compare the approximate solution with the solution using numerical techniques, and find that it generally works well.

$E(c_t^r - w_t^r)$. Note that all the linearization parameters are endogenous in the sense that they depend on the unconditional means of endogenous variables.

The loglinear budget constraint depends on the log of the portfolio return. (6).¹² Following Campbell and Viceira (1999), we write the log return on wealth as follows:

$$r_{p,t+1} = \alpha_t (r_{1,t+1} - r_f) + r_f + \frac{1}{2} \alpha_t (1 - \alpha_t) \sigma_u^2. \quad (12)$$

This is a discrete-time version of the log return on wealth in continuous-time, where Ito's Lemma can be applied to equation (6).

We now turn to the Euler equations for consumption (7) and (8). The log linear version of the Euler equation (7) in the employment state is simply a weighted average of the marginal rates of substitution across the two possible states at time $t + 1$:

$$0 = \bar{V} + \pi^e [\log \beta - \kappa_1 g + E_t (r_{i,t+1}) - \gamma E_t (\Delta c_{t+1}^e) + \kappa_1 \varepsilon_t] \quad (13)$$

$$+ (1 - \pi^e) [\log \beta + \kappa_2 + E_t (r_{i,t+1}) - \hat{\gamma} E_t (c_{t+1}^r - c_t^e) - \kappa_1 (c_t^e - h_t^p) + \kappa_1 \varepsilon_t]$$

where \bar{V} is a constant variance term given in Appendix A, and $\kappa_1 \equiv (1 - \theta)(1 - \gamma)$, $\kappa_2 \equiv -\kappa_1 [\sigma_\varepsilon^2/2 - \log(\theta/(1 - \theta))]$.

The corresponding equation for the retirement state is:

$$0 = \log \beta - \hat{\gamma} E_t (\Delta c_{t+1}^r) + E_t (r_{i,t+1}) + \frac{1}{2} \text{Var}_t (\hat{\gamma} \Delta c_{t+1}^r - r_{i,t+1}). \quad (14)$$

Finally, we present the loglinearized version of the optimality condition for consumption and leisure (9). Using the expressions for the marginal utilities implied by (1) we obtain:

$$\log \left(\frac{\theta}{1 - \theta} \right) + \log (1 - \exp(n_t)) = c_t - h_t^p - \varepsilon_t + \frac{\sigma_\varepsilon^2}{2}. \quad (15)$$

¹²Note that we use different superscripts to denote the log portfolio return in each state since the optimal portfolio policy α does not need to be the same in each state.

Thus this first order condition determines explicitly a relationship between current consumption, labor supply and the wage rate. Note that this equation depends on θ , but not on γ . The reason for this is simple: In our non-separable preference specification, γ affects optimal decisions for the composite good formed by C_t and $1 - N_t$ together, but it does not affect the intra-division between C_t and $1 - N_t$. The choice between consumption and leisure is solely governed by the relative importance of each good, which is captured by θ .

Equation (15) almost has a loglinear form, but it still has the nonlinear term $\log(1 - \exp(n_t))$. To proceed further, we approximate this term around the unconditional mean of n_t . This leads to the following loglinear equation for the optimal labor supply

$$n_t = Q_{n,1} + Q_{n,2}(c_t - h_t^p) + Q_{n,3}\varepsilon_t, \quad (16)$$

where $Q_{n,1}$, $Q_{n,2}$ and $Q_{n,3}$ are all constants given in Appendix A. It is important to note that

$$Q_{n,2} < 0 \quad \text{and} \quad Q_{n,3} > 0.$$

The sign of $Q_{n,2}$ implies that the investor reduces her labor supply when her consumption-wage ratio $(c_t - h_t^p)$ increases. A higher level of consumption relative to permanent wage implies a lower marginal utility of consumption, which in turn increases the consumption of leisure and reduce work. Of course, this equation for n_t is not a solution yet, since $c_t - h_t^p$ itself is endogenous and must be determined.

III An Explicit Solution

We now use the approximate framework developed in the previous section to find explicit solutions to the optimal consumption, labor supply and portfolio policies. We solve the model in steps. First, we solve the consumption-investment problem faced by the investor in the retirement state. Second, we work backwards and use the method of undetermined

coefficients to determine the optimal policies in the employment state given the optimal policies in the retirement state.

The method of undetermined coefficients requires to first making a conjecture about the form of the policy functions. Specifically, we conjecture that the optimal allocation to stocks in the employment state is time-invariant,

$$\alpha_t^e = \alpha^e, \tag{17}$$

and that the optimal log consumption-permanent wage ratio is a linear in the log wealth-permanent wage ratio and the transitory shock to wages,¹³

$$c_t^e - h_t^p = b_0^e + b_1^e (w_t^e - h_t^p) + b_2^e \varepsilon_t, \tag{18}$$

where b_0^e, b_1^e and b_2^e are constants that depend on the underlying structural parameters of the model.

Finally, we proceed to solve the model using this conjecture and verify it by determining α^e and the coefficients explicitly. Note that once we have solved for $c_t^e - h_t^p$, the optimal labor supply policy is pinned down by the static first order condition (16).

Appendix A contains all the derivations in solving the model and discusses convergence issues. We summarize here the main results. We discuss first the optimal consumption and labor supply decisions, because they are necessary to fully understand optimal portfolio decisions.

III.1 Consumption and Labor Supply

The Euler equations of the problem (7) and (8) imply that solving for optimal consumption in the employment state requires first solving for consumption in the retirement state. The

¹³In the case of exogenous income, Deaton (1991) shows that the consumption-income ratio can be expressed as a function of cash-on-hand to income ratio and the transitory shock to income growth, although he eventually adopts a numerical procedure to solve the model.

investor's optimization problem in the retirement state is the standard problem with no labor income and constant investment opportunities. Merton (1969, 1971) provides an exact solution to the problem in continuous-time, and Viceira (2001) provides a discrete-time analogue. This solution implies that the consumption-wealth ratio in the retirement state is constant over time:

$$c_t^r - w_t^r = b_0^r, \quad (19)$$

where $b_0^r = [k^r - \log \beta / \hat{\gamma} + (1 - 1/\hat{\gamma}) E(r_{p,t+1}^r) - (1 - \hat{\gamma})^2 \text{Var}(r_{p,t+1}^r) / 2\hat{\gamma}] \rho_c^r$. This constant captures the familiar effects of discounting, intertemporal substitution and precautionary savings on consumption.

In the employment state, our conjecture implies that the log consumption-wage ratio is linear in the log wealth-wage ratio and the transitory wage shock. Appendix A verifies this conjecture and shows that coefficients of the optimal consumption rule verify

$$0 < b_1^e < \hat{\gamma} / \gamma \leq 1 \quad \text{and} \quad 0 < b_2^e < 1. \quad (20)$$

Equation (20) says that the elasticity coefficient on the wealth-wage ratio, b_1^e , is positive and bounded above by $\hat{\gamma} / \gamma$. Since $\hat{\gamma} = 1 + \theta(\gamma - 1)$ and $0 < \theta \leq 1$, we have that $\hat{\gamma} / \gamma \leq 1$ for all $\gamma > 1$. This upper bound is tighter than the bound in the case with exogenous labor income. Viceira (2001) shows that it is 1. The intuition for this sharper bound is that the inclusion of leisure in the preferences implies that consumption good contributes less to the utility of the investor. This puts a bound on her maximal optimal response of consumption to changes in financial wealth. Moreover, it is straightforward to show that this bound is increasing in θ for $\gamma > 1$; i.e., $\partial(\hat{\gamma} / \gamma) / \partial \theta > 0$. That is, as leisure receives more weight in the investor's preferences, the bound on consumption response to changes in wealth relative to labor income becomes tighter.

More importantly, the upper bound for b_1^e implies, as in Viceira (2001), that shocks to financial wealth w_t are not fully absorbed into consumption, whereas they are fully absorbed in the retirement state—equation (19) implies trivially that $b_1^r = 1$. There are two reasons

for this: First, the presence of labor income gives the investor a buffer to weather financial downturns. This effect is present whether labor supply is flexible or not. Second, in the presence of flexible labor supply, the investor has an additional protection against adverse financial shocks. In the event of simultaneously negative shocks to stock returns and wages—which tend to occur with positive correlation between the shocks—the investor can increase her work effort to compensate a reduction in her wealth. Thus, we would expect labor supply to be generally higher when this correlation is positive, and the elasticity coefficient b_1^e to be *smaller* in the current model than in the model with exogenous labor income. The calibration exercise in section IV below confirms these two predictions.

The upper bound on b_1^e also has an important implication for the investor’s response to changes in the permanent wage component. From (3), shocks to permanent wages appear in h_t^p ; thus the propensity to consume out of permanent wage shocks is given by $1 - b_1^e$, which satisfies $0 < (1 - \hat{\gamma}/\gamma) < 1 - b_1^e < 1$. In other words, the investor increases consumption by *at least* $(1 - \hat{\gamma}/\gamma)\%$ in response to a 1% increase in permanent wage. This result is intimately linked to the optimal labor/leisure choice that we discuss below. Note that the marginal propensity to consume out of transitory wage shocks is also positive and less than 1.

The intercept b_0^e is given in equation (40) of Appendix A. It is a complicated function of the underlying parameters of the model but many of its components resemble those in the constant term in the policy function $c_t^r - w_t^r$ for the retirement state. However, there are additional factors affecting the intercept of the log consumption function in the employment state. In particular, the expected growth in the log wage rate (g) affects this intercept. One can show that an increase in g induces an increase in b_0^e . This result is not obvious, because with flexible labor supply there are opposite effects from an increase in the expected growth of wages. First, there is a positive income effect on consumption from a higher income growth rate. Second, there is a substitution effect since it is optimal to supply more labor when wages are high and this tends to reduce current consumption via the static first order

condition (15). The upper bound on b_1^e allows us to show that the income effect dominates the substitution effect.

We can now turn to the optimal labor supply decision. Equation (16) gives an exact relationship between optimal labor decision and consumption choice. Thus, the results in Proposition 1 immediately imply that the optimal labor supply is given by

$$n_t = J_1 + J_2 (w_t^e - h_t^p) + J_3 \varepsilon_t, \quad (21)$$

where

$$J_2 = b_1^e Q_{n,2} < 0 \quad \text{and} \quad J_3 = (1 - b_1^e) Q_{n,3} > 0. \quad (22)$$

Thus optimal log labor supply policy is also a linear function of the state variable and the transitory shock to the wage rate. The coefficients J_2 and J_3 determine the responses of labor supply to changes in these variables. There are three important properties of this policy rule. First, the negative sign of the coefficient J_2 —which follows from $b_1^e > 0$ and $Q_{n,2} < 0$ —implies that work effort responds positively to permanent wage shocks. Therefore, the investor increases work effort by $J_2\%$ for a 1% increase in permanent wage. Once again, we have income and substitution effects on labor supply of an increase in the wage rate. A permanent positive shock the wage rate produces a negative income effect on work effort (i.e., a positive effect on leisure). On the other hand, it also has a positive substitution effect on work effort, since the investor can earn a higher income and afford higher consumption by working more. But we have shown that a 1% permanent shock to wages raises consumption by $(1 - b_1^e)\%$, which is strictly less than 1. Since optimal leisure is chosen in relation to consumption by equating *price-weighted* marginal utilities (see equation [9] or [15]), the increase in consumption is not sufficient to bring an increase in leisure. In other words, the substitution effect dominates the income effect, and work effort is raised to take advantage of the higher wage. Equation (22) also shows that work effort responds positively to positive transitory shocks to the wage rate.

The macroeconomics literature has argued that the hours worked per household in the

US data have been relatively stable in spite of the upward trend in real wages. It is worth pointing out that our result is consistent with the balanced growth phenomenon. As equation (21) indicates, J_2 measures the impact of an increase in the permanent wage, holding financial wealth constant. The impact is to stimulate labor supply, and as a result, the investor accumulates wealth until $w_t^e - h_t^p$ is back at its steady-state value—Appendix C shows that this log ratio is stationary. At this point, work effort is back at steady state, and there is zero long-run effect on labor supply of a permanent increase in the wage rate.

III.2 Portfolio Choice

We have analyzed the solutions for optimal consumption and work effort in the previous subsection. We can now examine the optimal portfolio rule. In the retirement state, the optimal portfolio allocation to the risky asset is given by

$$\alpha_t^r = \frac{\mu + \sigma_u^2/2}{\widehat{\gamma}\sigma_u^2}.$$

The optimal portfolio rule in the retirement state is well-known from the work of Samuelson (1969) and Merton (1969, 1971, 1973). It is the rule that obtains when investment opportunities are constant and either there is no labor income.¹⁴ In this case, the optimal allocation to stocks is solely dictated by speculative motives. This allocation is proportional to the Sharpe ratio of stocks $(\mu + \sigma_u^2/2)/\sigma_u$, and inversely related to the volatility of stock returns (σ_u) and to the relative risk aversion of the indirect utility function over wealth ($\widehat{\gamma}$). Viceira (2001) notes that the relative risk aversion of the value function is equal to the product of the relative risk aversion of the direct utility function over consumption times the wealth elasticity of consumption. From equation (19), it is immediate to see that the the wealth elasticity of consumption in the retirement state is equal to one, so that the relative risk aversion of both utility functions is the same.¹⁵

¹⁴Or, alternatively, there is labor income, but human capital is a tradable asset.

¹⁵Note that in the retirement state the investor still derives utility over leisure, so that the relevant

The optimal portfolio rule in the employment state is given by the following expression:¹⁶

$$\alpha^e = \frac{1}{\pi^e \gamma b_1^e + (1 - \pi^e) \widehat{\gamma}} \left[\frac{\mu + \sigma_u^2/2}{\sigma_u^2} - \pi^e (\gamma (1 - b_1^e) + \kappa_1) \frac{\sigma_{u\xi}}{\sigma_u^2} - \pi^e (\gamma b_2^e + \kappa_1) \frac{\sigma_{u\varepsilon}}{\sigma_u^2} \right] \quad (23)$$

where

$$\kappa_1 \equiv (1 - \theta) (1 - \gamma) \quad \text{and} \quad \gamma (1 - b_1^e) + \kappa_1 > 0. \quad (24)$$

Equation (23) extends the result in Viceira (2001) to the case of flexible labor supply.¹⁷ It shows that the optimal portfolio rule in the employment state has three components. The first component is similar to the to the optimal rule in the retirement state. It reflects the speculative demand for stocks. However, this rule also includes two extra components that are linearly related to the covariance of stock returns with permanent and transitory shocks to wages times the continuation probability of the employment state (π^e). These components reflect that, if any of these covariances are non-zero, risky stocks can serve an extra role as a hedging instrument against unexpected wage fluctuations in the employment state. The total allocation to stocks is inversely related to the average relative risk aversion of the indirect utility function across states, $\pi^e \gamma b_1^e + (1 - \pi^e) \widehat{\gamma}$ (Viceira, 2001).¹⁸

Equation (24) implies that the sign of the hedging demand against permanent shocks to wages depends exclusively on the sign of the correlation between permanent shocks to wages and stock returns. When this correlation is positive, the sign of this hedging demand is unambiguously negative. Viceira (2001) shows a similar result in a model with fixed coefficient of relative risk aversion over consumption goods is still $\widehat{\gamma}$. The only difference with the employment state is that the investor is forced to consume one unit of leisure each period.

¹⁶Appendix A derives this expression.

¹⁷Note that $\theta = 1$ implies $\gamma = \widehat{\gamma}$ and $\kappa_1 = 0$, and equation (23) reduces to the equation for α^e given in Viceira (2001) for the case with fixed labor supply.

¹⁸To see why α^e depends on the average relative risk aversion, and not just on the relative risk aversion in the employment state, note that the effects of wealth of a portfolio decision made at time t are not known until the following period, when the return on stocks is realized. Since there is uncertainty about the state of labor income next period, the investor must take that into account.

labor supply. However, in our model with flexible labor supply, this result is not intuitively obvious, because there are two opposing effects acting on the hedging portfolio demand for stocks.

First, when this correlation is positive, the investor insures her consumption stream against wage fluctuations by reducing her position on stocks. This is the effect shown in Viceira (2001). In our model, this effect is embedded in $-\pi^e \gamma (1 - b_1^e) < 0$, the first component of the coefficient that multiplies $\sigma_{u\xi}$ in equation (23). Second, in our model there is a second effect that acts in opposite direction. With labor supply flexibility, the investor can avail herself of an extra buffer in times of financial downfalls simply by working more to make up the losses. This enables the investor to take on more aggressive positions on risky stocks and reduces the need for hedging. This effect is captured indirectly by b_1^e , but also directly by $-\pi^e \kappa_1$, which is zero when $\theta = 1$ —that is, when the investor does not value leisure—and positive when $\theta < 1$ and $\gamma > 1$. Equation (24) implies that the insurance effect always dominates the substitution effect between labor and leisure, and the overall hedging demand against permanent wage shocks is negative. Intuitively, the insurance effect dominates because by supplying more labor at the prevailing wage rate, the investor also subjects herself to more of the risks associated with the positive correlation between shocks to wages and unexpected stock returns. Thus increasing labor supply also increases the desire for hedging.

We have not been able to determine analytically the sign of the hedging demand against transitory shocks, but in all the calibration exercises we have performed, the sign is always negative. Equation (23) shows that the hedging demand against transitory shocks exhibits similar components as the hedging demand against permanent shocks to wages.

Another interesting implication of the model is the magnitude of the optimal portfolio allocation to stocks when wages are uncorrelated with stock returns, so that labor income risk is idiosyncratic. Since $\gamma b_1^e < \hat{\gamma}$, we have that $\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma} < \hat{\gamma}$ and $\alpha^e > \alpha^r$.

Viceira (2001) shows a similar result for the case with fixed labor supply. However, as we have argued in section III.1, b_1^e is likely to be smaller when labor income is endogenous, so that stock demand is likely to be larger relative to the fixed labor supply case. Results from the calibration exercise reported in the next section confirm this conjecture. Thus the introduction of the labor/leisure choice exacerbates the difference between the optimal portfolio allocations to stocks of employed investors and retired investors. This result stresses the importance of labor supply flexibility in the life-cycle pattern of asset allocation.

IV Calibration

The discussion in the previous section has mainly focused on the qualitative implications of labor supply flexibility on asset allocation. The closed-form solutions for the optimal policies have allowed us to analyze the mechanisms through which the endogeneity of labor decision affects consumption and portfolio choice. The coefficients in the policy functions are complicated functions of the underlying structural parameters and it is not easy to see analytically how the optimal policies will change as we alter these parameters. In this section we try to quantify these effects for investors with different risk tolerance and expected employment horizons through a calibration exercise. The investor in the model takes the stochastic properties of asset returns and the wage process as given. Section IV.1 describes the structural parameters of the model chosen in this calibration exercise, section IV.2 discusses the calibration results, section IV.3 analyzes more specifically the elasticities of labor supply, and, finally, section IV.4 carries out some robustness checks of the results.

IV.1 Structural Parameters

Following the standard practice in the literature, we take the parameters for the stock return process and the safe asset from the historical US experience. The annual average

excess return on equities in the US has been 6%. with a volatility of 18% (Campbell, Lo and MacKinlay 1997, Table 8.1). This implies an average log excess return of 4.21%, per year. The riskless rate is set to its long-run average of 2% per year.

One of the main goals in this calibration exercise is to compare optimal portfolio choice under both exogenous and endogenous labor income. To this end, we specify the parameters in the wage process to resemble the labor income process estimated in the consumption literature. This has the advantage that if the investor supplies all her time endowment inelastically, the wage process becomes the labor income process. We set the standard deviation of shocks to the permanent and transitory components to wages to be 5% and 10%, respectively. A larger standard deviation is chosen for the transitory component to reflect the fact that individuals often face more idiosyncratic fluctuations in their wages. We choose the correlations between shocks to stock returns and wages in accord with the empirical findings reported in Heaton and Lucas (2000). We set them to 25% and 0% for the permanent and transitory components, respectively.¹⁹ We also consider the case where both correlations are zero. This case is an useful benchmark, since it implies that labor income risk is idiosyncratic. We complete the specification of the wage process by setting to 1% per year the expected growth in the wage rate. This low value ensures that investors have a strong incentive to save for the future. We discuss the effect of higher values of growth rates in section IV.4.

Finally, we need to specify the parameters that determine investor preferences. For θ , the parameter that determines how investors value consumption relative to leisure, we use $\theta = 0.5$ —so that investors value equally consumption and leisure. Section IV.4 explores the implications of considering a higher value of $\theta = 0.75$ —so that investors value consumption more than leisure. For the time discount rate, we consider $\beta = 1/(1 + 0.15)$ and explore

¹⁹The effects of transitory shocks to wages on portfolio choice are very small quantitatively. We discuss where appropriate the effect of including a non-zero correlation between transitory shocks to wages and stock returns.

in section IV.4 the effects of a higher discount rate, $\beta = 1/(1 + 0.10)$ —so that investors are more patient. We distinguish investors by their risk tolerance and expected horizon till retirement. In our model, the coefficient of relative risk aversion over consumption is given by $\hat{\gamma}$, not γ . Thus, we consider values for $\hat{\gamma}$ and compute the value of γ implied by the values of $\hat{\gamma}$ and θ via the definition of $\hat{\gamma}$. Specifically, we consider $\hat{\gamma} = \{2, 3, \dots, 10\}$.²⁰ Finally, we consider a range of expected retirement horizons between 30 and 5 years, and calculate the implied π^e from the fact that expected horizon is related to the probability via $1/(1 - \pi^e)$.

IV.2 Optimal Portfolio Choice and Wealth Accumulation With and Without Labor Supply Flexibility.

Results on optimal consumption, portfolio and labor supply policies are reported in a series of three-dimensional plots. For each plot, the X-axis represents the expected horizon till retirement and the Y-axis represents risk aversion over consumption ($\hat{\gamma}$).

IV.2.1 Portfolio Allocation When Labor Income Risk Is Idiosyncratic

Figures 1 and 2 summarize the results on portfolio allocation when labor income shocks are uncorrelated with stock returns—idiosyncratic labor income risk. Each figure contains two columns of plots. Plots on the left column report results for the case where investors value leisure and supply labor optimally. Plots on the right column report results for the case where investors do not value leisure and supply their labor endowment inelastically. This is the problem considered by Viceira (2001).

Panels A through D in Figure 1 report the portfolio allocations to risky stocks in both

²⁰Mehra and Prescott (1985) consider values of relative risk aversion less than 10 as being plausible, although other authors have argued for much larger risk aversion (Kandel and Stambaugh 1991, and Kocherlakota 1990).

the retirement (Panels A and B) and employment state (Panels C and D).²¹ Panel A and B are identical, since the optimal asset allocation in the retirement state is independent of labor income. In the retirement state, the investor is long in both stocks and cash for the range of risk aversion considered. As risk tolerance decreases, her position in stocks is reduced. Indeed, her position is proportional to $1/\hat{\gamma}$ since asset allocation consists of purely speculative demand in the retirement state.

Section III.2 has shown that, when labor income risk is idiosyncratic, the optimal portfolio allocation to stocks in the employment state is unambiguously larger than the optimal allocation in the retirement state. A comparison of Panels A and B with Panels C and D in Figure 1 shows that this difference can be very large, especially for investors with long retirement horizons. Panels C and D also show that α^e declines as we consider investors who are increasingly risk averse, and as we consider investors with shorter expected retirement horizons. The intuition behind this horizon effect is now well understood in the literature.²² Human capital represents a larger fraction of total wealth for investors with long expected retirement horizons than for investors with shorter horizons. When labor income risk is idiosyncratic, human capital is more similar to an implicit holding of the riskless asset than to an implicit holding of stocks. Thus investors with long retirement horizons have large implicit holdings of the safe asset, and optimally shift the composition of their financial wealth toward stocks.

Panels C and D also show the effect of labor supply flexibility on the optimal demand for stocks. They show that α^e is systematically larger when labor income is endogenous (Panel C) than when labor income is exogenous (Panel D). This difference is particularly pronounced for risk tolerant investors with long horizons. For example, an investor with

²¹There is no horizon effect in the retirement state. For ease of comparison, however, we plot the same α^r for all expected horizons in Figure 1A and 1B.

²²See, for example, Cocco (1999), Cocco, Gomes and Maenhout (1998), Heaton and Lucas (1997b), Jagannathan and Kocherlakota (1996), and Viceira (2001).

$\hat{\gamma} = 2$ and 30 years of expected retirement horizon who can adjust optimally her labor supply is willing to hold in stocks twice as much financial wealth as an identical investor who is not allowed to optimally set her labor supply. When the labor supply decision is endogenous, the flexibility to work *more* becomes an extra buffer to potential financial downfalls, in addition to the buffer given by the mere presence of labor income. Thus negative shocks to financial wealth have even *smaller* impact on consumption than with exogenous labor income. Figure 2 confirms this intuition. Panel A in this figure plots the elasticity of consumption with respect to financial wealth (b_1^e) when labor income is endogenous, and Panel B considers the exogenous case. For all combinations of risk tolerance and expected horizon, this elasticity is much smaller when labor supply is flexible.²³ Thus, optimal consumption is much less sensitive to movements in financial wealth and, in particular, to adverse shocks.

IV.2.2 Portfolio Allocation When Stock Returns and Wages are Correlated

Figure 3 summarizes the results on portfolio allocation when permanent shocks to wages are correlated with stock returns. We set this correlation to 25% per year—that is, $\text{Corr}(u_t, \xi_t) = 0.25$. The structure of the figure is similar to that of Figure 1, with plots on the left column reporting results for the case of flexible labor supply, and plots on the right column reporting results for the case of fixed labor supply. Panels A and B plot the optimal portfolio allocation to stocks in the employment state (α^e), and Panels C and D plot the hedging component. Panels D and E plot the ratio of the hedging demand to the total demand. Since the correlation coefficient is positive, hedging demand is negative. However, the magnitude of the correlation is not enough to offset the horizon effect at any horizon, and the optimal

²³Note that this result does not follow necessarily from the one in section III.1 that shows that b_1^e has an upper bound in the flexible labor supply case which is strictly lower than its upper bound in the fixed labor supply case. This calibration exercise, as well as other exercises not reported here to save space, shows numerically that b_1^e with $\theta < 1$ is uniformly lower than b_1^e with $\theta = 1$ for all combinations of $\gamma > 1$ and expected retirement horizons.

portfolio allocations in the employment state are larger than the optimal portfolio allocations in the retirement state—which we show in Figure 1.

Panel C shows that the magnitude of the hedging demand is small relative to the total demand for stocks. Nevertheless, a comparison of Panel C and D and Panel E and F shows that the size of hedging demand is larger in the case with flexible labor supply, both in absolute value and relative to the size of total portfolio demand. Panel E shows that the hedge ratio increase (in absolute terms) with risk aversion and expected horizon. As we consider more conservative investors, both the myopic demand for stocks and the hedging demand decline, but the myopic demand declines more rapidly. On the other hand, as the expected retirement horizon shortens, the impact of a permanent shock on wages is reduced since the investor has a shorter period to receive labor income. As a consequence, hedging demand against permanent shocks declines as the retirement horizon shortens. For the choice of structural parameters considered, hedging demand against permanent shocks ξ accounts for about 23% of the total stock demand for stocks of highly risk averse investors with long expected retirement horizons.

Results not reported here to save space show that allowing for non-zero correlation between transitory shocks to wages and stock returns produces very small hedging demands as a portion of total portfolio demand—less than 2% when $\text{Corr}(u_t, \varepsilon_t) = 0.10$, even for very risk averse investors with long retirement horizons.²⁴ This is not surprising since, by their very nature, transitory shocks only have short-run impact whereas permanent shocks have long-lasting effects on consumption.

To summarize, the main lesson from Figures 1–3 is that when leisure/work choice is endogenous, consumption responds more sensitively to permanent shocks to wages and therefore the desire to hedge against labor income risk is higher. Investors can use labor flexibility

²⁴One might suspect that the small hedging demand against ε is due to the small correlation specified $\text{Corr}(u_t, \varepsilon_t) = 0.10$. Results with $\text{Corr}(u_t, \varepsilon_t) = \text{Corr}(u_t, \xi_t) = 0.25$ show that this conjecture is not true and transitory shocks only induce small hedging demand.

to reduce the impact of permanent shocks to wages by working more. The extra cushion offered by flexible labor supply makes consumption much less sensitive to financial wealth downfalls. Consequently, this drives up asset demand significantly.

IV.2.3 Wealth, Consumption and Labor Supply

We will now turn to analyze how the consumption-wage ratio and labor supply respond to changes in risk tolerance and expected horizon. Equations (18) and (21) express the log optimal consumption-wage ratio and log labor supply as a function of the log wealth-wage ratio—the state variable—and the transitory shock to the wage rate. Thus understanding consumption and labor supply requires first to understand wealth accumulation in relation to wages. Figure 4 reports the exponentiated mean log wealth-permanent wage ratio, mean log consumption-permanent wage ratio and mean log labor supply for both the case with zero correlations (left panels) and with non-zero correlations (right panels).

Panel A and B in Figure 4 show that the mean wealth-wage ratio increases as expected retirement horizon and risk tolerance decrease. The intuition behind this pattern is simple: Investors with shorter expected retirement horizons have a shorter period of time to build up their wealth for retirement; therefore they optimally choose to accumulate more wealth while they are working. On the other hand, when risk tolerance decreases, the precautionary savings motive becomes stronger.²⁵ Results not reported here for space consideration show that the mean log wealth-permanent income ratios reported in Panel A are systematically much lower than the corresponding mean ratios in the case of fixed labor supply (Viceira 2001). There is a simple explanation for this result: When leisure is valued, consumption

²⁵We have also experimented with increasing the uncertainty of the permanent wage component. Specifically, the standard deviation of ξ was raised from 5% to 7% and 9% per year. It was found that investors optimally increase their mean wealth-wage ratio in these alternative scenarios. This is consistent with the empirical evidence presented in Carroll and Samwick (1997). Using the PSID dataset, they find that consumers who face greater income uncertainty hold more wealth for precautionary reasons.

only receives a weight of θ in the utility function, even in the retirement stage. Thus, less wealth is accumulated for consumption after retirement. Finally, a comparison of Panel A and B shows that wealth accumulation is slightly lower when there are correlations.

Panel C and D of Figure 4 reports the exponentiated mean log consumption-wage ratio for the case with no correlations. For long expected retirement horizons, this mean increases with $\hat{\gamma}$, but for short retirement horizons, it decreases. On the other hand, it increases as expected retirement horizon decreases for all values of $\hat{\gamma}$. To understand this, it is useful to recall that our specification of preferences implies that $\hat{\gamma}$ represents three different dimensions of investor preferences: The coefficient of relative risk aversion over consumption ($\hat{\gamma}$), the reciprocal of the elasticity of intertemporal substitution over consumption ($1/\hat{\gamma}$), and the coefficient of relative prudence in the sense of Kimball (1990) $(1 + \hat{\gamma})^{26}$. Thus, as we consider larger values of $\hat{\gamma}$, we are considering investors who are less willing to substitute consumption intertemporally. This tends to push up current consumption. On the other hand, we are also considering investors who are more prudent. This tends to push up precautionary savings and suppress current consumption. Moreover, we are also considering more conservative investors who shift their portfolios toward the riskless asset. This tends to reduce the expected return on wealth and the variability of consumption. A lower mean expected return reduces sustainable consumption, whereas a lower variability reduces the precautionary savings motive. Depending on which effect dominates, the optimal consumption-wage ratio can exhibit different patterns. Panel C shows that at long expected horizon, the substitution effect dominates for risk tolerant investors since the precautionary savings motive is weak for these “aggressive” investors. As $\hat{\gamma}$ increases, precautionary savings start to manifest itself and this effect offsets the intertemporal substitution effect.

Panel E and F of Figure 4 show that mean optimal labor supply is a mirror image of the optimal mean log consumption-wealth ratio. Thus, it is a decreasing function of $\hat{\gamma}$ for investors with long expected retirement horizons, and an increasing function of $\hat{\gamma}$ for investors

²⁶Kimball (1990) defines relative prudence as $-U_{CCC} \cdot C/U_{CC}$.

with short horizons. This results from the static first order condition (9)—or its log version (15)—that links the optimal choice labor supply to the optimal choice of consumption. In deciding how much to work, investors balance the benefits of receiving labor income which can finance current and future consumption, and the disutility of having less leisure time. For example, as consumption increases, the marginal utility of consumption declines. Since leisure is determined by equating price-weighted marginal utilities, consumption of leisure good also rises and this implies a decrease of labor supply. Intuitively, as investors build up more wealth (Panel A and B), they can afford to work less.

A comparison of Panel E and F reveals the mean labor supply is generally larger in the case of positive correlation between permanent wage shocks and stock returns than in the case of uncorrelated labor income, except for highly risk tolerant investors with long expected retirement horizons. This is consistent with our prediction in section III.1. When the correlation is positive, wages tend to fall when stocks underperform and investors have to work harder to make up for losses in their financial portfolios. As a result, mean labor supply is higher in general in the case with correlated labor income.

IV.3 Analysis of Labor Supply Elasticities

The critical element which distinguishes the current model from most of the literature on portfolio choice with risky labor income is the endogeneity of the leisure/labor decision. We have argued that its impact on portfolio allocation is significant, because the investor can adjust her labor supply in response to shocks to wages and, in particular, to financial wealth. Section ?? have provided a characterization of the optimal labor supply policy—the signs of the elasticity coefficients with respect to financial wealth, permanent and transitory shocks. This section explores the size of these elasticities as a function of risk aversion and expected retirement horizon using a calibration exercise.

Figure 5 reports the results of the exercise for the case of positive correlation between

permanent shocks to wages and stock returns. Panel A of the figure reproduces Panel F of Figure 4 (exponentiated mean log labor supply) for convenience, and Panel B and C give the coefficients in the optimal labor supply function (21). Panel C shows that the elasticity of labor supply to financial wealth shocks (J_2) is larger (in absolute terms) for investors with shorter expected retirement horizons. Intuitively, investors with short expected horizons have a more urgent need to build up financial wealth and react to adverse shocks to their financial wealth by working harder. Since $(1 - J_2)$ is the elasticity of labor supply to permanent wage shocks, Panel C implies that this elasticity is smaller for investors with shorter retirement horizons. Intuitively, a permanent increase in the wage rate is less valuable to investors with short retirement horizons, and they are less willing to give up leisure in response to the shock. Results not reported here to save space show that this pattern across retirement horizons is reversed for the elasticity of labor supply to transitory shocks to wages: When shocks to wages are transitory, investors respond by working more than they would if the shocks were permanent.

IV.4 Robustness

In this section, we examine how changes in the specification of the underlying structural parameters affect the calibration results we have obtained so far. Specifically, we consider the effects of an increase in θ —the relative importance of consumption relative to leisure in the investor’s utility function—, and an increase in the time preference parameter β . Other experiments, not reported here, are briefly discussed at the end.

Figure 6 provides the optimal portfolio rule, consumption and labor supply when θ is raised from 0.5 to 0.8, so that the weight on consumption is larger than the weight on leisure in the investor’s utility function. This figure assumes that permanent wage shocks are positively correlated with stock returns. As one would expect, the optimal policies move away from the case $\theta = 0.5$ toward the case with fixed labor supply (or $\theta = 1$). Thus the

optimal portfolio allocation to stocks is smaller than the optimal allocation when $\theta = 0.5$, while the mean log consumption-wage ratio and mean log labor supply are larger—compare Panel A, C and D in Figure 6 to Panel A, C and E in Figure 3.. When investors are less concerned about leisure, they choose to work harder and consume.

Figure 7 examines another dimension of robustness—changing the investor’s time preference rate from 15% to 10%.²⁷ More patient investors care more about the future and they tend to save more. This is reflected in a lower mean log consumption-permanent wage ratio (Panel C of Figure 7) and a larger wealth accumulation.²⁸ Thus, although α^e in Panel A of the figure is smaller than α^e in Panel A of Figure 3, the actual amount invested in risky stocks is not necessarily smaller.

We have experimented with increasing the expected wage growth rate g from 1% to 2%, ..., 10% for an investor with $\hat{\gamma} = 3$, $\theta = 0.5$ and $\beta = 1/(1 + 0.15)$. In this scenario, not surprisingly, the investor’s portfolio allocation to stocks is increasing in the expected wage growth rate, due to the fact that the propensity to consume out of financial wealth is decreasing in g . We have also conducted an exercise in which the standard deviation of the permanent shocks is increased from 5% to 6%, 7%, ..., 10%. As the uncertainty in the permanent wage rate increases, investors reduce their exposure to risky stocks and build up more wealth, and the effects are more pronounced for more risk averse investors.

V Conclusion

This paper explores the effects on portfolio choice and savings of risky labor income when investors are able to modify their labor supply in response to shocks to their wages and

²⁷That is, we change $\beta = 1/(1 + 0.15)$ to $\beta = 1/(1 + 0.10)$.

²⁸A plot of the mean log wealth-permanent wage ratio, not shown here for space considerations, shows that investors build up more wealth as they become more patient.

financial wealth. In a model with riskless wage income, Bodie, Merton and Samuelson (1992) show that ignoring investors flexibility to change her labor supply in response to those shocks can understate their willingness to invest in risky assets. We show that this result holds in an arguably more realistic model, where wage income is risky and most of this risk is idiosyncratic. We also show that, when the wage rate is positively correlated with stock returns, the optimal portfolio allocation to stocks is unambiguously smaller than in the case in which this correlation is zero. This is true despite the fact that flexible labor supply provides investors with an extra buffer to protect consumption against unexpected negative shocks to labor income and/or financial wealth.

There are several possible directions for future research. One important agenda is the development of general equilibrium asset pricing models with flexible labor supply when markets are incomplete. As we have argued, the demand for stocks can be significantly higher than when labor supply is completely inelastic. When prices are endogenously determined, this increased demand will drive down returns, which may in turn reduce demand in equilibrium. The current partial equilibrium framework is not able to analyze these interesting interactions. One possible framework is to introduce permanent and transitory technological shocks in the production functions but still allow only one risky asset available to investors. All prices including wages can then be determined within the model.

Second, the general class of preferences proposed by Epstein-Zin (1989,1991) has the important feature that the notion of risk aversion is separated from the notion of intertemporal substitution. It would be very interesting to generalize it to allow for multiple goods in the preference structure, one of which can be leisure. This will allow us to pin down precisely how these preference parameters influence optimal decisions in a multi-goods setting. It would seem natural to interpret the consumption good in their framework as a composite good. Of course, the Euler equation for optimality that they provide will be different and have to be derived for each individual good.

Finally, introducing time-variation in expected returns along the line of Campbell, Chan and Viceira (1999) is also an interesting exercise.

VI References

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VII Appendix

Appendix A gives the derivation of the equations in the text. Appendix B describes a simple numerical procedure to solve for the coefficients. Appendix C discusses convergence issues and steady-state properties of the solution and Appendix D proves the properties of the elasticity coefficients in the consumption policy rule discussed in Proposition 1.

VII.1 Appendix A

Derivation of Equation (10)

The budget constraint in the employment state is given by:

$$W_{t+1}^e = (W_t^e + N_t H_t - C_t) R_{p,t+1}^e.$$

Dividing through by H_{t+1}^p and taking logs,

$$w_{t+1}^e - h_{t+1}^p = \log \left[\exp(w_t^e - h_t^p) + \exp \left(n_t + \varepsilon_t - \frac{\sigma_\varepsilon^2}{2} \right) - \exp(c_t - h_t^p) \right] - \Delta h_{t+1}^p + r_{p,t+1}^e,$$

where lower-case letters denotes log variables. Taking a first-order Taylor expansion of the RHS around the unconditional means $E(w_t^e - h_t^p)$, $E(c_t - h_t^p)$ and $(E(n_t) + \sigma_\varepsilon^2/2)$ yields

$$w_{t+1}^e - h_{t+1}^p \approx k^e + \rho_w^e (w_t^e - h_t^p) - \rho_c^e (c_t - h_t^p) + \rho_n^e n_t + \rho_n^\varepsilon \varepsilon_t - \Delta h_{t+1}^p + r_{p,t+1}^e,$$

where $\bar{N} \approx \exp(E(n_t))$, $\rho_n^e = \bar{N} \tilde{\rho}_n^e$, $\rho_n^\varepsilon = \bar{N} \tilde{\rho}_n^\varepsilon$, and $k^e, \chi, \rho_w^e, \rho_c^e$, and $\tilde{\rho}_n^e$ are constants arising from the loglinearization procedure and they are defined as follows:

$$\begin{aligned} k^e &\equiv \log(\chi) - \rho_w^e E(w_t^e - h_t^p) + \rho_c^e E(c_t - h_t^p) - \bar{N} \tilde{\rho}_n^e (E(n_t) + \sigma_\varepsilon^2), \\ \chi &\equiv \exp[E(w_t^e - h_t^p)] + \bar{N} - \exp[E(c_t - h_t^p)], \\ \rho_w^e &\equiv \chi^{-1} \exp[E(w_t^e - h_t^p)], \\ \rho_c^e &\equiv \chi^{-1} \exp[E(c_t - h_t^p)], \\ \tilde{\rho}_n^e &\equiv \chi^{-1}, \end{aligned}$$

as in equation (10). Alternatively, k^e and χ can be expressed as

$$\begin{aligned} k^e &= -(1 - \rho_w^e + \rho_c^e) \log(1 - \rho_w^e + \rho_c^e) - (1 - \rho_w^e + \rho_c^e) \sigma_\varepsilon^2 - \rho_w^e \log(\rho_w^e) + \rho_c^e \log(\rho_c^e), \\ \chi &= \bar{N} (1 - \rho_w^e + \rho_c^e)^{-1}. \end{aligned}$$

■

Derivation of Equation (12)

The derivation of the log return on the portfolio is a discrete time approximation of the continuous time counterpart.²⁹ We start with the following specification of the return processes for the risky stock (P_t) and the (instantaneous) riskless bond (B_t):

$$\begin{aligned}\frac{dP_t}{P_t} &= \mu dt + \sigma dW_t, \\ \frac{dB_t}{B_t} &= r_f dt,\end{aligned}$$

where μ is the drift and σ the diffusion for the stock price, r_f is the instantaneous return on the riskless bond, and W_t is a 1-dimensional standard Wiener process.

Since we are working with log returns, we apply Ito's Lemma to each asset:

$$\begin{aligned}d \log B_t &= \left(\frac{dB_t}{B_t} \right), \\ d \log P_t &= \left(\frac{dP_t}{P_t} \right) - \frac{1}{2} \sigma^2 dt.\end{aligned}$$

Let V_t denote the value of the portfolio. We will use $d \log V_t$ to approximate $r_{p,t+1}$. To this end, we need to find $d \log V_t$. By Ito's Lemma,

$$d \log V_t = \left(\frac{dV_t}{V_t} \right) - \frac{1}{2} \left(\frac{dV_t}{V_t} \right)^2.$$

We will now derive these 2 terms:

$$\begin{aligned}\frac{dV_t}{V_t} &= \alpha_t \left(\frac{dP_t}{P_t} \right) + (1 - \alpha_t) \left(\frac{dB_t}{B_t} \right) \\ &= \alpha_t \left(d \log P_t + \frac{1}{2} \sigma^2 dt \right) + (1 - \alpha_t) (d \log B_t) \\ &= \alpha_t (d \log P_t - d \log B_t) + d \log B_t + \frac{1}{2} \alpha_t \sigma^2 dt.\end{aligned}$$

With this expression, it is straight forward to get the second term:

$$\begin{aligned}\left(\frac{dV_t}{V_t} \right)^2 &= \alpha_t^2 (d \log P_t - d \log B_t)^2 + (d \log B_t)^2 + 2\alpha_t (d \log P_t - d \log B_t) (d \log B_t) \\ &\quad + \text{zero terms}, \\ &= \alpha_t^2 \sigma^2 + \text{zero terms}\end{aligned}$$

²⁹The derivation for the general case with multiple state variables and multiple risky assets (including the short term instrument) can be found in Campbell, Chan, and Viceira (1999).

where “zero terms” refer to terms which will vanish because they involve either $(dt)^2$ or $(dt)(dW_t)$.

Collecting these results and using our specification for expected return in (4), we have

$$\begin{aligned} r_{p,t+1} &= d \log V_t \\ &= \alpha_t (r_{1,t+1} - r_f) + r_f + \frac{1}{2} \sigma_u^2 \alpha_t (1 - \alpha_t) \end{aligned}$$

as in equation (12). ■

Derivation of Equation (16)

The static first order condition for leisure is given by equation (15):

$$\log \left(\frac{\theta}{1 - \theta} \right) + \log (1 - \exp(n_t)) = c_t - h_t^p - \varepsilon_t + \frac{\sigma_\varepsilon^2}{2}.$$

To obtain a simple linear relationship between n_t and $(c_t - h_t^p)$, we approximate the expression $\log(1 - \exp(n_t))$ around $E(n_t)$:

$$\log(1 - \exp(n_t)) \approx (\log(1 - \exp(E(n_t))) - Q_n E(n_t)) + Q_n n_t, \quad (25)$$

where

$$Q_n \equiv \frac{-\exp(E(n_t))}{1 - \exp(E(n_t))} \approx \frac{-\bar{N}}{1 - \bar{N}}.$$

Note that since N_t is bounded in the interval $[0, 1]$, and hence $Q_n < 0$.

Substituting this expression into the first order condition and rearranging, we immediately establish the claim:

$$n_t = Q_{n,1} + Q_{n,2} (c_t - h_t^p) + Q_{n,3} \varepsilon_t,$$

where

$$\begin{aligned} Q_{n,1} &\equiv \frac{1}{Q_n} \left[\frac{\sigma_\varepsilon^2}{2} - \log \left(\frac{\theta}{1 - \theta} \right) - \log(1 - \bar{N}) + Q_n \log(\bar{N}) \right], \\ Q_{n,2} &\equiv \frac{1}{Q_n} < 0, \\ Q_{n,3} &\equiv -\frac{1}{Q_n} > 0. \end{aligned}$$

■

Derivation of the Loglinearized Euler Equation (13) in the Employment State

From the Euler equation in (7),

$$E_t \left\{ \beta \left[\pi^e \frac{u_C(C_{t+1}^e, N_{t+1})}{u_C(C_t^e, N_t)} + (1 - \pi^e) \frac{u_C(C_{t+1}^r, 0)}{u_C(C_t^e, N_t)} \right] R_{i,t+1} \right\} = 1.$$

We will now derive the 2 components separately. Substituting in the expressions for the marginal utilities, the first component can be expressed as:

$$\begin{aligned} & E_t \left[\pi^e \beta \frac{u_C(C_{t+1}^e, N_{t+1})}{u_C(C_t^e, N_t)} R_{i,t+1} \right] \\ &= \pi^e E_t \left[\exp \left(\log \beta + r_{i,t+1} + \kappa_1 \log \left(\frac{1 - N_{t+1}}{1 - N_t} \right) - \hat{\gamma} \Delta c_{t+1}^e \right) \right], \end{aligned} \quad (26)$$

where $\kappa_1 = (1 - \theta)(1 - \gamma)$ and $\hat{\gamma} = 1 + \theta(1 - \gamma)$. Note that

$$\begin{aligned} \log \left(\frac{1 - N_{t+1}}{1 - N_t} \right) &= \log(1 - e^{n_{t+1}}) - \log(1 - e^{n_t}) \\ &\approx Q_n(n_{t+1} - n_t), \end{aligned}$$

where the second line follows from equation (25) above. Using this expression, the RHS of (26) becomes

$$\pi^e E_t \left[\exp(\log \beta + r_{i,t+1} + \kappa_1 Q_n(n_{t+1} - n_t) - \hat{\gamma} \Delta c_{t+1}^e) \right].$$

The next step is to use a second-order Taylor expansion to approximate the expression inside the expectation operator, around the conditional means $E_t(r_{i,t+1})$, $E_t(\Delta n_{t+1})$ and $E_t(\Delta c_{t+1}^e)$. After some tedious algebra, the loglinearized version of the first component is given by

$$\begin{aligned} & E_t \left[\pi^e \beta \frac{u_C(C_{t+1}^e, N_{t+1})}{u_C(C_t^e, N_t)} R_{i,t+1} \right] \\ &= \pi^e \left[1 + \log \beta + E_t(r_{i,t+1}) + \kappa_1 Q_n E_t(\Delta n_{t+1}) - \hat{\gamma} E_t(\Delta c_{t+1}^e) \right] \\ &\quad + \frac{\pi^e}{2} \text{Var}_t(r_{i,t+1} + \kappa_1 Q_n \Delta n_{t+1} - \hat{\gamma} \Delta c_{t+1}^e). \end{aligned} \quad (27)$$

Following the same derivation, the second component is given by

$$\begin{aligned} & E_t \left[(1 - \pi^e) \beta \frac{u_C(C_{t+1}^r, 0)}{u_C(C_t^e, N_t)} R_{i,t+1} \right] \\ &= (1 - \pi^e) E_t \left[\exp(\log \beta + r_{i,t+1} - \kappa_1 \log(1 - e^{n_t}) - \hat{\gamma}(c_{t+1}^r - c_t^e)) \right] \\ &= (1 - \pi^e) \left[1 + \log \beta - \kappa_1 (\log(1 - \exp(E(n_t))) - Q_n E(n_t)) \right] \\ &\quad + (1 - \pi^e) \left[E_t(r_{i,t+1}) - \kappa_1 Q_n n_t - \hat{\gamma}(c_{t+1}^r - c_t^e) + \frac{1}{2} \text{Var}_t(r_{i,t+1} - \hat{\gamma}(c_{t+1}^r - c_t^e)) \right], \end{aligned} \quad (28)$$

where the second equality follows from equation (25).

The final step in the derivation is to recognize that both (27) and (28) contain the choice variable n_t . Thus, we use the loglinearized static first order condition (16) to substitute out n_t . Taking first difference of (16) and its conditional expectation, we obtain

$$\begin{aligned} \Delta n_{t+1} &= Q_{n,2} (\Delta c_{t+1}^e - \Delta h_{t+1}^p) + Q_{n,3} \Delta \varepsilon_{t+1} \\ E_t(\Delta n_{t+1}) &= Q_{n,2} (E_t(\Delta c_{t+1}^e) - g) - Q_{n,3} \varepsilon_t. \end{aligned}$$

Substituting this into (27) and regrouping terms,

$$\begin{aligned} & E_t \left[\pi^e \beta \frac{u_C(C_{t+1}^e, N_{t+1})}{u_C(C_t^e, N_t)} R_{i,t+1} \right] \\ &= \pi \left[1 + \log \beta - \kappa_1 g + E_t(r_{i,t+1}) - \gamma E_t(\Delta c_{t+1}^e) + \kappa_1 \varepsilon_t \right] \\ & \quad + \frac{\pi^e}{2} \text{Var}_t(r_{i,t+1} - \gamma \Delta c_{t+1}^e - \kappa_1 \xi_{t+1} - \kappa_1 \varepsilon_{t+1}), \end{aligned}$$

where we have used the properties that $Q_n Q_{n,2} = 1$, $Q_n Q_{n,3} = -1$, and $\kappa_1 - \hat{\gamma} = -\gamma$.

Similarly, the second component is given by

$$\begin{aligned} & E_t \left[(1 - \pi^e) \beta \frac{u_C(C_{t+1}^r, 0)}{u_C(C_t^e, N_t)} R_{i,t+1} \right] \\ &= (1 - \pi^e) \left[1 + \log \beta + \kappa_2 + E_t(r_{i,t+1}) - \hat{\gamma} E_t(c_{t+1}^r - c_t^e) - \kappa_1 (c_t^e - h_t^p) + \kappa_1 \varepsilon_t \right] \\ & \quad + \frac{(1 - \pi^e)}{2} \text{Var}_t(r_{i,t+1} - \hat{\gamma} (c_{t+1}^r - c_t^e)), \end{aligned}$$

where

$$\kappa_2 \equiv -\kappa_1 \left[\frac{1}{2} \sigma_\varepsilon^2 - \log \left(\frac{\theta}{1 - \theta} \right) \right].$$

Combining these two components, we obtain the expression given in the text. ■

Derivation of the Optimal Portfolio Rule (23) in the Employment State

The optimal portfolio policy can be derived from the loglinearized Euler equation (13). The idea is that the optimal consumption choice is determined with relation to asset's return in the Euler equation. Thus, the strategy is to express excess return on the stock, adjusted for Jensen's Inequality effect, as function of its covariances with consumption growth and shocks. From this relation, we can then derive the optimal portfolio allocation.

First, taking the difference of the loglinearized Euler equation (13) with $i = 1$ and with $i = f$, we obtain

$$\begin{aligned} 0 &= E_t(r_{1,t+1} - r_f) + \frac{1}{2} \text{Var}_t(r_{1,t+1}) + \pi^e \text{Cov}_t(r_{1,t+1}, -\gamma \Delta c_{t+1}^e - \kappa_1 \xi_{t+1} - \kappa_1 \varepsilon_{t+1}) \\ & \quad + (1 - \pi^e) \text{Cov}_t(r_{1,t+1}, -\hat{\gamma} (c_{t+1}^r - c_t^e)). \end{aligned}$$

Given our distributional assumptions on the correlation of the shocks, we have

$$\begin{aligned} & E_t(r_{1,t+1} - r_f) + \frac{1}{2} \text{Var}_t(r_{1,t+1}) \\ &= \pi^e \text{Cov}_t(r_{1,t+1}, \gamma \Delta c_{t+1}^e) + (1 - \pi^e) \text{Cov}_t(r_{1,t+1}, \hat{\gamma} (c_{t+1}^r - c_t^e)) + \pi^e \kappa_1 \sigma_{u\xi} + \pi^e \kappa_1 \sigma_{u\varepsilon}. \end{aligned}$$

Now, it remains to derive the expressions for the conditional covariances: $\text{Cov}_t(r_{1,t+1}, \gamma \Delta c_{t+1}^e)$ and $\text{Cov}_t(r_{1,t+1}, \hat{\gamma} (c_{t+1}^r - c_t^e))$.

To derive the first covariance term, we substitute our conjecture of the optimal consumption policy $c_t^e - h_t^p = b_0^e + b_1^e (w_t^e - h_t^e) + b_2^e \varepsilon_{t+1}$ into the trivial identity

$$\Delta c_{t+1}^e = (c_{t+1}^e - h_{t+1}^p) - (c_t^e - h_t^p) + (h_{t+1}^p - h_t^p), \quad (29)$$

and use the loglinearized budget constraint (10) for $w_{t+1}^e - h_{t+1}^e$ and (16) to get

$$\begin{aligned} \Delta c_{t+1}^e &= b_1^e (-\Delta h_{t+1}^p + r_{p,t+1}) + b_2^e \varepsilon_{t+1} + \xi_{t+1} \\ &= b_1^e r_{p,t+1} + b_2^e \varepsilon_{t+1} + (1 - b_1^e) \xi_{t+1} \end{aligned}$$

where we have used the property that $\Delta h_{t+1}^p = g + \xi_{t+1}$, and have ignored time t terms and constants due to the fact that they vanish under the conditional covariance operator. Finally, given our portfolio return (12),

$$\begin{aligned} \text{Cov}_t(r_{1,t+1}, \gamma \Delta c_{t+1}^e) &= \text{Cov}_t(r_{1,t+1}, b_1^e r_{p,t+1} + b_2^e \varepsilon_{t+1} + (1 - b_1^e) \xi_{t+1}) \\ &= b_1^e \alpha_t^e \sigma_u^2 + b_2^e \sigma_{u\varepsilon} + (1 - b_1^e) \sigma_{u\xi}. \end{aligned}$$

Following the same logic, it can be shown that $\text{Cov}_t(r_{1,t+1}, c_{t+1}^r - c_t^e) = \alpha_t^e \sigma_u^2$. Collecting these results,

$$\begin{aligned} &E_t(r_{1,t+1} - r_f) + \frac{1}{2} \text{Var}_t(r_{1,t+1}) \\ &= \pi^e \gamma (b_1^e \alpha_t^e \sigma_u^2 + b_2^e \sigma_{u\varepsilon} + (1 - b_1^e) \sigma_{u\xi}) + (1 - \pi^e) \widehat{\gamma} \alpha_t^e \sigma_u^2 + \pi^e \kappa_1 (\sigma_{u\xi} + \sigma_{u\varepsilon}). \end{aligned}$$

Solving for α_t^e , we obtain the optimal portfolio rule

$$\alpha_t^e = \frac{1}{\pi^e \gamma b_1^e + (1 - \pi^e) \widehat{\gamma}} \left[\frac{\mu + \sigma_u^2/2}{\sigma_u^2} - \pi^e (\kappa_1 + \gamma b_2^e) \frac{\sigma_{u\varepsilon}}{\sigma_u^2} - \pi^e (\kappa_1 + \gamma (1 - b_1^e)) \frac{\sigma_{u\xi}}{\sigma_u^2} \right],$$

as claimed in the proposition. ■

Derivation of the Optimal Consumption Policy (??) in the Employment State

The derivation of the optimal consumption policy uses the technique of undetermined coefficients. It takes the conjectured solution as given and solves an “expectational” difference equation in the consumption-permanent wage ratio. It then proceeds to verify that the conjectured solution is indeed the solution by matching up the coefficients, in the sense to be made precise below.

The optimal consumption choice has to satisfy the Euler equation in (13). Thus, using the trivial identity (29) and similarly with $c_{t+1}^r - c_t^e$, we obtain:

$$\begin{aligned} E_t(\Delta c_{t+1}^e) &= E_t(c_{t+1}^e - h_{t+1}^p) - (c_t^e - h_t^p) + g, \\ E_t(c_{t+1}^r - c_t^e) &= E_t(c_{t+1}^r - h_{t+1}^p) - (c_t^e - h_t^p) + g \end{aligned}$$

Substitute these expressions into (13) with $i = p$, we have

$$\begin{aligned}
& \gamma \pi^e E_t (c_{t+1}^e - h_{t+1}^p) + \hat{\gamma} (1 - \pi^e) E_t (c_{t+1}^r - h_{t+1}^e) \\
= & \bar{V} + \pi^e (\gamma (c_t^e - h_t^p) - \gamma g + \log \beta + E_t (r_{p,t+1}) - \kappa_1 g + \kappa_1 \varepsilon_t) \\
& + (1 - \pi^e) ((\hat{\gamma} - \kappa_1) (c_t^e - h_t^p) - \hat{\gamma} g + \log \beta + \kappa_2 + E_t (r_{p,t+1}) + \kappa_1 \varepsilon_t),
\end{aligned} \tag{30}$$

where \bar{V} is defined as in (??) with $i = p$ and Δc_{t+1}^e and $(c_{t+1}^r - c_t^e)$ substituted out:

$$\begin{aligned}
\bar{V} = & \frac{\pi^e}{2} \text{Var}_t (r_{p,t+1} - \gamma (c_{t+1}^e - h_{t+1}^p) - (\gamma + \kappa_1) \xi_{t+1} - \kappa_1 \varepsilon_{t+1}) \\
& + \frac{(1 - \pi^e)}{2} \text{Var}_t (r_{p,t+1} - \hat{\gamma} (c_{t+1}^r - h_{t+1}^p) - \hat{\gamma} \xi_{t+1})
\end{aligned}$$

(30) is the expectational difference equation discussed above. The next step is to evaluate the LHS and RHS separately as functions of the state variables $(w_t - h_t^p)$ and ε_t . Once this is achieved, we match up the coefficients.

To this end, we first compute the LHS. Substituting the conjecture (??), the budget constraint (10) and (16),

$$\begin{aligned}
& \gamma \pi^e E_t (c_{t+1}^e - h_{t+1}^p) + \hat{\gamma} (1 - \pi^e) E_t (c_{t+1}^r - h_{t+1}^e) \\
= & \gamma \pi^e (b_0^e + b_1^e E_t (w_{t+1}^e - h_{t+1}^p)) + \hat{\gamma} (1 - \pi^e) (b_0^r + E_t (w_{t+1}^e - h_{t+1}^p)) \\
= & [(\pi^e \gamma b_0^e + (1 - \pi^e) \hat{\gamma} b_0^r) + (\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) \\
& \times (k^e + \bar{N} \rho_n^e Q_{n,1} - g + E_t (r_{p,t+1}) - (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_0^e)] \\
& + (\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) [\rho_w^e - (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_1^e] (w_t^e - h_t^p) \\
& + (\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) [- (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_2^e + (\bar{N} \rho_n^e + \bar{N} \rho_n^e Q_{n,3})] \varepsilon_t.
\end{aligned} \tag{31}$$

Thus, the LHS can be expressed as a linear combination of the wealth-wage ratio and the transitory shock.

Next, we will evaluate the RHS of the difference equation. Plugging the conjecture (??) and regrouping,

$$\begin{aligned}
& \text{RHS} \\
= & [- (\pi^e \gamma + (1 - \pi^e) \hat{\gamma}) g + \gamma b_0^e + \log \beta + E_t (r_{p,t+1}) - \pi^e \kappa_1 g + (1 - \pi^e) \kappa_2 + \bar{V}] \\
& + (\gamma b_1^e) (w_t^e - h_t^p) + (\kappa_1 + \gamma b_2^e) \varepsilon_t,
\end{aligned} \tag{32}$$

which again is a function of the state variables. It remains to compute $E_t(r_{p,t+1})$ and \bar{V} in the expressions above. Note that with our assumption of constant investment opportunity set,

$$E_t (r_{p,t+1}) = \alpha^e \mu + r_f + \frac{1}{2} \sigma_u^2 \alpha^e (1 - \alpha^e).$$

Moreover, it is straightforward to show that

$$\begin{aligned}\bar{V} &= \frac{\pi^2}{2} \left[(1 - \gamma b_1^e)^2 (\alpha_t^e)^2 \sigma_u^2 + (\gamma b_1^e - \gamma - \kappa_1)^2 \sigma_\xi^2 + (\gamma b_2^e + \kappa_1)^2 \sigma_\varepsilon^2 \right. \\ &\quad + 2(1 - \gamma b_1^e) (\gamma b_1^e - \gamma - \kappa_1) \alpha_t^e \sigma_{u\xi} - 2(1 - \gamma b_1^e) (\gamma b_2^e + \kappa_1) \alpha_t^e \sigma_{u\varepsilon} \\ &\quad \left. - 2(\gamma b_1^e - \gamma - \kappa_1) (\gamma b_2^e + \kappa_1) \sigma_{\xi u} \right] \\ &\quad - \frac{1 - \pi^e}{2} \left[(1 - \hat{\gamma})^2 (\alpha_t^e)^2 \sigma_u^2 \right],\end{aligned}$$

which is a constant, as claimed in the text.

Finally, we equate the coefficients in (31) and (32) to get a system of equations. For the coefficients on $(w_t^e - h_t^e)$, we have

$$(\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma} b_1^e) [\rho_w^e - (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_1^e] = \gamma b_1^e, \quad (33)$$

and for the coefficient on the transitory disturbance, we have

$$(\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) [- (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_2^e + (\rho_n^e + \bar{N} \rho_n^e Q_{n,3})] = \kappa_1 + \gamma b_2^e, \quad (34)$$

and finally the intercepts are equated

$$\begin{aligned}& (\pi^e \gamma b_0^e + (1 - \pi^e) \hat{\gamma} b_0^e + \pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) \\ & \times (k^e + \bar{N} \rho_n^e Q_{n,1} - g + E_t(r_{p,t+1}) - (\rho_c^e - \bar{N} \rho_n^e Q_{n,2}) b_0^e) \\ = & - (\pi^e \gamma + (1 - \pi^e) \hat{\gamma}) g + \gamma b_0^e + \log \beta + E_t(r_{p,t+1}) - \pi^e \kappa_1 g + (1 - \pi^e) \kappa_2 + \bar{V}.\end{aligned} \quad (35)$$

We note that this system of equations are functions of the coefficients in the conjecture solution and the underlying parameters of the model. In order for the conjecture solution to be consistent, it must be the case that they simultaneously satisfy these set of equations. Operationally, the first equation (33) only involves b_1^e . Thus, it uniquely pins down this parameter. Once we have b_1^e , the second equation (34) allows us to obtain b_2^e as a function of b_1^e and the underlying structural parameters. Finally, with b_0^e and b_1^e , the last equation (35) gives b_2^e . This iterative procedure was also found in Viceira (2001). It is perhaps interesting to find that introducing another choice variable into the model and thus complicating the problem considerably does not change this property.

Solving (33) gives a quadratic equation in b_1^e :

$$\mathcal{A} (b_1^e)^2 + \mathcal{B} b_1^e + \mathcal{C} = 0, \quad (36)$$

where

$$\begin{aligned}
\mathcal{A} &\equiv -(\rho_c^e - \bar{N}\rho_n^e Q_{n,2}) \pi^e \gamma, \\
\mathcal{B} &\equiv \rho_w^e \pi^e \gamma g - (1 - \pi^e) \hat{\gamma} (\rho_c^e - \bar{N}\rho_n^e Q_{n,2}) - \gamma, \\
\mathcal{C} &\equiv \rho_w^e (1 - \pi^e) \hat{\gamma}.
\end{aligned} \tag{37}$$

This equation has two solutions of opposite signs. We take the positive root since $b_1^e < 0$ implies consumption is always decreasing in wealth for *all* levels of wealth.

Solving (34) gives

$$b_2^e = \frac{(\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) (\bar{N}\rho_n^e + \bar{N}\rho_n^e Q_{n,3}) - \kappa_1}{\gamma + (\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) (\rho_c^e - \bar{N}\rho_n^e Q_{n,2})}. \tag{38}$$

Define

$$\begin{aligned}
G_1 &\equiv (\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) (k^e + \bar{N}\rho_n^e Q_{n,1} - g + E_t(r_{p,t+1})) + (1 - \pi^e) \hat{\gamma} b_0^e, \\
G_2 &\equiv -(\pi^e \gamma + (1 - \pi^e) \hat{\gamma}) g + \log \beta + E_t(r_{p,t+1}) - \pi^e \kappa_1 g + (1 - \pi^e) \kappa_2 + \bar{V}.
\end{aligned} \tag{39}$$

Then,

$$b_0^e = \frac{G_1 - G_2}{(\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}) (\rho_c^e - \bar{N}\rho_n^e Q_{n,2}) + \gamma(1 - \pi^e)}. \tag{40}$$

■

VII.2 Appendix B

The policy functions in (??), (21) and (23) involve loglinearization constants ρ_c^e , ρ_n^e and ρ_w^e . These constants in turn depend on $E(c_t - h_t^p)$, $E(n_t)$ and $E(w_t - h_t^p)$, respectively. We use a simple numerical procedure to solve for the fixed point. We start with some initial values for \bar{N} , ρ_c^e , ρ_n^e and ρ_w^e . Together with other structural parameters of the model, this allows us to compute the coefficients b_0^e , b_1^e and b_2^e according to (40), (36) and (38), and the portfolio rule. We can then calculate a new set of values for $E(w_t - h_t^p)$, $E(c_t - h_t^p)$ and $E(n_t)$. (See Appendix C for the computation of $E(w_t^e - h_t^p)$.) Finally, these new means can be used to get new \bar{N} , ρ_c^e , ρ_n^e and ρ_w^e according to their definitions. The recursion stops when the values of ρ_c^e , ρ_n^e and ρ_w^e converge.

VII.3 Appendix C

The optimal policies given in section III.1—III.2 are expressed as functions of the state variable— $w_t - h_t^p$. Therefore, we need to make sure that this state variable is itself stationary so that the analysis and the calibration exercise is valid. It is obvious from an inspection of (31) and (32) that the wealth process has an autoregressive component with the autoregressive coefficient given by

$$\frac{\gamma b_1^e}{\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}}. \quad (41)$$

Taking the unconditional expectation of the second line in (31) and (32), we immediately obtain the long-run mean:

$$E(w_t^e - h_t^p) = \frac{G_2 + (1 - \pi^e)(\gamma b_0^e - \hat{\gamma} b_0^r)}{(\hat{\gamma} - \gamma b_1^e)(1 - \pi^e)}, \quad (42)$$

where G_2 is defined in (39).

The numerical method of solving for the coefficients turns out to depend critically on the sign of the term G_2 . It is necessary that $G_2 < 0$. Deaton (1991) gives a similar condition in a model with liquidity constraints and exogenous labor income but no transitory disturbances; that condition was later extended by Carroll (1996) to the case with transitory shocks, and by Viceira (2001) to include portfolio choice. It is further extended here to allow for leisure/labor choice in the model. Accordingly, in our calibration exercise, we choose the underlying parameters in such a way that $G_2 < 0$.

VII.4 Appendix D

This section proves the properties of the coefficients in the consumption policy in the employment state. We first consider the coefficient b_1^e . It is easy to verify that the two roots to the quadratic equation (36) are of opposite signs. We take the positive solution as argued at the end of Appendix A. Now, since the wealth-wage ratio has to following a stationary process, the autoregressive coefficient in (41) has to satisfy

$$\frac{\gamma b_1^e}{\pi^e \gamma b_1^e + (1 - \pi^e) \hat{\gamma}} < 1.$$

Reordering the terms, we immediately have

$$b_1^e < \frac{\hat{\gamma}}{\gamma} \leq 1, \quad (43)$$

since $\hat{\gamma} \leq \gamma$, for $\gamma > 1$.

Now, consider b_2^e . From the definitions of the terms involved, it is obvious that $b_2^e > 0$. Define $\Lambda \equiv (\pi^e b_1^e + (1 - \pi^e) \hat{\gamma} / \gamma)$. Then,

$$b_2^e = \frac{\Lambda (\overline{N} \rho_n^e + \overline{N} \rho_n^e Q_{n,3}) - \kappa_1 / \gamma}{1 + \Lambda (\rho_c^e - \overline{N} \rho_n^e Q_{n,2})} < \frac{\Lambda (\overline{N} \rho_n^e + \overline{N} \rho_n^e Q_{n,3}) + (1 - b_1^e)}{1 + \Lambda (\rho_c^e - \overline{N} \rho_n^e Q_{n,2})},$$

where the inequality follows from the fact that $\kappa_1 - \hat{\gamma} = -\gamma$ and hence $-\kappa_1 / \gamma = 1 - \hat{\gamma} / \gamma < 1 - b_1^e$. It is sufficient to show that the term on the right is bounded by 1. But, proving

$$\frac{\Lambda (\overline{N} \rho_n^e + \overline{N} \rho_n^e Q_{n,3}) + (1 - b_1^e)}{1 + \Lambda (\rho_c^e - \overline{N} \rho_n^e Q_{n,2})} < 1$$

is equivalent to showing

$$-\frac{b_1^e}{\Lambda} < (\rho_c^e - \overline{N} \rho_n^e).$$

To show this latter condition, recall that the quadratic equation for b_1^e in (36)

$$\mathcal{A} (b_1^e)^2 + \mathcal{B} b_1^e + \mathcal{C} = 0,$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the coefficients defined in (37). Using the bound in (43),

$$-\mathcal{B} + \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}} < \frac{\hat{\gamma}}{\gamma} (2\mathcal{A}).$$

Rearranging and substituting the definitions of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ yields

$$\begin{aligned} 0 &> \frac{\hat{\gamma}}{\gamma} (\rho_c^e - \overline{N} \rho_n^e Q_{n,2}) + 1 - \rho_w^e \\ &= \frac{\hat{\gamma}}{\gamma} (\rho_c^e - \overline{N} \rho_n^e Q_{n,2}) - (\rho_c^e - \overline{N} \rho_n^e), \end{aligned}$$

where we have used the fact that $\overline{N} \rho_n^e = (1 - \rho_w^e + \rho_c^e)$. Therefore,

$$(\rho_c^e - \overline{N} \rho_n^e) > \frac{\hat{\gamma}}{\gamma} (\rho_c^e - \overline{N} \rho_n^e Q_{n,2}) > 0 > -\frac{b_1^e}{\Lambda},$$

where the second inequality follows from $Q_{n,2} < 0$ and $\rho_c^e, \overline{N}, \rho_n^e$ are all positive constants and the last inequality from $0 < b_1^e < \Lambda$. This completes our proof. ■

Figure 1: Optimal Portfolio Policies with Endogenous and Exogenous Labor Income

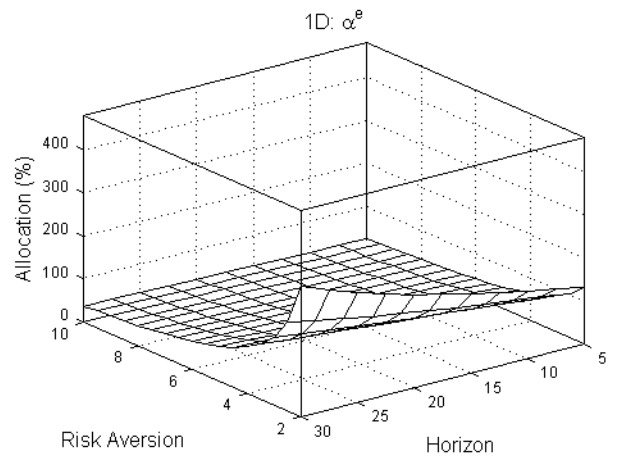
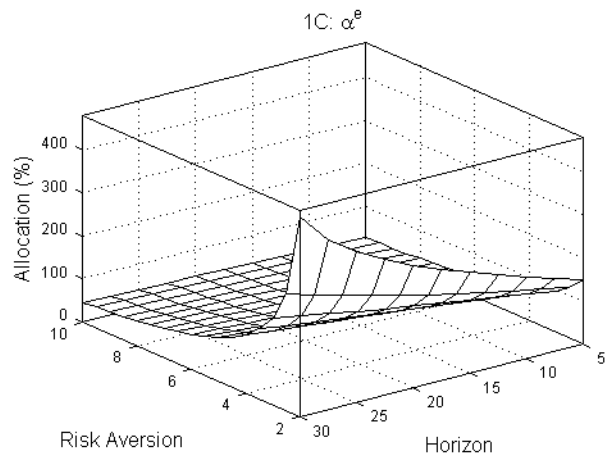
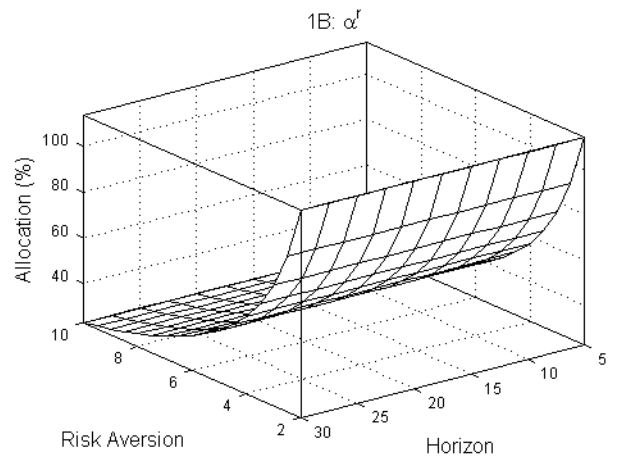
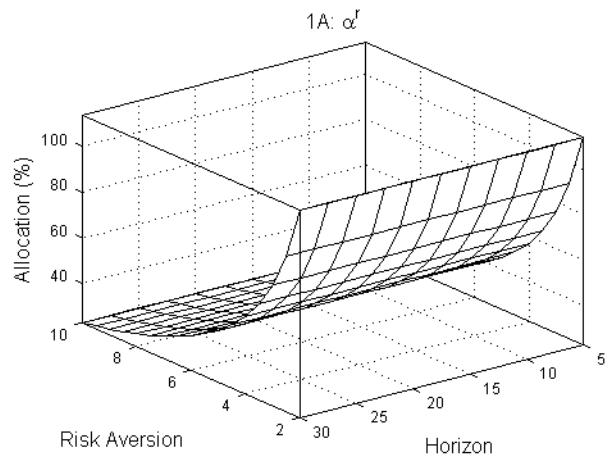


Figure 2: Wealth Elasticities of Consumption with Endogenous and Exogenous Labor Income

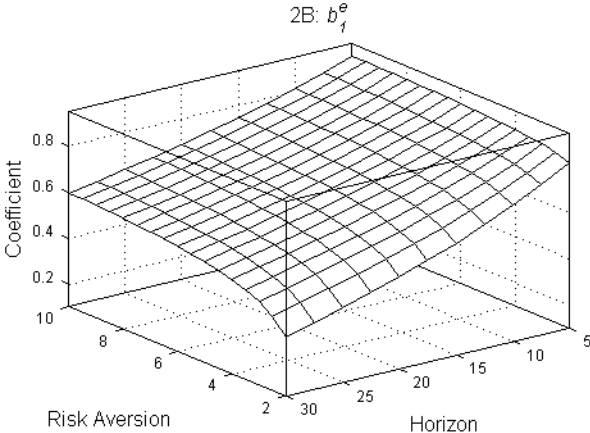
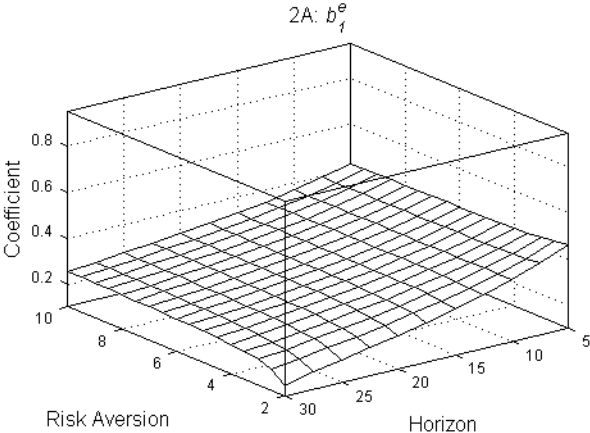


Figure 3: Optimal Portfolio Policies with Correlated Labor Income

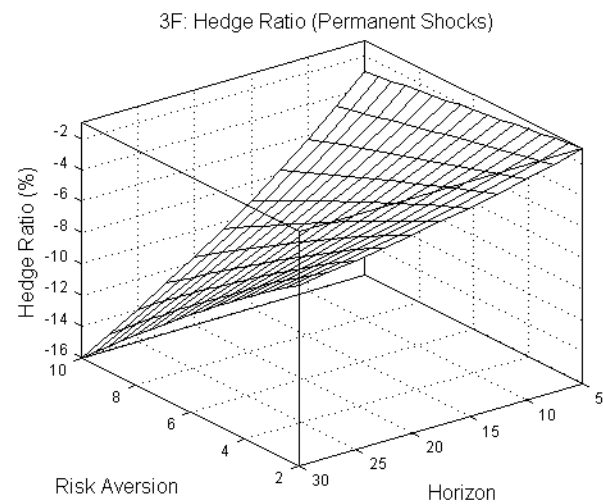
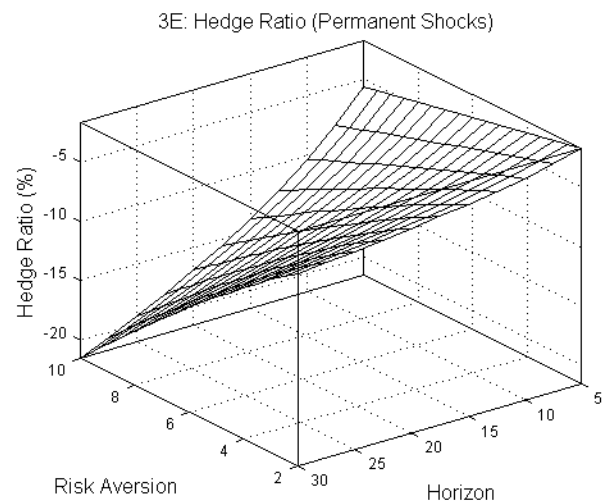
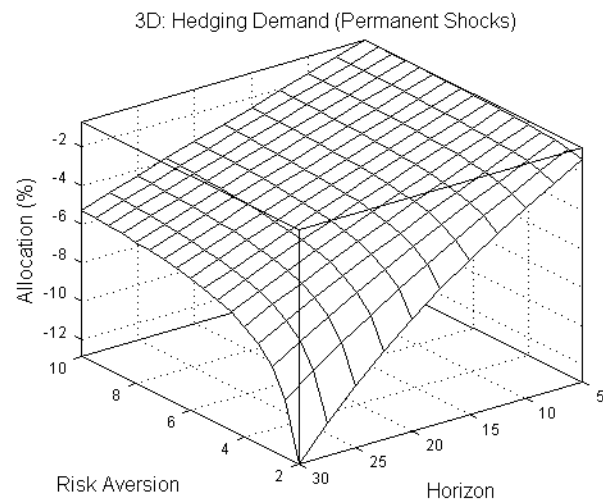
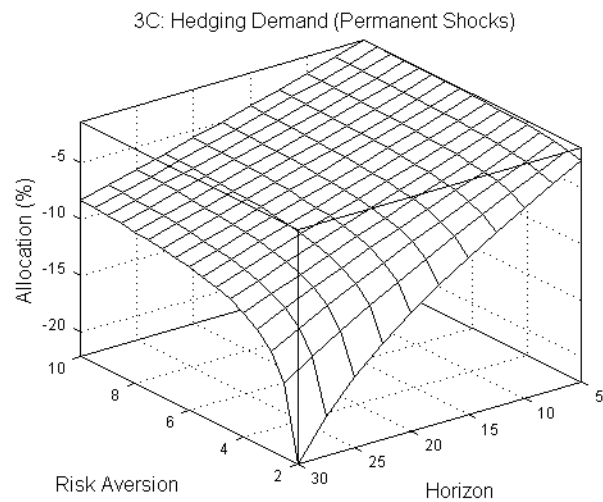
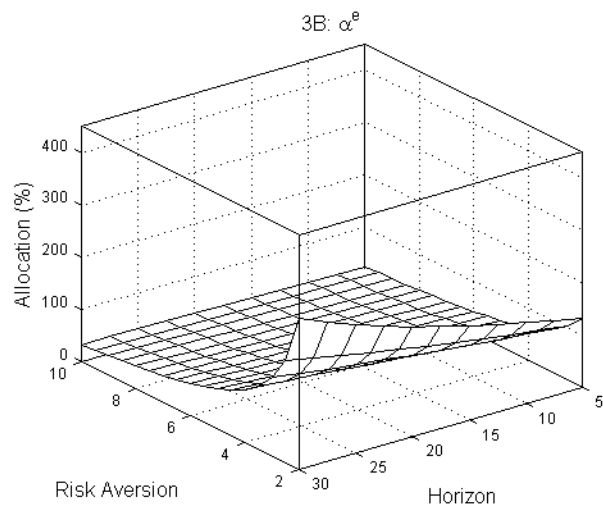
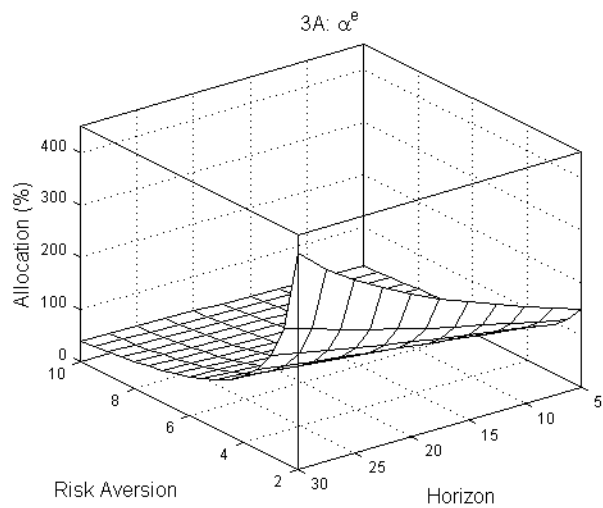


Figure 4: Wealth, Consumption and Labor Supply

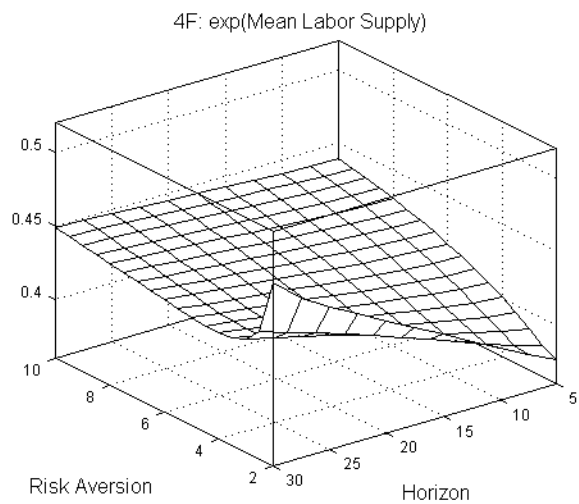
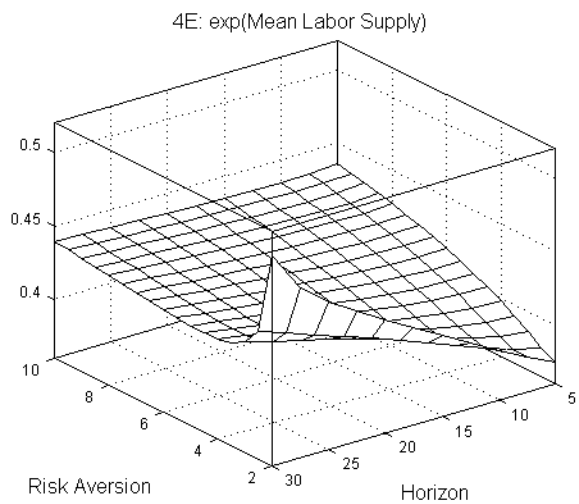
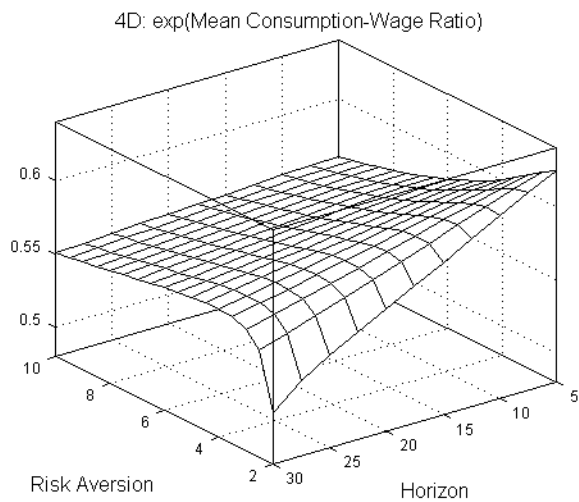
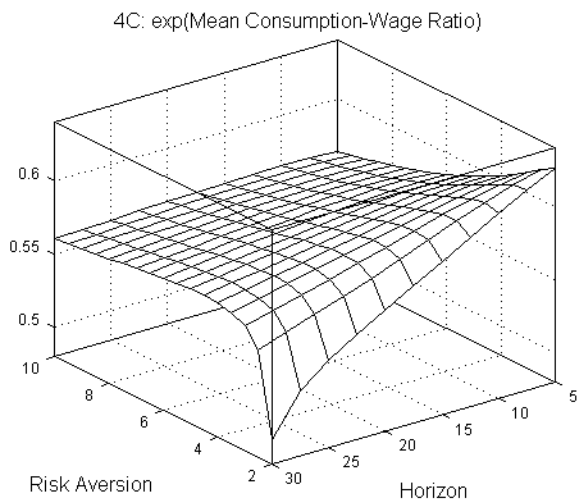
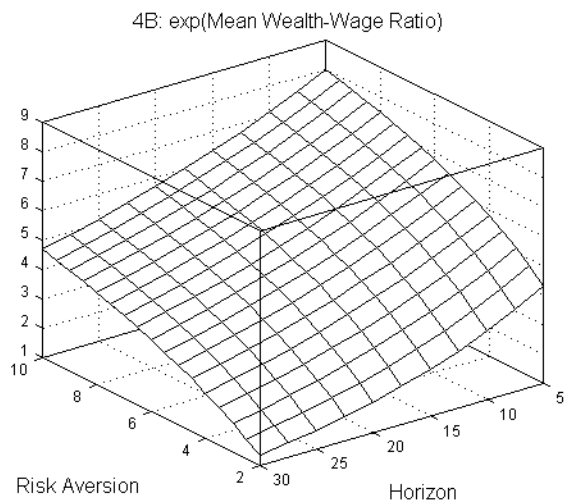
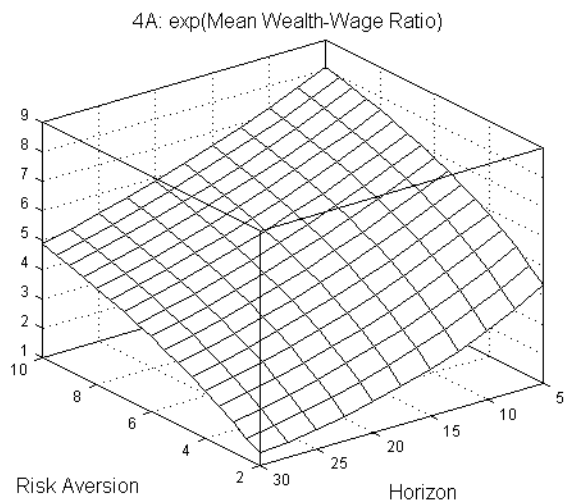


Figure 5: Elasticities of Labor Supply to Shocks

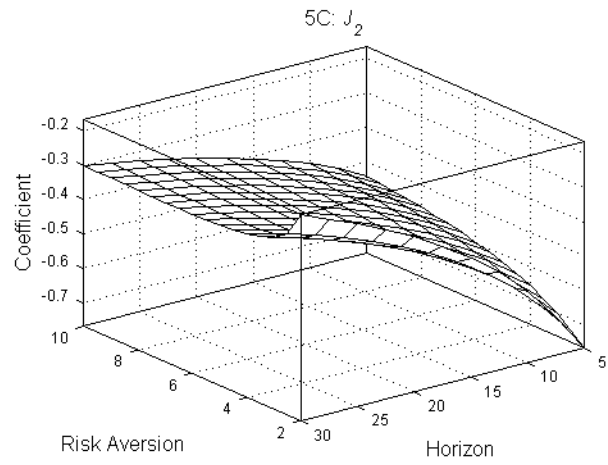
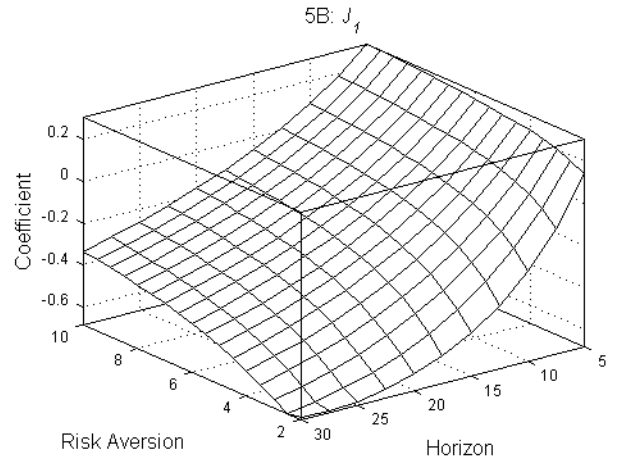
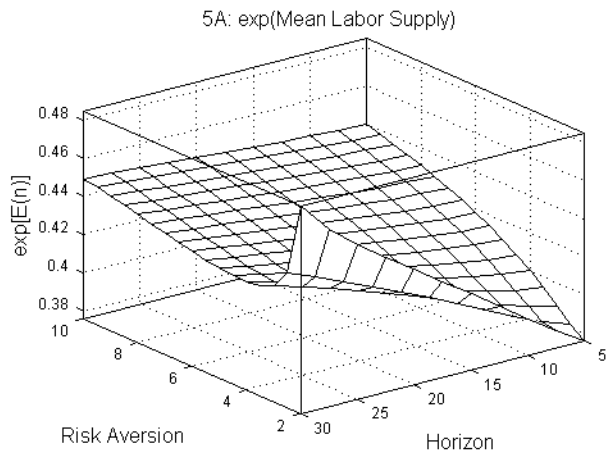


Figure 6: Effects of Varying θ

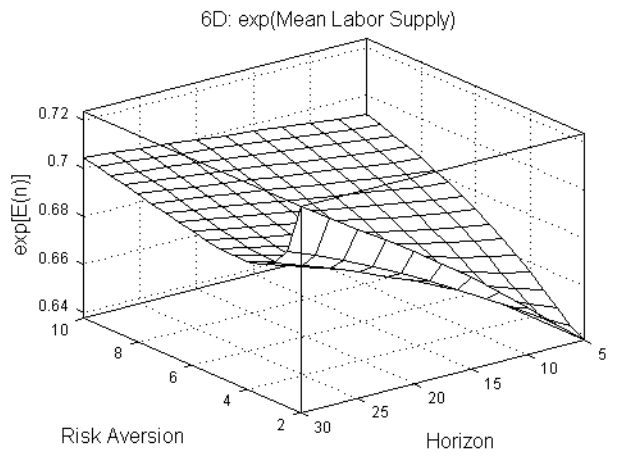
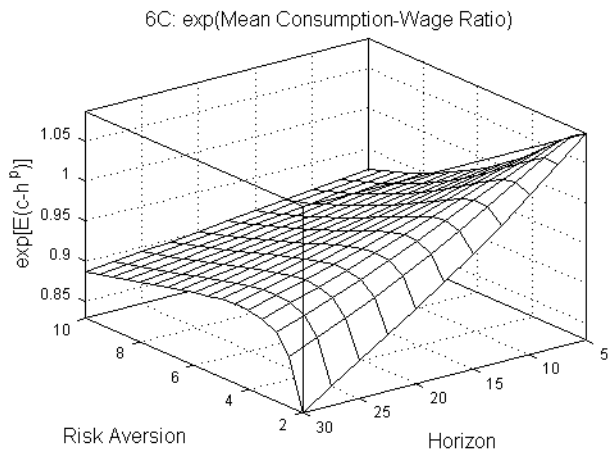
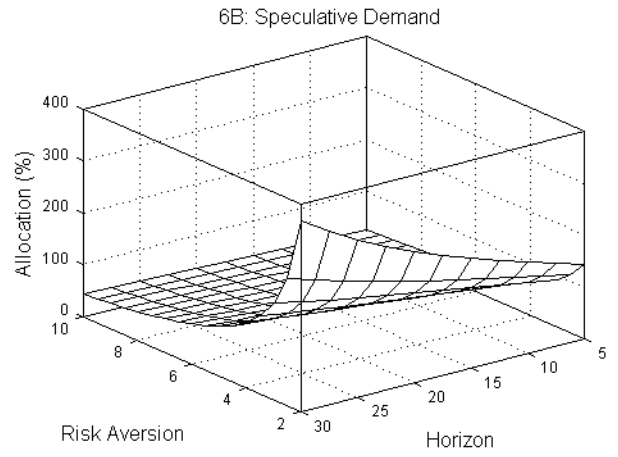
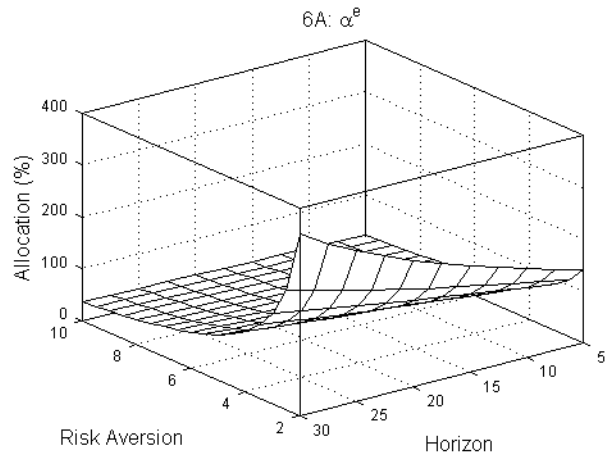


Figure 7: Effects of Time Discounting

