Appendix to
Who Should Buy Long-Term Bonds?

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1 Introduction

This Appendix contains mathematical derivations of the results in the paper "Who Should Buy Long-Term Bonds?" as well as tables with detailed results on portfolio choice, consumption and welfare for different scenarios. To avoid confusion between equations in the main text of the paper and equations in this Appendix, we number equations in the Appendix as (A1), (A2), etc.

2 Pricing Nominal Bonds

The pricing of default-free nominal bonds follows the same steps as the pricing of indexed bonds. The relevant stochastic discount factor to price nominal bonds is the nominal SDF $M^{\$}_{t+1}$, whose log is given by equation (10) in the paper: $m^{\$}_{t+1} = m_{t+1} - \pi_{t+1}$. Since both M_{t+1} and Π_{t+1} are jointly lognormal and homoskedastic, $M^{\$}_{t+1}$ is also lognormal. The log nominal return on a one-period nominal bond is $r^{\$}_{1,t+1} = -\log \mathrm{E}_t[M^{\$}_{t+1}]$, or

$$\begin{aligned} r^{\$}_{1,t+1} &= -\mathrm{E}_t\left[m^{\$}_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[m^{\$}_{t+1}\right] \\ &= x_t + z_t - \frac{1}{2}\left[\left(\beta_{mx}+\beta_{\pi x}\right)^2\sigma_x^2 + \beta_{\pi z}^2\sigma_z^2 + \left(1+\beta_{\pi m}\right)^2\sigma_m^2 + \sigma_\pi^2\right], \qquad \text{(A1)} \end{aligned}$$

a linear combination of the expected log real SDF and expected inflation.

The risk premium on a 1-period nominal bond over a 1-period real bond can be written as

$$\mathrm{E}_t\left[r^{\$}_{1,t+1} - \pi_{t+1} - r_{1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[\pi_{t+1}\right] = -\beta_{mx}\beta_{\pi x}\sigma_x^2 - \beta_{\pi m}\sigma_m^2, \qquad \text{(A2)}$$

which has the same form as equation (4) for equities.

The log price of an n-period nominal bond, $p^{\$}_{n,t}$, also has an affine structure. It is a linear combination of x_t and z_t whose coefficients are time-invariant, though they vary with the maturity of the bond. As shown in equation (11) in the paper, $-p^{\$}_{n,t} = A^{\$}_n + B^{\$}_{1,n}x_t +$

$B_{2,n}^{\$} z_t$,where

$$
\begin{aligned}
B_{1,n}^{\$} &= 1+\phi_x B_{1,n-1}^{\$} = \frac{1-\phi_x^n}{1-\phi_x} \\
B_{2,n}^{\$} &= 1+\phi_z B_{2,n-1}^{\$} = \frac{1-\phi_z^n}{1-\phi_z} \\
A_n^{\$} - A_{n-1}^{\$} &= (1-\phi_x)\,\mu_x B_{1,n-1}^{\$} + (1-\phi_z)\,\mu_z B_{2,n-1}^{\$} \\
&\quad -\frac{1}{2}\left(\beta_{mx}+\beta_{\pi_x}+B_{1,n-1}^{\$}+\beta_{zx}B_{2,n-1}^{\$}\right)^2 \sigma_x^2 \\
&\quad -\frac{1}{2}\left(\beta_{\pi z}+B_{2,n-1}^{\$}\right)^2 \sigma_z^2 - \frac{1}{2}\left(1+\beta_{\pi m}+\beta_{zm}B_{2,n-1}^{\$}\right)^2 \sigma_m^2 \qquad \text{(A3)} \\
&\quad -\frac{1}{2}\sigma_\pi^2,
\end{aligned}
$$

and $A_0^{\$} = B_{1,0}^{\$} = B_{2,0}^{\$} = 0$.

The excess return on a n-period bond over the one-period log nominal interest rate is

$$
\begin{aligned}
r_{n,t+1}^{\$} - r_{1,t+1}^{\$} &= p_{n-1,t+1}^{\$} - p_{n,t}^{\$} + p_{1,t}^{\$} \\
&= -\left(B_{1,n-1}^{\$}+B_{2,n-1}^{\$}\beta_{zx}\right)(\beta_{mx}+\beta_{\pi x})\,\sigma_x^2 - B_{2,n-1}^{\$}\beta_{\pi z}\sigma_z^2 - (1+\beta_{\pi m})\,\beta_{zm}B_{2,n-1}^{\$}\sigma_m^2 \\
&\quad -\frac{1}{2}\left(B_{1,n-1}^{\$}+B_{2,n-1}^{\$}\beta_{zx}\right)^2\sigma_x^2 - \frac{1}{2}\left(B_{2,n-1}^{\$}\right)^2\sigma_z^2 - \frac{1}{2}\beta_{zm}^2\left(B_{2,n-1}^{\$}\right)^2\sigma_m^2 \\
&\quad -\left(B_{1,n-1}^{\$}+B_{2,n-1}^{\$}\beta_{zx}\right)\varepsilon_{x,t+1} - B_{2,n-1}^{\$}\beta_{zm}\varepsilon_{m,t+1} - B_{2,n-1}^{\$}\varepsilon_{z,t+1}. \qquad \text{(A4)}
\end{aligned}
$$

The terms in $B_{2,n-1}^{\$}\beta_{zx}$ and $B_{2,n-1}^{\$}\beta_{zm}$ arise because shocks to expected inflation are correlated with shocks to the expected and unexpected log real SDF. Thus risk premia in the nominal term structure are different from risk premia in the real term structure because they include compensation for inflation risk. Like real risk premia, however, nominal risk premia are constant over time.

3 Solution of the Model When Only Indexed Bonds Are Available

Our guess (26) and the expression for the log excess return on the long-term bond given in equation (8) imply that $\mathrm{Cov}_t\left(r_{n,t+1}, c_{t+1} - w_{t+1}\right) = -B_{n-1} b_1 \sigma_x^2$. Our term structure model implies that $\mathrm{E}_t[r_{n,t+1} - r_{1,t+1}] = -B_{n-1}^2 \sigma_x^2/2 - \beta_{mx} B_{n-1} \sigma_x^2$ and $\mathrm{Var}_t(r_{n,t+1}) = B_{n-1}^2 \sigma_x^2$. Substituting these expressions into equation (25) in the paper we obtain:

$$\alpha_{n,t} = \alpha_n = \frac{-1}{\gamma B_{n-1}} \left(\beta_{mx} + \frac{1-\gamma}{1-\psi} b_1 \right), \tag{A5}$$

which does not depend on the future portfolio and consumption choices of the investor.

Given the optimal portfolio rule (A5) we can now solve for the parameters b_0 and b_1 of the consumption rule. The expected return on the wealth portfolio is a linear function of the state variable:

$$\mathrm{E}_t\left[r_{p,t+1}\right] = p_0 + x_t, \tag{A6}$$

where the intercept

$$p_0 \equiv -(\alpha_n B_{n-1})(\beta_{mx}\sigma_x^2) - (\alpha_n B_{n-1})^2 \sigma_x^2/2 - (\beta_{mx}^2 \sigma_x^2 + \sigma_m^2)/2 \tag{A7}$$

does not vary with t or n. The consumption intercept term given in (21) becomes

$$v_{p,t} = v_p \equiv -((1-\gamma)/(1-\psi))[b_1 - (1-\psi)(\alpha_n B_{n-1})]^2 \sigma_x^2/2, \tag{A8}$$

which also does not vary with t or n.

Substituting (A6), (A7), and (A8) into equation (20) in the paper, we get a linear expectational difference equation for $(c_t - w_t)$,

$$c_t - w_t = \rho\, \mathrm{E}_t\left[c_{t+1} - w_{t+1}\right] + \rho\,(1-\psi)\, x_t - \rho\left[(1-\psi)\, p_0 - \psi \log \delta - v_p + k\right], \tag{A9}$$

from which we can identify the coefficients of the consumption rule.

4 Implications of complete markets

This section proves the properties of the solution when markets are complete.

4.1 Bond Portfolio Return

The first property of the solution with complete markets refers to the return on the bond portfolio. With complete markets the investor can combine short- and long-term bonds so that the return on her bond portfolio is independent of the maturity of the long-term bond traded in the market. That is, she can synthesize her own optimal long-term bond, with the maturity optimal for her given her risk preferences. The return on the optimal bond portfolio is given by

$$\begin{aligned} r_{p,t+1} &= -\left(\frac{1}{2}\left(\alpha_n B_{n-1}\right)^2 + \alpha_n B_{n-1}\beta_{mx}\right)\sigma_x^2 - \frac{1}{2}\left(\beta_{mx}^2\sigma_x^2 + \sigma_m^2\right) \\ &\quad + x_t - \alpha_n B_{n-1}\varepsilon_{x,t+1}, \end{aligned} \tag{A10}$$

and only the product $\alpha_n B_{n-1}$ enters this expression. Our portfolio solution—see equation (29) in the paper—implies that $\alpha_n B_{n-1}$ does not depend on n.

4.2 Stochastic Discount Factor (SDF) and Intertemporal Marginal Rate of Substitution (IMRS)

The second property of the solution with complete markets refers to the consistency between the SDF assumed for the term-structure model and the IMRS of the investor in the portfolio choice model. If real-interest-rate variation is the only source of risk, then markets are complete with respect to all sources of risk. We can explore this case by setting $\sigma_m^2 = 0$ so that $\varepsilon_{m,t+1}$ drops from the definition of m_{t+1} in equation (1). In this case the SDF is unique. Since the IMRS of any investor can be used as a valid SDF, it follows that all investors must have the same IMRS which must equal the SDF we specified exogenously for our term structure model. This provides a check on the internal consistency of our solution. Using equation (14) to express the investor's IMRS as a function of consumption growth and the portfolio return, we must have

$$IMRS_{t+1} = \left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} R_{p,t+1}^{-(1-\theta)} = M_{t+1}. \tag{A11}$$

Taking logs and using our solution, it is straightforward to show that

$$\begin{aligned} \log\left(IMRS_{t+1}\right) &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{p,t+1} \\ &= -x_t - \beta_{mx} \varepsilon_{x,t+1} = m_{t+1}, \end{aligned} \tag{A12}$$

which is the required result.

4.3 The Cox-Huang Method with Recursive Utility

The third property of the solution with complete markets refers to an alternative derivation of the solution using a discrete-time version of the Cox-Huang approach. Cox and Huang (1989) have proposed an alternative solution method for intertemporal consumption and portfolio choice problems with complete markets. They work in continuous time and show that with complete markets, optimally invested wealth must satisfy a partial differential equation (PDE). Unfortunately this PDE does not generally have a closed-form solution. We now show that our solution methodology is equivalent to a discrete-time version of the Cox-Huang approach; our loglinear approximation allows us to solve the discrete-time equivalent of the Cox-Huang PDE in closed form.

We start by defining a new variable $W_t^* = W_t - C_t$—invested wealth—and note that from the budget constraint (13), the portfolio return equals $R_{p,t+1} = (W_{t+1}^* + C_{t+1})/W_t^*$. Then the Euler equation (15) implies that with recursive utility the optimal invested wealth-consumption ratio W_t^*/C_t satisfies

$$\left(\frac{W_t^*}{C_t}\right)^\theta = \mathrm{E}_t\left[\left(1+\frac{W_{t+1}^*}{C_{t+1}}\right)^\theta \delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{\theta\left(1-\frac{1}{\psi}\right)}\right]. \tag{A13}$$

The consumer's IMRS is $\delta^\theta (C_{t+1}/C_t)^{-\theta/\psi} R_{p,t+1}^\theta$, which under complete markets must equal the SDF M_{t+1}, so that

$$\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{\theta\left(1-\frac{1}{\psi}\right)} = \delta^{\theta\psi} M_{t+1}^{1-\psi} R_{p,t+1}^{(1-\theta)(1-\psi)}, \tag{A14}$$

and we can rewrite (A13) as

$$\left(\frac{W_t^*}{C_t}\right)^\theta = \mathrm{E}_t\left[\left(1+\frac{W_{t+1}^*}{C_{t+1}}\right)^\theta \delta^{\theta\psi} M_{t+1}^{1-\psi} R_{p,t+1}^{(1-\theta)(1-\psi)}\right]. \tag{A15}$$

This nonlinear expectational difference equation is the discrete-time equivalent of the Cox-Huang PDE.

Equation (A15) does not generally have a closed form solution, so it must be solved numerically or using an analytical approximation method. We can apply the same approximation that we have already used. Taking logs on both sides of (A15) and using the same approximation around the mean log consumption-wealth ratio that we use to loglinearize the budget constraint[2], we can write (A15) in log form as

$$\begin{aligned} c_t - w_t \;=\; & \rho\left[k - \psi \log\delta + \mathrm{E}_t\left(c_{t+1}-w_{t+1}\right) - \left(\frac{1-\psi}{\theta}\right)\mathrm{E}_t\, m_{t+1} \right. \\ & -\frac{(1-\theta)(1-\psi)}{\theta}\mathrm{E}_t\, r_{p.t+1} \\ & \left. -\frac{\theta}{2}\mathrm{Var}_t\left(\left(c_{t+1}-w_{t+1}\right)+\left(\frac{1-\psi}{\theta}\right)m_{t+1}+\frac{(1-\theta)(1-\psi)}{\theta} r_{p.t+1}\right)\right], \end{aligned} \tag{A16}$$

[2]The approximation is

$$\begin{aligned} \log\left(\frac{W_t^*}{C_t}\right) &= \log\left(\frac{W_t}{C_t}\left(1-\frac{C_t}{W_t}\right)\right) = -\left(c_t-w_t\right)+\log\left(1-\frac{C_t}{W_t}\right) \\ &\approx -\left(c_t-w_t\right)+k+\left(1-\frac{1}{\rho}\right)\left(c_t-w_t\right) = k-\frac{1}{\rho}\left(c_t-w_t\right), \end{aligned}$$

where ρ and k have been defined after equation (22).

which is linear. It is trivial to show that this equation has the same solution that we derive in the paper using our Euler-equation methodology.

5 On the Economic Definition of the Riskless Asset

We now show that in our model an individual who is infinitely risk-averse and infinitely reluctant to substitute consumption intertemporally chooses a portfolio of indexed bonds that is equivalent to a real perpetuity. That is, if a real perpetuity were available, the portfolio would be fully invested in that bond. To see this, we first note that, from equations (23) and (29), the interest-rate sensitivity of the optimal portfolio for an infinitely risk-averse individual is given by

$$\lim_{\gamma\to\infty}\frac{\partial r_{p,t+1}}{\partial\epsilon_{x,t+1}}=\lim_{\gamma\to\infty}\left(-\alpha_n B_{n-1}\right)=-\frac{\rho}{1-\rho\phi_x}. \tag{A17}$$

A real perpetuity pays a fixed coupon of one unit of consumption per period forever. The log coupon on the bond is therefore $d_{c,t}=0\ \forall t$. Campbell, Lo and MacKinlay (1997, p. 408), following Shiller (1979), show that a log-linear approximation to the log yield on a real perpetuity is

$$y_{c,t}\approx(1-\rho_c)\,\mathrm{E}_t\sum_{j=0}^{\infty}\rho_c^j r_{c,t+1+j}, \tag{A18}$$

where $y_{c,t}$ is the log yield on the real perpetuity, $r_{c,t+1}$ is the log return on the perpetuity, and ρ_c is a log-linearization constant defined as $\rho_c\equiv 1-\exp\{\mathrm{E}(-p_{c,t})\}$, where $p_{c,t}$ is the log "cum-dividend" price of the perpetuity including its current coupon.[3]

The return on the real perpetuity must verify the pricing relation $1=\mathrm{E}_t[M_{t+1}R_{c,t+1}]$.

[3] Campbell, Lo, and MacKinlay give an alternative definition of ρ_c in relation to the "ex-dividend" price of the consol excluding its current coupon. This is more natural in a bond pricing context, but less convenient here because the form of the budget constraint (see equation [13] in the paper) implies that we are measuring wealth inclusive of current consumption, that is, on a "cum-dividend" basis.

Assuming that $R_{c,t+1}$ is lognormally distributed, we must have that

$$\begin{aligned} \mathrm{E}_t\left[r_{c,t+1}\right] &= x_t - \frac{1}{2}\operatorname{Var}_t\left(m_{t+1}\right) - \frac{1}{2}\operatorname{Var}_t\left(r_{c,t+1}\right) + \operatorname{Cov}_t\left(m_{t+1}, r_{c,t+1}\right) \\ &= x_t + \omega_c^2, \end{aligned} \tag{A19}$$

where ω_c^2 is a positive constant. Equations (A18) and (A19) imply that

$$y_{c,t} \approx \omega_c^2 + \left(1 - \frac{1-\rho_c}{1-\rho_c\phi_x}\right)\mu_x + \left(\frac{1-\rho_c}{1-\rho_c\phi_x}\right)x_t. \tag{A20}$$

But Campbell, Lo and MacKinlay (1997, p. 408, eq. 10.1.19) also show that

$$\begin{aligned} r_{c,t+1} &\approx \left(\frac{1}{1-\rho_c}\right)y_{c,t} - \left(\frac{\rho_c}{1-\rho_c}\right)y_{c,t+1} \qquad \text{(A21)} \\ &= \omega_c^2 + \left(\frac{\rho_c}{1-\rho_c\phi_x}\right)\left(1-\phi_x\right)\mu_x + \left(\frac{1}{1-\rho_c\phi_x}\right)x_t - \left(\frac{\rho_c}{1-\rho_c\phi_x}\right)x_{t+1}, \end{aligned}$$

which in turn implies that the interest-rate sensitivity of a real perpetuity is given by

$$\frac{\partial r_{c,t+1}}{\partial \epsilon_{x,t+1}} = -\frac{\rho_c}{1-\rho_c\phi_x}. \tag{A22}$$

Equations (A17) and (A22) differ only by the log-linearization constants ρ and ρ_c. These two constants are the same for an individual who is infinitely reluctant to substitute consumption intertemporally ($\psi = 0$). Such an individual consumes the annuity value of wealth, the consumption stream that can be sustained indefinitely by the initial level of wealth. But the annuity value of a real perpetuity is just its dividend of one. Thus for this investor $C/W = 1/P_c$, which implies $\mathrm{E}[c - w] = \mathrm{E}[-p_c]$, and thus, from the definitions of the log-linearization parameters, $\rho = \rho_c$. The infinitely risk-averse investor who is infinitely reluctant to substitute intertemporally holds a real perpetuity that finances a riskless consumption stream over the infinite future.

6 Optimal Rules in the Multivariate Case

We now derive the optimal rules when there are equities as well as long-term nominal and indexed bonds available to the investor. For simplicity, we assume that there is only one bond of each class available to the investor. For realism, we also assume that the one-period bond is nominal. As in the case where there are only two indexed bonds available, the investor maximizes (12) subject to the intertemporal budget constraint $W_{t+1} = R_{p,t+1}(W_t - C_t)$. The only difference from the case with two indexed bonds is in the number and classes of assets that enter $R_{p,t+1}$.

6.1 Log Real Portfolio Return

We derive the log real return on the wealth portfolio as a discrete-time approximation to its continuous-time counterpart. We begin by specifying

$$\frac{dV_t}{V_t} = \alpha_1 \frac{dP_{e,t}}{P_{e,t}} + \alpha_2 \frac{dP_{n,t}}{P_{n,t}} + \alpha_3 \frac{d\left(P^{\$}_{n,t}/I_t\right)}{P^{\$}_{n,t}/I_t} + \left(1 - \sum_{i=1}^{3} \alpha_i\right) \frac{d\left(P^{\$}_{1,t}/I_t\right)}{P^{\$}_{1,t}/I_t},$$

where V_t is the value in real dollars of the portfolio at time t, $P_{e,t}$ is the price of equity in real dollars, $P_{n,t}$ is the price of the n-period indexed bond in real dollars, $P^{\$}_{n,t}$ is the price of the n-period nominal bond in nominal dollars, $P^{\$}_{1,t}$ is the price of the 1-period nominal bond in nominal dollars and I_t is the price index used to deflate prices in nominal dollars. Itô's Lemma implies that

$$\begin{aligned}
\frac{d\left(P^{\$}_{n,t}/I_t\right)}{P^{\$}_{n,t}/I_t} &= \left(\frac{dP^{\$}_{n,t}}{P^{\$}_{n,t}} - \frac{dI_t}{I_t}\right) + \left(\frac{dI_t}{I_t}\right)^2 - \left(\frac{dP^{\$}_{n,t}}{P^{\$}_{n,t}}\right)\left(\frac{dI_t}{I_t}\right) \\
&= \left(\frac{dP^{\$}_{n,t}}{P^{\$}_{n,t}} - \frac{dI_t}{I_t}\right) + \left(\sigma^2_\pi - \sigma_{r^{\$}_n,\pi}\right),
\end{aligned}$$

where the notational equivalence between the first and second lines of the equation is obvious. A similar expression obtains for $d(P^{\$}_{1,t}/I_t)/(P^{\$}_{1,t}/I_t)$.[4] Therefore,

$$\begin{aligned}
\frac{dV_t}{V_t} &= \alpha_1 \frac{dP_{e,t}}{P_{e,t}} + \alpha_2 \frac{dP_{n,t}}{P_{n,t}} + \alpha_3 \left(\frac{dP^{\$}_{n,t}}{P^{\$}_{n,t}} - \frac{dI_t}{I_t}\right) + \left(1 - \sum_{i=1}^{3}\alpha_i\right)\left(\frac{dP^{\$}_{1,t}}{P^{\$}_{1,t}} - \frac{dI_t}{I_t}\right) \qquad \text{(A23)} \\
&\quad + \alpha_3\left(\sigma^2_\pi - \sigma_{r^{\$}_n,\pi}\right) + \left(1 - \sum_{i=1}^{3}\alpha_i\right)\sigma^2_\pi.
\end{aligned}$$

[4]Note that in the expression for $d(P^{\$}_{1,t}/I_t)/(P^{\$}_{1,t}/I_t)$ we have $\sigma_{r^{\$}_1,\pi} = 0$, because the one-period nominal bond is riskless in nominal terms. For the same reason, we also have $\sigma^2_{r^{\$}_1} = 0$.

We now compute the log portfolio return, $d\log V_t$. We will use $d\log V_t$ to approximate $r_{p,t+1} \equiv \log(R_{p,t+1})$. From Itô's Lemma we have that

$$d\log V_t = \left(\frac{dV_t}{V_t}\right) - \frac{1}{2}\left(\frac{dV_t}{V_t}\right)^2.$$

Similarly,

$$d\log P_{e,t} = \frac{dP_{e,t}}{P_{e,t}} - \frac{1}{2}\left(\frac{dP_{e,t}}{P_{e,t}}\right)^2,$$

or

$$\frac{dP_{e,t}}{P_{e,t}} = d\log P_{e,t} + \frac{1}{2}\left(\frac{dP_{e,t}}{P_{e,t}}\right)^2.$$

Similar expressions obtain for $d\log P_{e,t}$, $d\log P_{n,t}$, $d\log P^{\$}_{n,t}$ and $d\log P^{\$}_{1,t}$. Substituting back into (A23) and simplifying we get:

$$\begin{aligned} d\log V_t &= \alpha_1\left(d\log P_{e,t} - \left(d\log P^{\$}_{1,t} - d\log I_t\right)\right) + \alpha_2\left(d\log P_{n,t} - \left(d\log P^{\$}_{1,t} - d\log I_t\right)\right) \\ &\quad +\alpha_3\left(d\log P^{\$}_{n,t} - d\log P^{\$}_{1,t}\right) + \left(d\log P^{\$}_{1,t} - d\log I_t\right) \qquad \text{(A24)} \\ &\quad +\frac{1}{2}\alpha_1(1-\alpha_1)\left(\sigma_e^2 + \sigma_\pi^2 + 2\sigma_{e,\pi}\right) + \frac{1}{2}\alpha_2(1-\alpha_2)\left(\sigma_{r_n}^2 + \sigma_\pi^2 + 2\sigma_{r_n,\pi}\right) \\ &\quad +\frac{1}{2}\alpha_3(1-\alpha_3)\sigma^2_{r_n^{\$}} - \alpha_1\alpha_2\sigma_{e,r_n} - \alpha_1\alpha_3\left(\sigma_{e,r_n^{\$}} + \sigma_{e,\pi}\right) - \alpha_2\alpha_3\left(\sigma_{r_n,r_n^{\$}} + \sigma_{r_n,\pi}\right). \end{aligned}$$

The discrete-time version of (A24) obtains immediately after making the following equivalences: $d\log V_t \equiv r_{p,t+1}$, $d\log P_{e,t} \equiv r_{e,t+1}$, $d\log P_{n,t} \equiv r_{n,t+1}$, $d\log P^{\$}_{n,t} \equiv r^{\$}_{n,t+1}$, $d\log P^{\$}_{1,t} \equiv r^{\$}_{1,t+1}$ and $d\log I_t \equiv \pi_{t+1}$; $\sigma_e^2 \equiv \text{Var}_t(r_{e,t+1})$, $\sigma^2_{r_n} \equiv \text{Var}_t(r_{n,t+1})$, $\sigma^2_{r_n^{\$}} \equiv \text{Var}_t(r^{\$}_{n,t+1})$, $\sigma^2_\pi \equiv \text{Var}_t(\pi_{t+1})$, $\sigma^2_{r_n} \equiv \text{Var}_t(r_{n,t+1})$, $\sigma_{e,\pi} \equiv \text{Cov}_t(r_{e,t+1}, \pi_{t+1})$, $\sigma_{r_n,\pi} \equiv \text{Cov}_t(r_{n,t+1}, \pi_{t+1})$, $\sigma_{e,r_n} \equiv \text{Cov}_t(r_{e,t+1}, r_{n,t+1})$, $\sigma_{e,r_n^{\$}} \equiv \text{Cov}_t(r_{e,t+1}, r^{\$}_{n,t+1})$ and $\sigma_{r_n,,r_n^{\$}} \equiv \text{Cov}_t(r_{n,t+1}, r^{\$}_{n,t+1})$. Since all second moments are constant, we can trivially write the log real return on the wealth portfolio:

$$\begin{aligned} r_{p,t+1} &= \alpha_1\left(r_{n,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1}\right)\right) + \alpha_2\left(r_{n,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1}\right)\right) \qquad \text{(A25)} \\ &\quad +\alpha_3\left(r^{\$}_{n,t+1} - r^{\$}_{1,t+1}\right) + \left(r^{\$}_{1,t+1} - \pi_{t+1}\right) + k_p, \end{aligned}$$

where k_p is a constant.

6.2 Log Euler Equations

6.2.1 Log Euler equations for individual assets

We first log-linearize the Euler equation (14) in the paper for each asset:

$$\begin{aligned} 0 &= \theta \log \delta - \frac{\theta}{\psi} \mathrm{E}_t \Delta c_{t+1} - (1-\theta) \mathrm{E}_t r_{p,t+1} + \mathrm{E}_t r_{i,t+1} \\ &+ \frac{1}{2} \mathrm{Var}_t \left[-\frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta) r_{p,t+1} + r_{i,t+1} \right], \end{aligned} \tag{A26}$$

for $r_{i,t+1} = r_{e,t+1}$, $r_{n,t+1}$, $(r^{\$}_{n,t+1} - \pi_{t+1})$ and $(r^{\$}_{1,t+1} - \pi_{t+1})$.

Combining the log Euler equation above, the trivial equality

$$\begin{aligned} \Delta c_{t+1} &= (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1} \\ &= r_{p,t+1} + (c_{t+1} - w_{t+1}) - \frac{1}{\rho} (c_t - w_t) + k, \end{aligned}$$

where the second line follows from substituting in the log budget constraint, and the equation for the log return on the wealth portfolio we obtain the following trio of equations for the expected log excess return on equities, the n-period indexed bond and the n-period nominal bond:

$$\begin{aligned} & \mathrm{E}_t \left[r_{i,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1} \right) \right] + \frac{1}{2} \left(\mathrm{Var}_t \left(r_{i,t+1} \right) - \mathrm{Var}_t \left(\pi_{t+1} \right) \right) \\ =\ & \mathrm{Cov}_t \left(r_{i,t+1} + \pi_{t+1}, \frac{\theta}{\psi} (c_{t+1} - w_{t+1}) + \left(\frac{\theta}{\psi} + (1-\theta) \right) r_{p,t+1} \right), \end{aligned}$$

for $r_{i,t+1} = r_{e,t+1}$, $r_{n,t+1}$and $(r^{\$}_{n,t+1} - \pi_{t+1})$.

Simplifying these equations we obtain:

$$\begin{aligned} & \mathrm{E}_t \left[r_{i,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1} \right) \right] + \frac{1}{2} \left(\mathrm{Var}_t \left(r_{i,t+1} \right) - \mathrm{Var}_t \left(\pi_{t+1} \right) \right) \\ =\ & -\frac{1-\gamma}{1-\psi} \mathrm{Cov}_t \left(r_{i,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1} \right), c_{t+1} - w_{t+1} \right) \\ & + \gamma \, \mathrm{Cov}_t \left(r_{i,t+1} - \left(r^{\$}_{1,t+1} - \pi_{t+1} \right), \left(r^{\$}_{1,t+1} - \pi_{t+1} \right) \right) \end{aligned} \tag{A27}$$

$$+\gamma\alpha_i \operatorname{Var}_t\left(r_{i,t+1}-\left(r^{\$}_{1,t+1}-\pi_{t+1}\right)\right)$$
$$+\gamma\sum_{j\neq i}\alpha_j \operatorname{Cov}_t\left(r_{i,t+1}-\left(r^{\$}_{1,t+1}-\pi_{t+1}\right), r_{j,t+1}-\left(r^{\$}_{1,t+1}-\pi_{t+1}\right)\right),$$

for $r_{i,t+1}, r_{j,t+1} = r_{e,t+1}$, $r_{n,t+1}$and $(r^{\$}_{n,t+1}-\pi_{t+1})$.

Equation (A27) defines a system of equations whose unknown is the vector of portfolio allocations $\boldsymbol{\alpha} = (\alpha_1,\, \alpha_2,\, \alpha_3)'$. The solution to the system is

$$\boldsymbol{\alpha} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\mathbf{a}, \tag{A28}$$

where $\mathbf{a}$ is given in equations (31)-(34) in the paper, and $\boldsymbol{\Sigma}$ is the variance-covariance matrix of log excess returns, with element ij given by

$$\boldsymbol{\Sigma}_{ij}=\operatorname{Cov}_t\left(r_{i,t+1}-\left(r^{\$}_{1,t+1}-\pi_{t+1}\right), r_{j,t+1}-\left(r^{\$}_{1,t+1}-\pi_{t+1}\right)\right).$$

This is a matrix of constants given our assumptions on the term structure of interest rates.

6.2.2 Log Euler equation for consumption growth

The log Euler equation for the return on the wealth portfolio is given by:

$$0=\theta\log\delta-\frac{\theta}{\psi}\operatorname{E}_t\Delta c_{t+1}+\theta\operatorname{E}_t r_{p,t+1}+\frac{1}{2}\operatorname{Var}_t\left(-\frac{\theta}{\psi}\Delta c_{t+1}+\theta r_{p,t+1}\right),$$

or

$$\operatorname{E}_t\Delta c_{t+1}=\psi\log\delta+v_{p,t}+\psi\operatorname{E}_t r_{p,t+1}, \tag{A29}$$

where

$$v_{p,t}=-\frac{1}{2}\frac{1-\gamma}{1-\psi}\operatorname{Var}_t\left(\Delta c_{t+1}-\psi r_{p,t+1}\right). \tag{A30}$$

6.3 Optimal Rules

Note that (A28) is recursive in nature because it depends on future portfolio consumption decisions through the covariance of the asset return with next period's log consumption-wealth ratio (see [A27]). A complete solution to the model requires deriving consumption and portfolio rules that depend only contemporaneously on the state variables. We do this by guessing the form of the optimal consumption rule and showing that this guess verifies the log Euler equations. It turns out that in the model with nominal bonds, the guess that solves the model is the same as in the only-indexed-bonds case. That is, the optimal log consumption-wealth ratio is linear in x_t and it does not depend on expected inflation z_t:

$$c_t - w_t = b_0 + b_1 x_t.$$

Substituting this guess into (A28) and computing all second moments it is easy to verify that $\boldsymbol{\alpha}$ is not time-varying and that

$$\begin{aligned} \mathrm{E}_t\, r_{p,t+1} &= p_{0,n} + x_t, \\ \mathrm{Var}_t\left(r_{p,t+1}\right) &\equiv \mathrm{Var}\left(r_{p,t+1}\right), \end{aligned}$$

where $p_{0,n}$ and $\mathrm{Var}(r_{p,t+1})$ are also time-invariant. Similarly, it is easy to check that the intercept of the log Euler equation for consumption growth (A30) is also time-invariant:

$$v_{p,t} \equiv v_{p,n}.$$

Substituting the log budget constraint into the trivial equality $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$ and taking expectations we get:

$$\begin{aligned} \mathrm{E}_t\, \Delta c_{t+1} &= \mathrm{E}_t\, r_{p,t+1} + \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \frac{1}{\rho}\left(c_t - w_t\right) + k \\ &= c_{0,n} + \left(1 + \left(\phi - 1/\rho\right) b_1\right) x_t, \end{aligned} \tag{A31}$$

where the second line also follows from the guess on the log consumption-wealth ratio.

Similarly, substituting the expressions for $\mathrm{E}_t\, r_{p,t+1}$, $v_{p,t}$ and $\mathrm{E}_t\, \Delta c_{t+1}$ into the log Euler equation for consumption growth (A29), we obtain another expression for $\mathrm{E}_t\, \Delta c_{t+1}$. By equating this expression to the right hand side of (A31), we can identify b_0 and b_1, whose expressions are similar to equations (27) and (28) in the paper.

7 Tables

TABLE A1

Term Structure Model Estimation
(Based on CPI data)

	1952.I - 1996.III		1983.I - 1996.III	
Parameter	est.	s.e.	est.	s.e.
μ_x	0.0573	0.0298	0.0194	0.0693
μ_z	0.0094		0.0087	
ϕ_x	0.8688	0.0057	0.9862	0.0042
ϕ_z	0.9992	0.0012	0.8599	0.0216
β_{mx}	-74.9797	41.6949	-28.6919	114.0025
β_{zx}	0.0752	0.0516	-0.4114	0.1886
β_{zm}	-0.0012	0.0006	0.0008	0.0024
$\beta_{\pi x}$	0.5198	0.3050	-0.0267	0.9790
$\beta_{\pi m}$	-0.0088	0.0034	0.0008	0.0193
$\beta_{\pi z}$	1.4320	0.2940	-1.5412	1.5047
β_{ex}	-3.4957	3.4123	-9.3629	6.3014
β_{em}	0.3013	0.0979	0.5089	1.3528
σ_x	0.0025	0.0001	0.0027	0.0006
σ_m	0.2694	0.0927	0.1351	0.3579
σ_z	0.0013	0.0001	0.0016	0.0002
σ_π	0.0071	0.0004	0.0072	0.0018
log-lik.	26.3327		26.8222	
no. obs.	179		55	
$E[r_{1,t+1}]$	1.39%		2.93%	
$E[r^{\$}_{1,t+1}]$	5.50%		6.40%	
$\sigma(r_{1,t+1})$	1.01%		3.25%	
$\sigma(r^{\$}_{1,t+1})$	6.70%		3.09%	
$E[\pi_{t+1}]$	3.77%		3.49%	
$\sigma_t(\pi_{t+1})$	1.57%		1.52%	

TABLE A2

Sample and Implied Moments of the Term Structure
(Based on CPI data)

	Moment		1952.I - 1996.III			1983.I - 1996.III		
			1 yr.	3 yr.	10 yr.	1 yr.	3 yr.	10 yr.
	A: Nominal Term Structure							
(1)	$E[r^{\$}_{n,t+1} - r^{\$}_{1,t+1}] + \sigma^2(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})/2$	sample	0.397	0.651	0.915	0.706	2.111	5.675
		implied	0.559	1.290	1.967	0.155	0.658	2.278
		(S.E.)	(0.272)	(0.655)	(1.071)	(0.570)	(2.373)	(8.307)
(2)	$\sigma(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})$	sample	1.615	4.615	11.365	1.135	4.220	12.612
		implied	1.634	4.501	11.566	1.312	4.627	14.896
		(S.E.)	(0.084)	(0.263)	(0.707)	(0.296)	(1.171)	(3.664)
(3)	$SR^{\$} = (1)/(2)$	sample	0.246	0.141	0.080	0.622	0.500	0.450
		implied	0.342	0.287	0.170	0.118	0.142	0.153
		(S.E.)	(0.172)	(0.148)	(0.091)	(0.450)	(0.535)	(0.579)
(4)	$E[y^{\$}_{n,t+1} - y^{\$}_{1,t+1}]$	sample	0.440	0.802	1.185	0.527	1.267	2.067
		implied	0.294	0.742	1.174	0.071	0.276	0.766
		(S.E.)	(0.142)	(0.392)	(0.748)	(0.283)	(1.148)	(4.373)
(5)	$\sigma(y^{\$}_{n,t+1} - y^{\$}_{1,t+1})$	sample	0.222	0.409	0.613	0.177	0.341	0.545
		implied	0.182	0.488	0.826	0.140	0.380	0.803
		(S.E.)	(0.010)	(0.026)	(0.048)	(0.019)	(0.045)	(0.123)
(6)	$\sigma(y^{\$}_{n-1,t+1} - y^{\$}_{n,t+1})$	sample	0.518	0.412	0.288	0.370	0.379	0.322
		implied	0.548	0.412	0.297	0.440	0.422	0.383
		(S.E.)	(0.028)	(0.024)	(0.018)	(0.098)	(0.106)	(0.094)
	B: Real Term Structure							
(7)	$E[r_{n,t+1} - r_{1,t+1}] + \sigma^2(r_{n,t+1} - r_{1,t+1})/2$	implied	0.490	1.075	1.345	0.245	0.851	2.513
		(S.E.)	(0.247)	(0.563)	(0.710)	(0.904)	(3.139)	(9.289)
(8)	$\sigma(r_{n,t+1} - r_{1,t+1})$	implied	1.309	2.994	3.788	1.590	5.521	16.295
		(S.E.)	(0.071)	(0.184)	(0.275)	(0.369)	(1.263)	(3.708)
(9)	$SR = (7)/(8)$	implied	0.374	0.374	0.374	0.154	0.154	0.154
		(S.E.)	(0.198)	(0.198)	(0.198)	(0.590)	(0.590)	(0.590)
(10)	$E[y_{n,t+1} - y_{1,t+1}]$	implied	0.253	0.665	1.100	0.118	0.381	0.858
		(S.E.)	(0.130)	(0.347)	(0.581)	(0.455)	(1.620)	(5.177)
(11)	$\sigma(y_{n,t+1} - y_{1,t+1})$	implied	0.182	0.486	0.816	0.067	0.235	0.738
		(S.E.)	(0.010)	(0.026)	(0.045)	(0.020)	(0.070)	(0.204)
(12)	$\sigma(y_{n-1,t+1} - y_{n,t+1})$	implied	0.441	0.276	0.099	0.531	0.502	0.418
		(S.E.)	(0.024)	(0.017)	(0.007)	(0.123)	(0.115)	(0.095)
	C: Equities							
(13)	$E[r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1})]$ $+\sigma^2(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))/2$	sample		6.910			8.738	
		implied		8.988			4.527	
		(S.E.)		(3.131)			(9.212)	
(14)	$\sigma(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))$	sample		15.917			14.646	
		implied		15.896			14.748	
		(S.E.)		(0.708)			(1.069)	
(15)	$SR = (13)/(14)$	sample		0.434			0.597	
		implied		0.565			0.307	
		(S.E.)		(0.184)			(0.621)	

TABLE A3

Term Structure Model Estimation
(Based on PCE data)

	1952.I - 1996.III		1983.I - 1996.III	
Parameter	est.	s.e.	est.	s.e.
μ_x	0.0550	0.0299	0.0229	0.0656
μ_z	0.0095		0.0086	
ϕ_x	0.8682	0.0058	0.9872	0.0043
ϕ_z	0.9997	0.0013	0.8423	0.0226
β_{mx}	-71.8668	40.4470	-30.2376	104.0367
β_{zx}	0.0776	0.0575	-0.3893	0.1872
β_{zm}	-0.0011	0.0005	0.0006	0.0020
$\beta_{\pi x}$	0.5088	0.2535	-0.1645	0.6498
$\beta_{\pi m}$	-0.0063	0.0027	-0.0001	0.0147
$\beta_{\pi z}$	1.3775	0.2913	-1.6001	1.2798
β_{ex}	-3.5306	3.3441	-8.7665	5.0261
β_{em}	0.3031	0.1010	0.4452	1.1675
σ_x	0.0025	0.0001	0.0027	0.0006
σ_m	0.2657	0.0938	0.1557	0.4103
σ_z	0.0013	0.0001	0.0016	0.0002
σ_π	0.0063	0.0004	0.0060	0.0017
log-lik.	26.4541		26.9266	
no. obs.	179		55	
$E[r_{1,t+1}]$	1.45%		3.00%	
$E[r^{\$}_{1,t+1}]$	5.50%		6.40%	
$\sigma(r_{1,t+1})$	1.01%		3.34%	
$\sigma(r^{\$}_{1,t+1})$	11.12%		3.22%	
$E[\pi_{t+1}]$	3.79%		3.42%	
$\sigma_t(\pi_{t+1})$	1.37%		1.32%	

TABLE A4

Sample and Implied Moments of the Term Structure (Based on PCE data)

	Moment		1952.I - 1996.III			1983.I - 1996.III		
			1 yr.	3 yr.	10 yr.	1 yr.	3 yr.	10 yr.
	A: Nominal Term Structure							
(1)	$E[r^{\$}_{n,t+1} - r^{\$}_{1,t+1}] + \sigma^2(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})/2$	sample	0.397	0.651	0.915	0.706	2.111	5.675
		implied	0.536	1.238	1.900	0.169	0.716	2.463
		(S.E.)	(0.269)	(0.651)	(1.089)	(0.511)	(2.189)	(7.629)
(2)	$\sigma(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})$	sample	1.615	4.615	11.365	1.135	4.220	12.612
		implied	1.631	4.483	11.546	1.327	4.713	15.280
		(S.E.)	(0.077)	(0.251)	(0.711)	(0.300)	(1.231)	(4.112)
(3)	$SR^{\$} = (1)/(2)$	sample	0.246	0.141	0.080	0.622	0.500	0.450
		implied	0.329	0.276	0.165	0.127	0.152	0.161
		(S.E.)	(0.169)	(0.146)	(0.092)	(0.402)	(0.489)	(0.525)
(4)	$E[y^{\$}_{n,t+1} - y^{\$}_{1,t+1}]$	sample	0.440	0.802	1.185	0.527	1.267	2.067
		implied	0.282	0.711	1.122	0.078	0.302	0.843
		(S.E.)	(0.141)	(0.389)	(0.748)	(0.249)	(1.052)	(4.045)
(5)	$\sigma(y^{\$}_{n,t+1} - y^{\$}_{1,t+1})$	sample	0.222	0.409	0.613	0.177	0.341	0.545
		implied	0.182	0.487	0.819	0.145	0.384	0.791
		(S.E.)	(0.010)	(0.027)	(0.050)	(0.022)	(0.047)	(0.114)
(6)	$\sigma(y^{\$}_{n-1,t+1} - y^{\$}_{n,t+1})$	sample	0.518	0.412	0.288	0.370	0.379	0.322
		implied	0.547	0.410	0.297	0.445	0.430	0.392
		(S.E.)	(0.026)	(0.023)	(0.018)	(0.099)	(0.111)	(0.105)
	B: Real Term Structure							
(7)	$E[r_{n,t+1} - r_{1,t+1}] + \sigma^2(r_{n,t+1} - r_{1,t+1})/2$	implied	0.469	1.070	1.351	0.255	0.890	2.658
		(S.E.)	(0.242)	(0.551)	(0.695)	(0.808)	(2.815)	(8.389)
(8)	$\sigma(r_{n,t+1} - r_{1,t+1})$	implied	1.308	2.985	3.770	1.581	5.511	16.464
		(S.E.)	(0.076)	(0.194)	(0.287)	(0.368)	(1.298)	(4.135)
(9)	$SR = (7)/(8)$	implied	0.358	0.358	0.358	0.161	0.161	0.161
		(S.E.)	(0.193)	(0.193)	(0.193)	(0.533)	(0.533)	(0.533)
(10)	$E[y_{n,t+1} - y_{1,t+1}]$	implied	0.242	0.635	1.049	0.123	0.400	0.923
		(S.E.)	(0.127)	(0.339)	(0.568)	(0.407)	(1.452)	(4.684)
(11)	$\sigma(y_{n,t+1} - y_{1,t+1})$	implied	0.182	0.486	0.815	0.064	0.226	0.716
		(S.E.)	(0.010)	(0.027)	(0.048)	(0.017)	(0.060)	(0.179)
(12)	$\sigma(y_{n-1,t+1} - y_{n,t+1})$	implied	0.440	0.275	0.099	0.528	0.501	0.423
		(S.E.)	(0.025)	(0.018)	(0.007)	(0.123)	(0.118)	(0.106)
	C: Equities							
(13)	$E[r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1})]$ $+\sigma^2(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))/2$	sample		6.910			8.738	
		implied		8.879			5.108	
		(S.E.)		(3.177)			(11.897)	
(14)	$\sigma(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))$	sample		15.917			14.646	
		implied		15.895			14.719	
		(S.E.)		(0.718)			(1.402)	
(15)	$SR = (13)/(14)$	sample		0.434			0.597	
		implied		0.559			0.347	
		(S.E.)		(0.187)			(0.820)	

TABLE A5

Optimal Percentage Allocation to n-Period Bond

$\alpha_n \times 100$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	1285	1286	1287	1288	1289	1289						
	1	988	988	988	988	988	988						
	2	541	541	540	540	539	539						
	5	272	272	272	272	272	272						
	10	182	183	183	184	184	184						
	5000	93	94	95	95	95	96						
Nominal Only	0.75							193	193	193	193	193	193
	1							147	147	147	147	147	147
	2							78	78	78	78	78	78
	5							37	37	37	37	37	37
	10							23	23	23	23	23	24
	5000							+	9	-	-	-	-
Indexed and Nominal	0.75	841	841	842	842	842	843	108	108	108	108	108	108
	1	654	654	654	654	654	654	81	81	81	81	81	81
	2	375	374	374	374	374	374	40	40	40	40	40	40
	5	206	206	207	207	207	207	16	16	16	16	16	16
	10	150	150	151	151	151	152	8	8	8	8	8	8
	5000	94	95	96	96	96	96	0	0	0	0	0	0
Both Nominal (3,10y)	0.75	2600	2602	2606	2608	2609	2609	-744	-744	-745	-746	-746	-746
	1	1998	1998	1998	1998	1998	1998	-572	-572	-572	-572	-572	-572
	2	1092	1091	1089	1088	1088	1088	-315	-315	-314	-314	-314	-314
	5	547	547	547	546	546	546	-160	-160	-160	-160	-160	-160
	10	365	365	366	366	367	367	-108	-109	-109	-109	-109	-109
	5000	+	184	-	-	-	-	+	-57	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ for these parameter values and "+" for the case where ρ converged to a negative value.

TABLE A6

Percentage Hedging Demand Over Total Demand

$$\alpha_{n,hedging}(\gamma,\psi)/\alpha_n(\gamma,\psi) = [1 - \alpha_n(1,\psi)/(\gamma\alpha_n(\gamma,\psi))] \times 100$$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	-2.5	-2.4	-2.3	-2.2	-2.2	-2.2						
	1	0.0	0.0	0.0	0.0	0.0	0.0						
Indexed	2	8.7	8.6	8.5	8.5	8.4	8.4						
Only	5	27.4	27.4	27.5	27.5	27.5	27.5						
	10	45.9	46.0	46.1	46.2	46.3	46.3						
	5000	99.8	99.8	99.8	99.8	99.8	99.8						
	0.75							-1.6	-1.6	-1.6	-1.6	-1.6	-1.6
	1							0.0	0.0	0.0	0.0	0.0	0.0
Nominal	2							5.8	5.9	6.0	6.0	6.1	6.1
Only	5							19.7	19.9	20.4	20.7	20.7	20.8
	10							35.4	35.9	36.7	37.2	37.3	37.5
	5000							+	99.7	-	-	-	-
	0.75	-3.8	-3.7	-3.6	-3.6	-3.6	-3.6	0.1	0.1	0.2	0.2	0.2	0.2
Indexed	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
and	2	12.6	12.6	12.6	12.6	12.5	12.5	-0.5	-0.5	-0.5	-0.6	-0.6	-0.6
Nominal	5	36.5	36.6	36.7	36.8	36.8	36.8	-2.2	-2.1	-1.9	-1.8	-1.8	-1.8
	10	56.4	56.5	56.7	56.8	56.8	56.8	-5.2	-4.8	-4.2	-3.9	-3.8	-3.7
	5000	99.9	99.9	99.9	99.9	99.9	99.9	103.6	104.1	105.0	105.7	106.0	106.3
	0.75	-2.5	-2.4	-2.2	-2.1	-2.1	-2.1	-2.7	-2.6	-2.4	-2.3	-2.3	-2.3
Both	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Nominal	2	8.5	8.4	8.3	8.2	8.2	8.2	9.2	9.1	8.9	8.9	8.8	8.8
(3,10y)	5	26.9	26.9	26.9	26.9	26.9	26.9	28.5	28.5	28.5	28.5	28.5	28.5
	10	45.2	45.3	45.4	45.5	45.5	45.5	47.2	47.3	47.4	47.5	47.5	47.5
	5000	+	99.8	-	-	-	-	+	99.8	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ for these parameter values and "+" for the case where ρ converged to a negative value.

TABLE A7

Percentage Mean Optimal Consumption-Wealth Ratio and Percentage Volatility of Consumption Growth

Model	R.R.A.	$E[C_t/W_t] \times 100$ E.I.S.						$\sigma\left(\Delta c_{t+1} - E_t\left[\Delta c_{t+1}\right]\right) \times 100$ E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	0.45	0.98	1.77	2.24	2.40	2.55	24.94	24.35	23.55	23.11	22.96	22.83
	1	0.61	0.98	1.53	1.86	1.97	2.08	19.31	18.71	17.85	17.38	17.22	17.07
Indexed	2	0.84	0.98	1.17	1.29	1.33	1.37	10.84	10.24	9.35	8.83	8.66	8.49
Only	5	0.99	0.98	0.96	0.95	0.95	0.95	5.74	5.16	4.27	3.74	3.56	3.39
	10	1.03	0.98	0.89	0.84	0.82	0.81	4.04	3.46	2.58	2.05	1.87	1.69
	5000	1.08	0.98	0.82	0.73	0.70	0.67	2.34	1.77	0.90	0.36	0.18	0.00
	0.75	1.00	0.98	0.94	0.92	0.91	0.90	11.73	11.45	11.08	10.88	10.82	10.76
	1	1.04	0.98	0.88	0.83	0.81	0.79	9.08	8.80	8.44	8.26	8.20	8.15
Nominal	2	1.09	0.98	0.80	0.69	0.65	0.62	5.13	4.84	4.52	4.40	4.37	4.35
Only	5	1.14	0.98	0.73	0.58	0.53	0.48	2.80	2.50	2.27	2.30	2.34	2.39
	10	1.18	0.98	0.67	0.49	0.43	0.36	2.06	1.76	1.62	1.77	1.86	1.97
	5000	+	0.98	-	-	-	-	+	1.09	-	-	-	-
	0.75	0.71	0.98	1.38	1.62	1.70	1.79	19.54	18.98	18.20	17.75	17.60	17.46
Indexed	1	0.80	0.98	1.24	1.40	1.45	1.50	15.22	14.65	13.84	13.37	13.22	13.06
and	2	0.94	0.98	1.03	1.06	1.07	1.09	8.73	8.17	7.33	6.83	6.67	6.51
Nominal	5	1.02	0.98	0.91	0.86	0.85	0.84	4.86	4.29	3.44	2.93	2.76	2.60
	10	1.05	0.98	0.86	0.80	0.78	0.75	3.59	3.02	2.16	1.64	1.47	1.30
	5000	1.10	0.98	0.81	0.71	0.68	0.65	2.36	1.79	0.91	0.38	0.20	0.03
	0.75	0.39	0.98	1.85	2.38	2.55	2.73	25.58	24.99	24.19	23.76	23.62	23.49
Both	1	0.56	0.98	1.60	1.97	2.10	2.22	19.80	19.19	18.34	17.87	17.72	17.57
Nominal	2	0.81	0.98	1.22	1.37	1.42	1.47	11.10	10.50	9.62	9.10	8.93	8.76
(3,10y)	5	0.96	0.98	1.00	1.01	1.01	1.02	5.89	5.30	4.43	3.90	3.73	3.56
	10	1.02	0.98	0.92	0.88	0.87	0.86	4.16	3.59	2.73	2.22	2.05	1.89
	5000	+	0.98	-	-	-	-	+	1.92	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ for these parameter values and "+" for the case where ρ converged to a negative value.

TABLE A8

Optimal Percentage Allocation to n-Period Bond

Under Borrowing and Short-Sale Constraints

$\alpha_n \times 100$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	100	100	100	100	100	100						
	1	100	100	100	100	100	100						
Indexed	2	100	100	100	100	100	100						
Only	5	100	100	100	100	100	100						
	10	100	100	100	100	100	100						
	5000	93	94	95	95	95	96						
	0.75							100	100	100	100	100	100
	1							100	100	100	100	100	100
Nominal	2							78	78	78	78	78	78
Only	5							37	37	37	37	37	37
	10							23	23	23	23	23	24
	5000							+	9	-	-	-	-
	0.75	1	1	1	1	1	1	99	99	99	99	99	99
Indexed	1	26	26	26	26	26	26	74	74	74	74	74	74
and	2	63	63	63	63	63	63	37	37	37	37	37	37
Nominal	5	85	85	85	85	85	85	14	15	15	15	15	15
	10	93	93	93	93	93	93	7	7	7	7	7	7
	5000	94	95	96	96	96	96	0	0	0	0	0	0
	0.75	0	0	0	0	0	0	100	100	100	100	100	100
Both	1	31	31	31	31	31	31	69	69	69	69	69	69
Nominal	2	87	87	87	87	87	87	13	13	13	13	13	13
(3,10y)	5	100	100	100	100	100	100	0	0	0	0	0	0
	10	100	100	100	100	100	100	0	0	0	0	0	0
	5000	+	48	-	-	-	-	+	0	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ for these parameter values and "+" for the case where ρ converged to a negative value.

TABLE A9

Percentage Mean Value Function

$\boldsymbol{E[V_t] \times 100}$

		Unconstrained						**Constrained**					
Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	10.01	4.93	3.19	2.76	2.65	2.55	0.73	0.72	0.70	0.69	0.69	0.68
	1	4.07	3.03	2.39	2.18	2.13	2.08	0.73	0.72	0.70	0.69	0.69	0.68
	2	1.51	1.46	1.41	1.38	1.38	1.37	0.73	0.72	0.70	0.69	0.69	0.68
	5	0.95	0.95	0.95	0.95	0.95	0.95	0.73	0.72	0.70	0.69	0.69	0.68
	10	0.82	0.82	0.81	0.81	0.81	0.81	0.73	0.72	0.70	0.69	0.69	0.68
	5000	0.72	0.71	0.69	0.68	0.67	0.67	0.72	0.71	0.69	0.68	0.67	0.67
Nominal Only	0.75	0.90	0.92	0.90	0.90	0.90	0.90	0.81	0.81	0.80	0.80	0.79	0.79
	1	0.81	0.81	0.80	0.80	0.79	0.79	0.78	0.78	0.77	0.76	0.76	0.75
	2	0.69	0.66	0.65	0.63	0.63	0.62	0.69	0.66	0.65	0.63	0.63	0.62
	5	0.61	0.58	0.54	0.51	0.49	0.48	0.61	0.58	0.54	0.51	0.49	0.48
	10	0.56	0.54	0.46	0.41	0.39	0.36	0.56	0.54	0.46	0.41	0.39	0.36
	5000	+	0.00	-	-	-	-	+	0.00	-	-	-	-
Indexed and Nominal	0.75	2.57	2.23	1.95	1.84	1.81	1.79	0.81	0.81	0.80	0.80	0.79	0.79
	1	1.78	1.68	1.58	1.53	1.52	1.50	0.79	0.79	0.78	0.77	0.77	0.76
	2	1.10	1.09	1.09	1.09	1.09	1.09	0.76	0.75	0.74	0.73	0.73	0.72
	5	0.85	0.85	0.84	0.84	0.84	0.84	0.74	0.73	0.72	0.71	0.70	0.70
	10	0.78	0.78	0.77	0.76	0.76	0.75	0.74	0.73	0.71	0.70	0.69	0.69
	5000	0.68	0.68	0.67	0.66	0.65	0.65	0.68	0.68	0.67	0.66	0.65	0.65
Both Nominal (3,10y)	0.75	15.20	5.89	3.52	2.97	2.84	2.73	0.81	0.81	0.80	0.80	0.79	0.79
	1	5.25	3.53	2.63	2.35	2.28	2.22	0.79	0.78	0.77	0.77	0.76	0.76
	2	1.72	1.63	1.54	1.49	1.48	1.47	0.75	0.74	0.73	0.72	0.72	0.71
	5	1.02	1.02	1.02	1.02	1.02	1.02	0.72	0.66	0.68	0.67	0.66	0.66
	10	0.86	0.86	0.86	0.86	0.86	0.86	0.66	0.55	0.61	0.59	0.58	0.57
	5000	+	0.00	-	-	-	-	+	0.00	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ for these parameter values and "+" for the case where ρ converged to a negative value.

TABLE A10

Optimal Percentage Allocation to Equities

and to n-Period Bond

$\alpha \times 100$

Model	R.R.A.	Equities E.I.S.						n-Period Bond E.I.S.					
(A) Unconstrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	-	443	443	443	443	443	-	1082	1088	1091	1091	1092
	1	-	332	332	332	332	332	-	835	835	835	835	835
	2	166	166	166	166	166	166	467	464	461	459	458	458
	5	66	66	66	66	66	66	243	242	240	239	239	238
	10	33	33	33	33	33	33	168	168	167	167	167	166
	5000	0	0	0	0	0	0	93	94	95	95	95	96
Nominal Only	0.75	-	470	470	470	470	470	-	25	25	26	26	26
	1	-	352	352	352	352	352	-	21	21	21	21	21
	2	175	175	175	175	175	175	16	15	15	15	15	15
	5	69	69	69	69	69	69	12	12	12	12	12	12
	10	33	33	33	33	33	33	11	11	11	11	11	11
	5000	+	-2	-	-	-	-	+	10	-	-	-	-
(B) Constrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
	2	100	100	100	100	100	100	0	0	0	0	0	0
	5	60	60	60	60	60	60	40	40	40	40	40	40
	10	30	30	30	30	30	30	70	70	70	70	70	70
	5000	0	0	0	0	0	0	93	94	95	95	95	96
Nominal Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
	2	100	100	100	100	100	100	0	0	0	0	0	0
	5	69	69	69	69	69	69	12	12	12	12	12	12
	10	33	33	33	33	33	33	11	11	11	11	11	11
	5000	+	0	-	-	-	-	+	10	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE A11

Optimal Percentage Allocation to Equities and to n-Period Bond Sample Period: 1983-1996

$\alpha \times 100$

Model	R.R.A.	Equities E.I.S.						n-Period Bond E.I.S.					
(A) Unconstrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	262	262	262	262	262	262	-8	-1	5	8	9	10
	1	196	196	196	196	196	196	21	21	21	21	21	21
	2	98	98	98	98	98	98	57	54	50	48	47	47
	5	39	39	39	39	39	39	76	74	72	70	69	69
	10	20	20	20	20	20	20	82	81	79	79	78	78
	5000	0	0	0	0	0	0	87	88	88	88	88	88
Nominal Only	0.75	259	259	259	259	259	259	-6	1	7	10	11	12
	1	195	195	195	195	195	195	24	24	24	24	24	24
	2	99	99	99	99	99	99	61	58	53	51	50	50
	5	41	41	41	41	41	41	80	78	76	74	73	73
	10	22	22	22	22	22	22	86	85	84	84	84	83
	5000	+	3	-	-	-	-	+	92	-	-	-	-
(B) Constrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	94	94	94	94	94	94	6	6	6	6	6	6
	2	51	53	55	57	57	58	49	47	45	43	43	42
	5	26	28	30	32	32	32	74	72	70	69	68	68
	10	18	19	20	20	20	20	82	81	79	79	78	78
	5000	0	0	0	0	0	0	87	88	88	88	88	88
Nominal Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	96	96	96	96	96	96	4	4	4	4	4	4
	2	50	52	54	56	56	57	50	48	46	44	44	43
	5	24	25	27	28	28	29	76	75	73	72	72	71
	10	16	16	17	17	17	17	84	84	83	83	83	83
	5000	+	3	-	-	-	-	+	92	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.