# A Appendix

## A.1 Additional Empirical Results

### A.1.1 Estimates of Covariances

| Parameter Estimates | | |
|---|---|---|
| Parameter | Estimate | Std Error |
| $\sigma_{xm} \times 10^2$ | $-5.76$ | 2.93 |
| $\sigma_{Xm} \times 10^7$ | 1.02 | 0.68 |
| $\sigma_{\Lambda m} \times 10^7$ | $-0.47$ | 0.30 |
| $\sigma_{\xi m} \times 10^2$ | $-5.23$ | 2.38 |
| $\sigma_{\xi \pi} \times 10^2$ | $-4.40$ | 16.80 |
| $\sigma_{\psi m} \times 10^3$ | 2.62 | 1.33 |
| $\sigma_{m\pi} \times 10^2$ | $-0.30$ | 11.10 |

### A.1.2 Additional Figures


**Figure A.1: Decomposing predicted TIPS variance and predicted stock-TIPS covariance.**

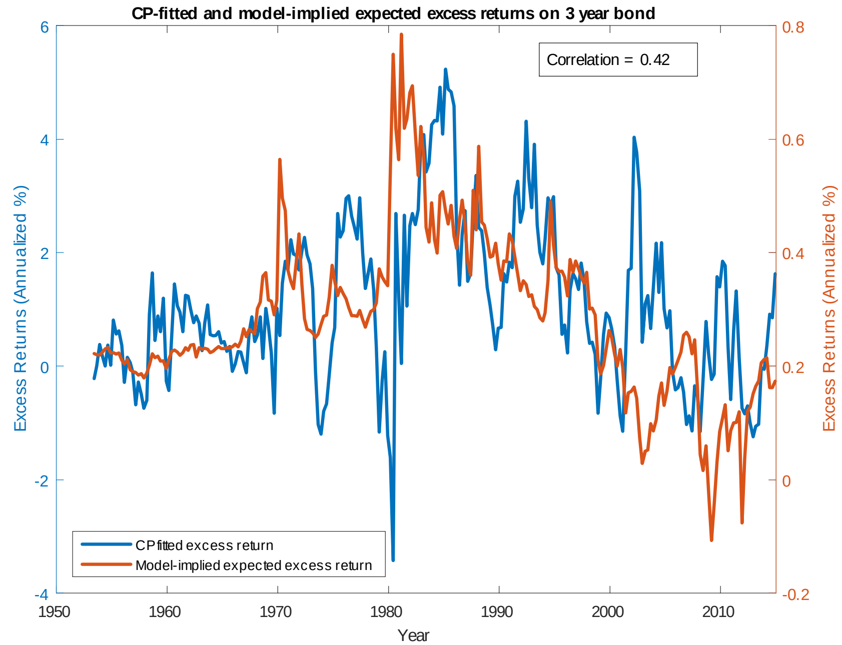


**Figure A.2: Time series of CP fitted and model-implied 3-year nominal bond excess returns.** The excess returns are over the 3-month Treasury bill rate.

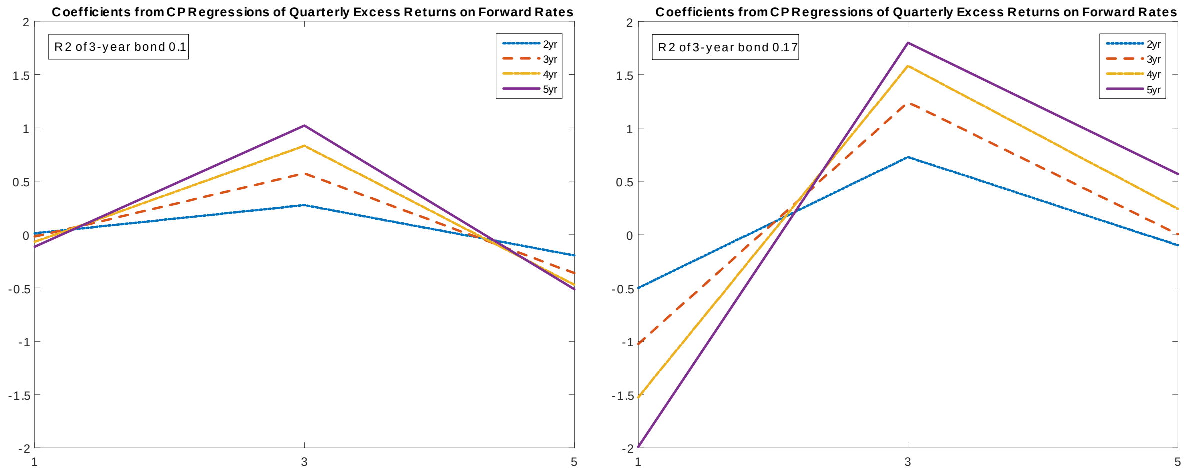


**Figure A.3: Coefficients from simulated Cochrane-Piazzesi regressions of yearly excess returns on forward rates (excess return is over 1-year rate).** In the left figure, the reported coefficients are the averages of coefficients from repeated regressions using 5000 simulated data series. The figure on the left is based on regressing yearly excess return (over 1-yr yield) on a bond on 1-year yield, 3-year forward rate, and 5-year forward rate. The reported R2 is the average R2 from the simulated regressions of excess returns on a 3-year bond on the single simulated CP factor. In the right figure, the reported coefficients are coefficients from the Cochrane-Piazzesi regressions using actual data. The figure on the left is based on regressing yearly excess return (over 1-yr yield) on a bond on 1-year yield, 3-year forward rate, and 5-year forward rate. The reported R2 is the R2 from the actual regression of excess return on a 3-year bond on the single CP factor.

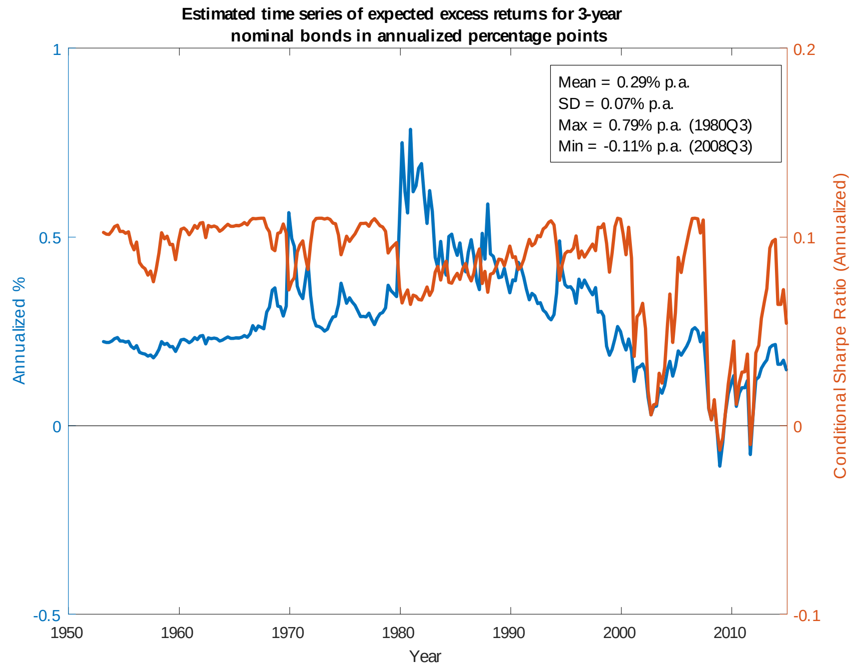


**Figure A.4: Estimated time series of expected excess returns for 10-year nominal bonds in annualized percentage points**. The excess return is over 3-month Treasury bill rate. The Sharpe ratio is computed as the conditional expected excess return over conditional standard deviation..

## A.2 Derivations for the Full Model

This section of the appendix reports the solution for a more general version of the model where we allow the volatility of the stochastic discount factor (SDF) to vary over time. The volatility of the SDF is controlled by the state variable $z_t$, which we model as following an AR(1) process. The solutions to the simplified model presented in the main text of the paper obtain when we set $z_t = 1$ and constant.

### A.2.1 State Variable Processes

The state variables in the model follow the processes:

$$\begin{aligned} -m_{t+1} &= x_t + \frac{1}{2} z_t^2 \sigma_m^2 + z_t \varepsilon_{m,t+1} \\ x_{t+1} &= \mu_x (1-\phi_x) + \phi_x x_t + \psi_t \varepsilon_{x,t+1} + \varepsilon_{X,t+1} \\ z_{t+1} &= \mu_z (1-\phi_z) + \phi_z z_t + \varepsilon_{z,t+1} \end{aligned}$$

$$\begin{aligned} \pi_{t+1} &= \lambda_t + \xi_t + \frac{1}{2} \psi_t^2 \sigma_\pi^2 + \psi_t \varepsilon_{\pi,t+1} \\ \lambda_{t+1} &= \lambda_t + \psi_t \varepsilon_{\lambda,t+1} + \varepsilon_{\Lambda,t+1} \\ \xi_{t+1} &= \phi_\xi \xi_t + \psi_t \varepsilon_{\xi,t+1} \\ \psi_{t+1} &= \mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t + \varepsilon_{\psi,t+1} \end{aligned}$$

### A.2.2 Pricing Equations

**Real Term Structure** The price of a single-period zero-coupon real bond satisfies

$$P_{1,t} = E_t \left[\exp\left\{m_{t+1}\right\}\right] = -x_t - \frac{1}{2} z_t^2 \sigma_m^2 + \frac{1}{2} z_t^2 \sigma_m^2 = -x_t$$

We conjecture that the price function is exponential affine in $x_t$ and $z_t$ with the form

$$P_{n,t} = \exp\left\{A_n + B_{x,n} x_t + B_{z,n} z_t + B_{\psi,n} \psi_t + C_{z,n} z_t^2 + C_{\psi,n} \psi_t^2 + C_{z\psi,n} z_t \psi_t\right\}.$$

The standard pricing equation implies

$$\begin{aligned} P_{n,t} &= E_t \left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t \left[\exp\left\{ \begin{array}{c} A_{n-1} + B_{x,n-1} x_{t+1} + B_{z,n-1} z_{t+1} + B_{\psi,n-1} \psi_{t+1} + C_{z,n-1} z_{t+1}^2 + C_{\psi,n-1} \psi_{t+1}^2 \\ + C_{z\psi,n-1} z_{t+1} \psi_{t+1} - x_t - \frac{1}{2} z_t^2 \sigma_m^2 - z_t \varepsilon_{m,t+1} \end{array} \right\}\right] \quad (1) \\ &= \exp\left\{ \begin{array}{c} A_{n-1} + B_{x,n-1} \left((1-\phi_x)\mu_x + \phi_x x_t\right) + B_{z,n-1} \left((1-\phi_z)\mu_z + \phi_z z_t\right) + B_{\psi,n-1} \left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi \psi_t\right) + C_{z,n-1} \left(\mu_z (1-\phi_z) + \phi_z z_t\right)^2 \\ + C_{\psi,n-1} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)^2 + C_{z\psi,n-1} \left(\mu_z (1-\phi_z) + \phi_z z_t\right) \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) - x_t - \frac{1}{2} z_t^2 \sigma_m^2 \end{array} \right\} \\ &\quad \times E_t \left[\exp\left\{\mathbf{d}_1' \boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}_{t+1}' \mathbf{D}_2 \boldsymbol{\omega}_{t+1}\right\}\right] \end{aligned}$$

where $\boldsymbol{\omega}'_{t+1} = (\varepsilon_{X,t+1}, \varepsilon_{m,t+1}, \varepsilon_{x,t+1}, \varepsilon_{z,t+1}, \varepsilon_{\psi,t+1})\ \tilde{}\ N(0, \boldsymbol{\Sigma}_{\omega})$,

$$\mathbf{d}_1 = \begin{pmatrix} B_{x,n-1} \\ -z_t \\ B_{x,n-1}\psi_t \\ B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t) \\ B_{\psi,n-1} + 2C_{\psi,n-1}(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) \end{pmatrix}$$

$$\mathbf{D}_2 = \begin{pmatrix} 0 & \cdots & & & 0 \\ \vdots & \ddots & & & \\ & & & & \\ & & & C_{z,n-1} & \frac{1}{2}C_{z\psi,n-1} \\ 0 & & & \frac{1}{2}C_{z\psi,n-1} & C_{\psi,n-1} \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$\begin{aligned} E_t\left[\exp\left\{\mathbf{d}'_1\boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}'_{t+1}\mathbf{D}_2\boldsymbol{\omega}_{t+1}\right\}\right] &= \frac{|\boldsymbol{\Sigma}_\omega|^{-1/2}}{\left|\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right|^{1/2}}\exp\left\{\frac{1}{2}\mathbf{d}_1\left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}\mathbf{d}'_1\right\} \\ &= \exp\left\{-\frac{1}{2}\log|\boldsymbol{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}\mathbf{d}_1\mathbf{G}\mathbf{d}'_1\right\} \end{aligned}$$

where $\mathbf{G} = \left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}$. Let $g_{ij}$ be the $ij$-th element of $\mathbf{G}$. Then expanding and collecting terms gives

$$p_{n,t} = \left[\begin{array}{c} A_{n-1} + B_{x,n-1}((1-\phi_x)\mu_x + \phi_x x_t) + B_{z,n-1}((1-\phi_z)\mu_z + \phi_z z_t) + B_{\psi,n-1}\left((1-\phi_\psi)\mu_\psi + \phi_\psi\psi_t\right) \\ +C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)^2 + C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)^2 + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) \\ -x_t - \frac{1}{2}z_t^2\sigma_m^2 - \frac{1}{2}\log|\boldsymbol{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{22}z_t^2 + \frac{1}{2}g_{33}B_{x,n-1}^2\psi_t^2 \\ +\frac{1}{2}g_{44}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right)^2 \\ +\frac{1}{2}g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right)^2 \\ -g_{12}B_{x,n-1}z_t + g_{13}B_{x,n-1}^2\psi_t + g_{14}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ +g_{15}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ -g_{23}B_{x,n-1}z_t\psi_t - g_{24}z_t\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ -g_{25}z_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ +g_{34}B_{x,n-1}\psi_t\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ +g_{35}B_{x,n-1}\psi_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ +g_{45}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ \times\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \end{array}\right]$$

Thus, equating coefficients across equation (1) yields

$$
\begin{aligned}
A_n &= \left[ \begin{array}{c}
A_{n-1} + B_{x,n-1}\left(1-\phi_x\right)\mu_x + B_{z,n-1}\left(1-\phi_z\right)\mu_z + B_{\psi,n-1}\left(1-\phi_\psi\right)\mu_\psi \\
+C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\
-\frac{1}{2}\log\left|\mathbf{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 \\
+\frac{1}{2}g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2 \\
+g_{14}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + g_{15}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g_{45}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array} \right] \\
B_{x,n} &= B_{x,n-1}\phi_x - 1 \\
B_{z,n} &= \left[ \begin{array}{c}
B_{z,n-1}\phi_z + 2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z + 2g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z \\
+g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z - g_{12}B_{x,n-1} + 2g_{14}B_{x,n-1}C_{z,n-1}\phi_z + g_{15}B_{x,n-1}C_{z\psi,n-1}\phi_z \\
-g_{24}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) - g_{25}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g_{45}\left[ \begin{array}{c}
2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array} \right]\phi_z
\end{array} \right] \\
B_{\psi,n} &= \left[ \begin{array}{c}
B_{\psi,n-1}\phi_\psi + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi + g_{13}B_{x,n-1}^2 + g_{14}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{15}B_{x,n-1}C_{\psi,n-1}\phi_\psi \\
+g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi \\
+2g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi \\
+g_{34}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + g_{35}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g_{45}\left[ \begin{array}{c}
2C_{\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C_{z\psi,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array} \right]\phi_\psi
\end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
C_{z,n} &= \left[C_{z,n-1}\phi_z^2 - \frac{1}{2}\sigma_m^2 + \frac{1}{2}g_{22} + 2g_{44}C_{z,n-1}^2\phi_z^2 + \frac{1}{2}g_{55}C_{z\psi,n-1}^2\phi_z^2 - 2g_{24}C_{z,n-1}\phi_z - g_{25}C_{z\psi,n-1}\phi_z + 2g_{45}C_{z,n-1}C_{z\psi,n-1}\phi_z^2\right] \\
C_{\psi,n} &= \left[C_{\psi,n-1}\phi_\psi^2 + \frac{1}{2}g_{33}B_{x,n-1}^2 + \frac{1}{2}g_{44}C_{z\psi,n-1}^2\phi_\psi^2 + 2g_{55}C_{\psi,n-1}^2\phi_\psi^2 + g_{34}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{35}B_{x,n-1}C_{\psi,n-1}\phi_\psi + 2g_{45}C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2\right] \\
C_{z\psi,n} &= \left[ \begin{array}{c}
C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{44}C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{55}C_{\psi,n-1}C_{z\psi,n-1}\phi_z\phi_\psi - g_{23}B_{x,n-1} - g_{24}C_{z\psi,n-1}\phi_\psi - 2g_{25}C_{\psi,n-1}\phi_\psi \\
+2g_{34}B_{x,n-1}C_{z,n-1}\phi_z + g_{35}B_{x,n-1}C_{z\psi,n-1}\phi_z + g_{45}C_{z\psi,n-1}^2\phi_\psi\phi_z
\end{array} \right]
\end{aligned}
$$

**Nominal Term Structure** The price of a single-period zero-coupon nominal bond satisfies

$$
P_{1,t}^{\$} = E_t\left[\exp\left\{m_{t+1} - \pi_{t+1}\right\}\right] = \exp\left\{-x_t - \lambda_t - \xi_t + z_t\psi_t\sigma_{m\pi}\right\}
$$

since $z_t\varepsilon_{m,t+1}$ and $\psi_t\varepsilon_{\pi,t+1}$ are jointly conditional normal.

We now guess that the price function is exponential linear-quadratic in the state variables with the following form:

$$
P_{n,t}^{\$} = \exp\left\{A_n^{\$} + B_{x,n}^{\$}x_t + B_{z,n}^{\$}z_t + B_{\lambda,n}^{\$}\lambda_t + B_{\xi,n}^{\$}\xi_t + B_{\psi,n}^{\$}\psi_t + C_{z,n}^{\$}z_t^2 + C_{\psi,n}^{\$}\psi_t^2 + C_{z\psi,n}^{\$}z_t\psi_t\right\}
$$

The standard pricing equation then implies

$$
\begin{aligned}
P^{\$}_{n,t} &= E_t\left[\exp\left\{p^{\$}_{n-1,t+1}+m_{t+1}-\pi_{t+1}\right\}\right] \qquad (2)\\
&= E_t\left[\exp\left\{\begin{array}{c} A^{\$}_{n-1}+B^{\$}_{x,n-1}x_{t+1}+B^{\$}_{z,n-1}z_{t+1}+B^{\$}_{\lambda,n-1}\lambda_{t+1}+B^{\$}_{\xi,n-1}\xi_{t+1}+B^{\$}_{\psi,n-1}\psi_{t+1} \\ +C^{\$}_{z,n-1}z^2_{t+1}+C^{\$}_{\psi,n-1}\psi^2_{t+1}+C^{\$}_{z\psi,n-1}z_{t+1}\psi_{t+1} \\ -x_t-\frac{1}{2}z_t^2\sigma_m^2-z_t\varepsilon_{m,t+1}-\lambda_t-\xi_t-\frac{1}{2}\psi_t^2\sigma_\pi^2-\psi_t\varepsilon_{\pi,t+1}\end{array}\right\}\right]\\
&= \exp\left\{\begin{array}{c} A^{\$}_{n-1}+B^{\$}_{x,n-1}\left(\mu_x\left(1-\phi_x\right)+\phi_x x_t\right)+B^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+B^{\$}_{\lambda,n-1}\left(\mu_\lambda+\lambda_t\right)+B^{\$}_{\xi,n-1}\phi_\xi\xi_t+B^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ +C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)^2+C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)^2+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ -x_t-\frac{1}{2}z_t^2\sigma_m^2-\lambda_t-\xi_t-\frac{1}{2}\psi_t^2\sigma_\pi^2\end{array}\right\}\\
&\quad\times E_t\left[\exp\left\{\mathbf{d}_1^{\$\prime}\boldsymbol{\omega}^{\$}_{t+1}+\boldsymbol{\omega}^{\$\prime}_{t+1}\mathbf{D}^{\$}_2\boldsymbol{\omega}^{\$}_{t+1}\right\}\right]
\end{aligned}
$$

where $\boldsymbol{\omega}^{\$\prime}_{t+1}=\left(\varepsilon_{X,t+1},\varepsilon_{\Lambda,t+1},\varepsilon_{\lambda,t+1},\varepsilon_{m,t+1},\varepsilon_{\pi,t+1},\varepsilon_{x,t+1},\varepsilon_{\xi,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right)\tilde{}N\left(0,\boldsymbol{\Sigma}^{\$}_\omega\right)$,

$$
\mathbf{d}^{\$}_1=\begin{pmatrix} B^{\$}_{x,n-1} \\ B^{\$}_{\lambda,n-1} \\ B^{\$}_{\lambda,n-1}\psi_t \\ -z_t \\ -\psi_t \\ B^{\$}_{x,n-1}\psi_t \\ B^{\$}_{\xi,n-1}\psi_t \\ B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\end{pmatrix}
$$

$$
\mathbf{D}^{\$}_2=\begin{pmatrix} 0 & \cdots & & 0 \\ & & & \vdots \\ \vdots & \ddots & & \\ & & C^{\$}_{z,n-1} & \frac{1}{2}C^{\$}_{z\psi,n-1} \\ 0 & \cdots & \frac{1}{2}C^{\$}_{z\psi,n-1} & C^{\$}_{\psi,n-1}\end{pmatrix}
$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$
E_t\left[\exp\left\{\mathbf{d}_1^{\$\prime}\boldsymbol{\omega}^{\$}_{t+1}+\boldsymbol{\omega}^{\$\prime}_{t+1}\mathbf{D}^{\$}_2\boldsymbol{\omega}^{\$}_{t+1}\right\}\right]=\exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_\omega\right|+\frac{1}{2}\log\left|\mathbf{G}^{\$}\right|+\frac{1}{2}\mathbf{d}^{\$}_1\mathbf{G}^{\$}\mathbf{d}^{\$\prime}_1\right\}
$$

where $\mathbf{G}^{\$} = \left(\mathbf{\Sigma}_{\omega}^{\$-1} - 2\mathbf{D}_{2}^{\$}\right)^{-1}$. Let $g_{ij}^{\$}$ be the $ij$-th element of $\mathbf{G}$. Then expanding and collecting terms gives $g^{\$}$

$$p_{n,t}^{\$} = \left[\begin{array}{c}
A_{n-1}^{\$} + B_{x,n-1}^{\$}\left(\mu_x\left(1-\phi_x\right) + \phi_x x_t\right) + B_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + B_{\lambda,n-1}^{\$}\lambda_t + B_{\xi,n-1}^{\$}\phi_\xi\xi_t + B_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \\
+C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)^2 + C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \\
-x_t - \frac{1}{2}z_t^2\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 - \frac{1}{2}\log\left|\mathbf{\Sigma}_\omega^{\$}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}g_{11}^{\$}B_{x,n-1}^{\$2} + \frac{1}{2}g_{22}^{\$}B_{\lambda,n-1}^{\$2} + \frac{1}{2}g_{33}^{\$}B_{\lambda,n-1}^{\$2}\psi_t^2 + \frac{1}{2}g_{44}^{\$}z_t^2 \\
+\frac{1}{2}g_{55}^{\$}\psi_t^2 + \frac{1}{2}g_{66}^{\$}B_{x,n-1}^{\$2}\psi_t^2 + \frac{1}{2}g_{77}^{\$}B_{\xi,n-1}^{\$2}\psi_t^2 + \frac{1}{2}g_{88}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)^2 \\
+\frac{1}{2}g_{99}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)^2 + g_{12}B_{x,n-1}^{\$}B_{\lambda,n-1}^{\$} + g_{13}B_{x,n-1}^{\$}B_{\lambda,n-1}^{\$}\psi_t \\
-g_{14}B_{x,n-1}^{\$}z_t - g_{15}B_{x,n-1}^{\$}\psi_t + g_{16}B_{x,n-1}^{\$2}\psi_t + g_{17}B_{x,n-1}^{\$}B_{\xi,n-1}^{\$}\psi_t \\
+g_{18}B_{x,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\
+g_{19}B_{x,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) + g_{23}^{\$}B_{\lambda,n-1}^{\$2}\psi_t - g_{24}^{\$}B_{\lambda,n-1}^{\$}z_t - g_{25}^{\$}B_{\lambda,n-1}^{\$}\psi_t \\
+g_{26}^{\$}B_{\lambda,n-1}^{\$}B_{x,n-1}^{\$}\psi_t + g_{27}^{\$}B_{\lambda,n-1}^{\$}B_{\xi,n-1}^{\$}\psi_t + g_{28}^{\$}B_{\lambda,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\
+g_{29}^{\$}B_{\lambda,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) - g_{34}^{\$}B_{\lambda,n-1}^{\$}z_t\psi_t - g_{35}^{\$}B_{\lambda,n-1}^{\$}\psi_t^2 \\
+g_{36}^{\$}B_{\lambda,n-1}^{\$}B_{x,n-1}^{\$}\psi_t^2 + g_{37}^{\$}B_{\lambda,n-1}^{\$}B_{\xi,n-1}^{\$}\psi_t^2 + g_{38}^{\$}B_{\lambda,n-1}^{\$}\psi_t\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\
+g_{39}^{\$}B_{\lambda,n-1}^{\$}\psi_t\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) + g_{45}^{\$}z_t\psi_t - g_{46}^{\$}B_{x,n-1}^{\$}z_t\psi_t \\
-g_{47}^{\$}B_{\xi,n-1}^{\$}z_t\psi_t - g_{48}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)z_t \\
-g_{49}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)z_t - g_{56}^{\$}B_{x,n-1}^{\$}\psi_t^2 - g_{57}^{\$}B_{\xi,n-1}^{\$}\psi_t^2 \\
-g_{58}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)\psi_t \\
-g_{59}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\psi_t + g_{67}^{\$}B_{x,n-1}^{\$}B_{\xi,n-1}^{\$}\psi_t^2 \\
+g_{68}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)B_{x,n-1}^{\$}\psi_t \\
+g_{69}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)B_{x,n-1}^{\$}\psi_t \\
+g_{78}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)B_{\xi,n-1}^{\$}\psi_t \\
+g_{79}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)B_{\xi,n-1}^{\$}\psi_t \\
+g_{89}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\
\times\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)
\end{array}\right]$$

Thus, the coefficients of the pricing equation satisfy

$$
A^{\$}_{n} = \left[\begin{array}{c}
A^{\$}_{n-1} + B^{\$}_{x,n-1}\mu_x\left(1-\phi_x\right) + B^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + B^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C^{\$}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\
-\frac{1}{2}\log\left|\mathbf{\Sigma}^{\$}_{\omega}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}g^{\$}_{11}B^{\$2}_{x,n-1} + \frac{1}{2}g^{\$}_{22}B^{\$2}_{\lambda,n-1} + g_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + g_{18}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+g_{19}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) + \frac{1}{2}g^{\$}_{88}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 \\
+\frac{1}{2}g^{\$}_{99}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2 + g^{\$}_{28}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\lambda,n-1} \\
+g^{\$}_{29}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1} \\
+g^{\$}_{89}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]
$$

$$
\begin{aligned}
B^{\$}_{x,n} &= B^{\$}_{x,n-1}\phi_x - 1 \\
B^{\$}_{\lambda,n} &= B^{\$}_{\lambda,n-1} - 1 \\
B^{\$}_{\xi,n} &= B^{\$}_{\xi,n-1}\phi_\xi - 1
\end{aligned}
$$

$$
B^{\$}_{z,n} = \left[\begin{array}{c}
\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\phi_z - g_{14}B^{\$}_{x,n-1} + 2g_{18}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + g_{19}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z - g^{\$}_{24}B^{\$}_{\lambda,n-1} \\
+2g^{\$}_{88}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\phi_z + g^{\$}_{99}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\phi_z \\
+2g^{\$}_{28}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z + g^{\$}_{29}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z - g^{\$}_{48}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
-g^{\$}_{49}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g^{\$}_{89}\left(2C^{\$}_{z,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) + C^{\$}_{z\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\right)\phi_z
\end{array}\right]
$$

$$
B^{\$}_{\psi,n} = \left[\begin{array}{c}
\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\phi_\psi + g_{13}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} - g_{15}B^{\$}_{x,n-1} + g_{16}B^{\$2}_{x,n-1} + g_{17}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + g_{18}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_\psi \\
+2g_{19}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi g^{\$}_{23}B^{\$2}_{\lambda,n-1} - g^{\$}_{25}B^{\$}_{\lambda,n-1} + g^{\$}_{26}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + g^{\$}_{27}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} + g^{\$}_{28}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{29}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi \\
+g^{\$}_{38}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + g^{\$}_{39}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g^{\$}_{88}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{99}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\phi_\psi \\
-g^{\$}_{58}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) - g^{\$}_{59}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+g^{\$}_{68}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{x,n-1} + g^{\$}_{69}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{x,n-1} \\
+g^{\$}_{78}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\xi,n-1} + g^{\$}_{79}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\xi,n-1} \\
+g^{\$}_{89}\left(2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{\psi,n-1} + \left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\right)\phi_\psi
\end{array}\right]
$$

$$C^{\$}_{z,n} = \left[C^{\$}_{z,n-1}\phi_z^2 - \frac{1}{2}\sigma_m^2 + \frac{1}{2}g^{\$}_{44} + 2g^{\$}_{88}C^{\$2}_{z,n-1}\phi_z^2 + \frac{1}{2}g^{\$}_{99}C^{\$2}_{z\psi,n-1}\phi_z^2 - 2g^{\$}_{48}C^{\$}_{z,n-1}\phi_z - g^{\$}_{49}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{89}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z^2\right]$$

$$C^{\$}_{\psi,n} = \left[\begin{array}{c} \frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1} + g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + 2g^{\$}_{39}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi + g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} - g^{\$}_{35}B^{\$}_{\lambda,n-1} \\ +\frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1} + C^{\$}_{\psi,n-1}\phi_\psi^2 - \frac{1}{2}\sigma_\pi^2 + \frac{1}{2}g^{\$}_{55} + \frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1} + \frac{1}{2}g^{\$}_{88}C^{\$2}_{z\psi,n-1}\phi_\psi^2 + 2g^{\$}_{99}C^{\$2}_{\psi,n-1}\phi_\psi^2 - g^{\$}_{56}B^{\$}_{x,n-1} - g^{\$}_{57}B^{\$}_{\xi,n-1} \\ -g^{\$}_{58}C^{\$}_{z\psi,n-1}\phi_\psi - 2g^{\$}_{59}C^{\$}_{\psi,n-1}\phi_\psi + g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{69}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi \\ +g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{79}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_\psi + 2g^{\$}_{89}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_\psi^2 \end{array}\right]$$

$$C^{\$}_{z\psi,n} = \left[\begin{array}{c} g^{\$}_{39}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z - g^{\$}_{34}B^{\$}_{\lambda,n-1} + C^{\$}_{z\psi,n-1}\phi_z\phi_\psi + g^{\$}_{45} - g^{\$}_{46}B^{\$}_{x,n-1} - g^{\$}_{47}B^{\$}_{\xi,n-1} + 2g^{\$}_{88}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_\psi \\ +2g^{\$}_{99}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_\psi - g^{\$}_{48}C^{\$}_{z\psi,n-1}\phi_\psi - 2g^{\$}_{49}C^{\$}_{\psi,n-1}\phi_\psi - 2g^{\$}_{58}C^{\$}_{z,n-1}\phi_z - g^{\$}_{59}C^{\$}_{z\psi,n-1}\phi_z \\ +2g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + g^{\$}_{69}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{z,n-1}\phi_z + g^{\$}_{79}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_z + g^{\$}_{89}\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\phi_z\phi_\psi \end{array}\right]$$

where $B^{\$}_{x,1} = -1, B^{\$}_{\lambda,1} = -1$, $B^{\$}_{\xi,1} = -1$, $C^{\$}_{z\psi,1} = \sigma_{m\pi}$ and all other coefficients are zero at $n = 1$.

### A.2.3 Expected Excess Returns

**Real Bond Premia** The log expected gross excess return on an $n$-period zero-coupon real bond is

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right] = \log E_t\left[\exp\left\{p_{n-1.t+1}-p_{n,t}\right\}\right]-x_t$$

$$= \begin{bmatrix} A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) \\ +C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\ +\left(B_{x,n-1}\phi_x-B_{x,n}-1\right)x_t+\left(C_{z,n-1}\phi_z^2-C_{z,n}\right)z_t^2+\left(C_{\psi,n-1}\phi_\psi^2-C_{\psi,n}\right)\psi_t^2+\left(C_{z\psi,n-1}\phi_z\phi_\psi-C_{z\psi,n}\right)z_t\psi_t \\ +\left(B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{bmatrix}$$

$$+\log E_t\left[\exp\left\{\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1}+B_{x,n-1}\varepsilon_{X,t+1}+C_{z,n-1}\varepsilon_{z,t+1}^2+C_{\psi,n-1}\varepsilon_{\psi,t+1}^2+C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right\}\right]$$

since the shocks are conditionally jointly normal. Note that the coefficient recursion implies that $B_{x,n}=B_{x,n-1}\phi_x-1$ so that the terms involving $x_t$ drop out. Following Campbell, Chan, and Viceira (2003), we calculate the expectation by completing the square. Let $\boldsymbol{\nu}'=\left(\varepsilon_{X,t+1},\varepsilon_{x,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right)\tilde{}N\left(0,\boldsymbol{\Sigma}_v\right)$,

$$\mathbf{f}_1=\begin{pmatrix} B_{x,n-1} \\ B_{x,n-1}\psi_t \\ \left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \end{pmatrix}$$

$$\mathbf{F}_2=\begin{pmatrix} 0 & 0 & 0 \\ \\ 0 & C_{z,n-1} & \frac{1}{2}C_{z\psi,n-1} \\ 0 & \frac{1}{2}C_{z\psi,n-1} & C_{\psi,n-1} \end{pmatrix}$$

Then

$$E_t\left[\exp\left\{\mathbf{f}_1'\boldsymbol{\nu}+\boldsymbol{\nu}'\mathbf{F}_2\boldsymbol{\nu}\right\}\right]=\exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}\mathbf{f}_1\mathbf{H}\mathbf{f}_1'\right\}$$

where $\mathbf{H}=\left(\boldsymbol{\Sigma}_\nu^{-1}-2\mathbf{F}_2\right)^{-1}$.

Let $h_{ij}$ be the $ij$-th element of $\mathbf{H}$. Then expanding and collecting terms gives

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right]=\left[\begin{array}{c} A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) \\ +C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\ +\left(C_{z,n-1}\phi_z^2-C_{z,n}\right)z_t^2+\left(C_{\psi,n-1}\phi_\psi^2-C_{\psi,n}\right)\psi_t^2+\left(C_{z\psi,n-1}\phi_z\phi_\psi-C_{z\psi,n}\right)z_t\psi_t \\ +\left(B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \\ -\frac{1}{2}\log\left|\mathbf{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}h_{11}B_{x,n-1}^2+\frac{1}{2}h_{22}B_{x,n-1}^2\psi_t^2 \\ +\frac{1}{2}h_{33}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2 \\ +\frac{1}{2}h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)^2 \\ +h_{12}B_{x,n-1}^2\psi_t+h_{13}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ +h_{14}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\ +h_{23}B_{x,n-1}\psi_t\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ +h_{24}B_{x,n-1}\psi_t\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\ +h_{34}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \times\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \end{array}\right]$$

Thus, we can write

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right]=\kappa_n+\eta_{z,n}z_t+\eta_{\psi,n}\psi_t+\beta_{z,n}z_t^2+\beta_{\psi,n}\psi_t^2+\beta_{z\psi,n}z_t\psi_t$$

where the coefficients are given by

$$\kappa_n=\left[\begin{array}{c} A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ +C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)-\frac{1}{2}\log\left|\mathbf{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}h_{11}B_{x,n-1}^2+\frac{1}{2}h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 \\ +\frac{1}{2}h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2+h_{13}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +h_{14}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +h_{34}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \end{array}\right]$$

$$\eta_{z,n}=\left[\begin{array}{c} B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z+2h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z \\ +h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z+2h_{13}B_{x,n-1}C_{z,n-1}\phi_z+h_{14}B_{x,n-1}C_{z\psi,n-1}\phi_z \\ +h_{34}\left[2C_{z,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)+C_{z\psi,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\right]\phi_z \end{array}\right]$$

$$\eta_{\psi,n}=\left[\begin{array}{c} B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi+h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi \\ +2h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi+h_{12}B_{x,n-1}^2\psi_t+h_{13}B_{x,n-1}C_{z\psi,n-1}\phi_\psi\psi_t+2h_{14}B_{x,n-1}C_{\psi,n-1}\phi_\psi\psi_t \\ +h_{23}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)+h_{24}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +h_{34}\left[2C_{\psi,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)+C_{z\psi,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\right]\phi_\psi \end{array}\right]$$

$$\beta_{z,n} = \left[ \left( C_{z,n-1}\phi_z^2 - C_{z,n} \right) + 2h_{33}C_{z,n-1}^2\phi_z^2 + \frac{1}{2}h_{44}C_{z\psi,n-1}^2\phi_z^2 + 2h_{34}C_{z,n-1}C_{z\psi,n-1}\phi_z^2 \right]$$

$$\beta_{\psi,n} = \left[ \left( C_{\psi,n-1}\phi_\psi^2 - C_{\psi,n} \right) + \frac{1}{2}h_{33}C_{z\psi,n-1}^2\phi_\psi^2 + 2h_{44}C_{\psi,n-1}^2\phi_\psi^2 + h_{23}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2h_{24}B_{x,n-1}C_{\psi,n-1}\phi_\psi + h_{34}2C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2 \right]$$

$$\beta_{z\psi,n} = \left[ \begin{array}{c} \left( C_{z\psi,n-1}\phi_z\phi_\psi - C_{z\psi,n} \right) + 2h_{33}C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi + 2h_{44}C_{\psi,n-1t}C_{z\psi,n-1}\phi_z\phi_\psi + 2h_{23}B_{x,n-1}C_{z,n-1}\phi_z \\ +h_{24}B_{x,n-1}C_{z\psi,n-1}\phi_z + h_{34}C_{z\psi,n-1}^2\phi_\psi\phi_z \end{array} \right]$$

**Nominal Bond Premia** The log conditional expected real return on a 1-period zero-coupon nominal bond is

$$E_t \left[ r_{1,t+1}^{\$} - \pi_{t+1} \right] = -\sigma_{m,\pi} z_t \psi_t$$

The log conditional expected gross excess return on an $n$-period zero-coupon nominal bond is

$$\begin{aligned}
\log E_t \left[ \frac{P_{n-1,t+1}^{\$}}{P_{n,t}^{\$}} \right] - E_t \left[ r_{1,t+1}^{\$} \right] &= \log E_t \left[ \exp \left\{ p_{n-1.t+1}^{\$} - p_{n,t}^{\$} \right\} \right] - x_t - \lambda_t - \xi_t + \sigma_{m,\pi} z_t \psi_t \\
&= \left[ \begin{array}{c} A_{n-1}^{\$} - A_n^{\$} + B_{x,n-1}^{\$}\mu_x (1-\phi_x) + B_{z,n-1}^{\$}\mu_z (1-\phi_z) + B_{\psi,n-1}^{\$}\mu_\psi \left(1-\phi_\psi\right) \\ +C_{z,n-1}^{\$}\mu_z^2 (1-\phi_z)^2 + C_{\psi,n-1}^{\$}\mu_\psi^2 \left(1-\phi_\psi\right)^2 + C_{z\psi,n-1}^{\$}\mu_z (1-\phi_z)\mu_\psi \left(1-\phi_\psi\right) \\ + \left( B_{x,n-1}^{\$}\phi_x - B_{x,n}^{\$} - 1 \right) x_t + \left( B_{\lambda,n-1}^{\$} - B_{\lambda,n}^{\$} - 1 \right) \lambda_t + \left( B_{\xi,n-1}^{\$}\phi_\xi - B_{\xi,n}^{\$} - 1 \right) \xi_t \\ + \left( C_{z,n-1}^{\$}\phi_z^2 - C_{z,n}^{\$} \right) z_t^2 + \left( C_{\psi,n-1}^{\$}\phi_\psi^2 - C_{\psi,n}^{\$} \right) \psi_t^2 + \left( \sigma_{m,\pi} + C_{z\psi,n-1}^{\$}\phi_z\phi_\psi - C_{z\psi,n}^{\$} \right) z_t\psi_t \\ + \left( B_{z,n-1}^{\$}\phi_z - B_{z,n}^{\$} + 2C_{z,n-1}^{\$}\mu_z (1-\phi_z)\phi_z + C_{z\psi,n-1}^{\$}\mu_\psi \left(1-\phi_\psi\right)\phi_z \right) z_t \\ + \left( B_{\psi,n-1}^{\$}\phi_\psi - B_{\psi,n}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi \left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}^{\$}\mu_z (1-\phi_z)\phi_\psi \right) \psi_t \end{array} \right] \\
&\quad + \log E_t \left[ \exp \left\{ \begin{array}{c} B_{x,n-1}^{\$}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}^{\$}\varepsilon_{X,t+1} + B_{\lambda,n-1}^{\$}\psi_t\varepsilon_{\lambda,t+1} + B_{\lambda,n-1}^{\$}\varepsilon_{\Lambda,t+1} + B_{\xi,n-1}^{\$}\psi_t\varepsilon_{\xi,t+1} \\ +C_{z,n-1}^{\$}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}^{\$}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}^{\$}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left( B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z (1-\phi_z) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \right) \varepsilon_{z,t+1} \\ + \left( B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z (1-\phi_z) + \phi_z z_t\right) \right) \varepsilon_{\psi,t+1} \end{array} \right\} \right]
\end{aligned}$$

Note that the coefficient recursions imply that $B_{x,n}^{\$} = B_{x,n-1}^{\$}\phi_x - 1$, $B_{\lambda,n}^{\$} = B_{\lambda,n-1}^{\$} - 1$,and $B_{\xi,n}^{\$} = B_{\xi,n-1}^{\$}\phi_\xi - 1$, so that the terms involving $x_t$, $\lambda_t$, and $\xi_t$ drop out. Following Campbell, Chan, and Viceira (2003), we calculate the expectation by completing the square. Let

$$\boldsymbol{\nu}^{\$\prime} = \left(\varepsilon_{X,t+1},\varepsilon_{\Lambda,t+1},\varepsilon_{x,t+1},\varepsilon_{\lambda,t+1},\varepsilon_{\xi,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right) \tilde{} N\left(0,\boldsymbol{\Sigma}_{v}^{\$}\right),$$

$$\mathbf{f}_1^{\$} = \begin{pmatrix} B_{x,n-1}^{\$} \\ B_{\lambda,n-1}^{\$} \\ B_{x,n-1}^{\$}\psi_t \\ B_{\lambda,n-1}^{\$}\psi_t \\ B_{\xi,n-1}^{\$}\psi_t \\ \left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \end{pmatrix}$$

$$\mathbf{F}_2^{\$} = \begin{pmatrix} 0 & \cdots & & 0 \\ & \ddots & & \\ \vdots & & & \\ & & C_{z,n-1}^{\$} & \frac{1}{2}C_{z\psi,n-1}^{\$} \\ 0 & & \frac{1}{2}C_{z\psi,n-1}^{\$} & C_{\psi,n-1}^{\$} \end{pmatrix}$$

Then

$$E_t\left[\exp\left\{\mathbf{f}_1^{\$\prime}\boldsymbol{\nu}^{\$} + \boldsymbol{\nu}^{\$\prime}\mathbf{F}_2^{\$}\boldsymbol{\nu}^{\$}\right\}\right] = \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_{\nu}^{\$}\right| + \frac{1}{2}\log\left|\mathbf{H}^{\$}\right| + \frac{1}{2}\mathbf{f}_1^{\$}\mathbf{H}^{\$}\mathbf{f}_1^{\$\prime}\right\}$$

where $\mathbf{H}^{\$} = \left(\boldsymbol{\Sigma}_{\nu}^{\$-1} - 2\mathbf{F}_2^{\$}\right)^{-1}$.

Let $h_{ij}^{\$}$ be the $ij$-th element of $\mathbf{H}^{\$}$. Then expanding and collecting terms gives

$$
\log E_t\left[\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}\right]-E_t\left[r^{\$}_{1,t+1}\right]=\left[\begin{array}{c}
A^{\$}_{n-1}-A^{\$}_{n}+B^{\$}_{x,n-1}\mu_x\left(1-\phi_x\right)+B^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+B^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2\\
+C^{\$2}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)+\left(C^{\$}_{z,n-1}\phi_z^2-C^{\$}_{z,n}\right)z_t^2+\left(C^{\$}_{\psi,n-1}\phi_\psi^2-C^{\$}_{\psi,n}\right)\psi_t^2\\
+\left(\sigma_{m,\pi}+C^{\$}_{z\psi,n-1}\phi_z\phi_\psi-C^{\$}_{z\psi,n}\right)z_t\psi_t+\left(B^{\$}_{z,n-1}\phi_z-B^{\$}_{z,n}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t\\
+\left(B^{\$}_{\psi,n-1}\phi_\psi-B^{\$}_{\psi,n}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t-\frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_{\nu}\right|+\frac{1}{2}\log\left|\mathbf{H}^{\$}\right|+\frac{1}{2}h^{\$}_{11}B^{\$2}_{x,n-1}\\
+\frac{1}{2}h^{\$}_{22}B^{\$2}_{\lambda,n-1}+\frac{1}{2}h^{\$}_{33}B^{\$2}_{x,n-1}\psi_t^2+\frac{1}{2}h^{\$}_{44}B^{\$2}_{\lambda,n-1}\psi_t^2+\frac{1}{2}h^{\$}_{55}B^{\$2}_{\xi,n-1}\psi_t^2\\
+\frac{1}{2}h^{\$}_{66}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2\\
+\frac{1}{2}h^{\$}_{77}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)^2\\
+h^{\$}_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}+h^{\$}_{13}B^{\$2}_{x,n-1}\psi_t+h^{\$}_{14}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t+h^{\$}_{15}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t\\
+h^{\$}_{16}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{17}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{23}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t+h_{24}B^{\$2}_{\lambda,n-1}\psi_t+h^{\$}_{25}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t\\
+h^{\$}_{26}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{27}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)+h^{\$}_{34}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t^2\\
+h^{\$}_{35}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t^2+h^{\$}_{36}B^{\$}_{x,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{37}B^{\$}_{x,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)+h^{\$}_{45}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t^2\\
+h^{\$}_{46}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{47}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{56}B^{\$}_{\xi,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{57}B^{\$}_{\xi,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{67}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
\times\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)
\end{array}\right]
$$

Thus, we can write

$$
\log E_t\left[\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}\right]-E_t\left[r^{\$}_{1,t+1}\right]=\kappa^{\$}_{n}+\eta^{\$}_{z,n}z_t+\eta^{\$}_{\psi,n}\psi_t+\beta^{\$}_{z,n}z_t^2+\beta^{\$}_{\psi,n}\psi_t^2+\beta^{\$}_{z\psi,n}z_t\psi_t
$$

where the coefficients are given by

$$\kappa_n^{\$} = \left[\begin{array}{c}
A^{\$}_{n-1} - A^{\$}_{n} + B^{\$}_{x,n-1}\mu_x\left(1-\phi_x\right) + B^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + B^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C^{\$2}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\
+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) - \frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_{\nu}\right| + \frac{1}{2}\log\left|\mathbf{H}^{\$}\right| + \frac{1}{2}h^{\$}_{11}B^{\$2}_{x,n-1} + \frac{1}{2}h^{\$}_{22}B^{\$2}_{\lambda,n-1} \\
+\frac{1}{2}h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 + \frac{1}{2}h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2 \\
+h^{\$}_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{16}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+h^{\$}_{17}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{26}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+h^{\$}_{27}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{67}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]$$

$$\eta^{\$}_{z,n} = \left[\begin{array}{c}
B^{\$}_{z,n-1}\phi_z - B^{\$}_{z,n} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z \\
+2h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\phi_z \\
+h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\phi_z \\
+2h^{\$}_{16}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{17}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z 2h^{\$}_{26}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{27}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z \\
+h^{\$}_{67}\left(\begin{array}{c}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\phi_z \\ +2C^{\$}_{z,n-1}\phi_z\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\end{array}\right)
\end{array}\right]$$

$$\eta^{\$}_{\psi,n} = \left[\begin{array}{c}
\left(B^{\$}_{\psi,n-1}\phi_\psi - B^{\$}_{\psi,n} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right) + h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\phi_\psi \\
+2h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\phi_\psi + h^{\$}_{13}B^{\$2}_{x,n-1} + h^{\$}_{14}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{15}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} \\
+h^{\$}_{16}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2h^{\$}_{17}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi + h^{\$}_{23}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + h^{\$}_{24}B^{\$2}_{\lambda,n-1} + h^{\$}_{25}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} + h^{\$}_{26}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2h^{\$}_{27}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi \\
+h^{\$}_{36}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{37}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{46}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{47}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{56}B^{\$}_{\xi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{57}B^{\$}_{\xi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{67}\left[\begin{array}{c}2C^{\$}_{\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +C^{\$}_{z\psi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\end{array}\right]\phi_\psi
\end{array}\right]$$

$$\beta^{\$}_{z,n} = \left[C^{\$}_{z,n-1}\phi_z^2 - C^{\$}_{z,n} + 2h^{\$}_{66}C^{\$2}_{z,n-1}\phi_z^2 + \frac{1}{2}h^{\$}_{77}C^{\$2}_{z\psi,n-1}\phi_z^2 + 2h^{\$}_{67}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z^2\right]$$

$$\beta^{\$}_{\psi,n} = \left[ \begin{array}{c} C^{\$}_{\psi,n-1}\phi^2_{\psi} - C^{\$}_{\psi,n} + \frac{1}{2}h^{\$}_{33}B^{\$2}_{x,n-1} + \frac{1}{2}h^{\$}_{44}B^{\$2}_{\lambda,n-1} + \frac{1}{2}h^{\$}_{55}B^{\$2}_{\xi,n-1} + \frac{1}{2}h^{\$}_{66}C^{\$2}_{z\psi,n-1}\phi^2_{\psi} + 2h^{\$}_{77}C^{\$2}_{\psi,n-1}\phi^2_{\psi} \\ +h^{\$}_{34}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{35}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + h^{\$}_{36}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{37}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} \\ +h^{\$}_{45}B^{\$}_{\xi,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{46}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{47}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} \\ +h^{\$}_{56}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{57}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + 2h^{\$}_{67}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi^2_{\psi} \end{array} \right]$$

$$\beta^{\$}_{z\psi,n} = \left[ \begin{array}{c} \sigma_{m,\pi} + C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} - C^{\$}_{z\psi,n} + 2h^{\$}_{66}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} + 2h^{\$}_{77}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} + 2h^{\$}_{35}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z \\ +2h^{\$}_{36}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{37}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2h^{\$}_{46}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{47}B^{\$}_{\lambda,n-1}C_{z\psi,n-1}\phi_z + 2h^{\$}_{56}B^{\$}_{\xi,n-1}C^{\$}_{z,n-1}\phi_z \\ +h^{\$}_{57}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_z + h^{\$}_{67}\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\phi_{\psi}\phi_z \end{array} \right]$$

### A.2.4 Observation Equations

**Stock Returns** We model the unexpected stock return as

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

We impose that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$. The standard pricing equation then implies that the expected equity return satisfies

$$\begin{aligned} 1 &= E_t\left[\exp\left(r_{e,t+1} + m_{t+1}\right)\right] \\ &= \exp\left(E_t r_{e,t+1} - x_t - \frac{1}{2}z_t^2\sigma_m^2\right)\exp\left(\begin{array}{c} \frac{1}{2}\beta_{ex}^2\sigma_x^2 + \frac{1}{2}\beta_{eX}^2\sigma_X^2 + \frac{1}{2}\beta_{em}^2\sigma_m^2 + \frac{1}{2}z_t^2\sigma_m^2 \\ +\beta_{ex}\beta_{em}\sigma_{xm} - \beta_{ex}z_t\sigma_{xm} + \beta_{eX}\beta_{em}\sigma_{X,m} - \beta_{eX}z_t\sigma_{Xm} - \beta_{em}z_t\sigma_m^2 \end{array}\right) \end{aligned}$$

so that

$$r_{e,t+1} = -\frac{1}{2}\beta_{ex}^2\sigma_x^2 - \frac{1}{2}\beta_{eX}^2\sigma_X^2 - \frac{1}{2}\beta_{em}^2\sigma_m^2 - \beta_{ex}\beta_{em}\sigma_{xm} - \beta_{eX}\beta_{em}\sigma_{X,m} + x_t + \left(\beta_{ex}\sigma_{xm} + \beta_{eX}\sigma_{Xm} + \beta_{em}\sigma_m^2\right)z_t + \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

and

$$E_t\left[r_{e,t+1} - r_{1,t+1}\right] + \frac{1}{2}Var_t\left[r_{e,t+1} - r_{1,t+1}\right] = \left(\beta_{ex}\sigma_{xm} + \beta_{eX}\sigma_{Xm} + \beta_{em}\sigma_m^2\right)z_t$$

**Stock-Real Bond Return Covariance** As we saw above, the holding period return on an $n$-period real bond is

$$\begin{aligned} r_{n,t+1} - r_{1,t+1} &= p_{n-1,t+1} - p_{n,t} - r_{1,t+1} \\ &= \left[\begin{array}{c} A_{n-1} - A_n + B_{x,n-1}\mu_x\left(1-\phi_x\right) + B_{z,n-1}\mu_z\left(1-\phi_z\right) + B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ +C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) + \left(B_{x,n-1}\phi_x - B_{x,n} - 1\right)x_t + \left(C_{z,n-1}\phi_z^2 - C_{z,n}\right)z_t^2 + \left(C_{\psi,n-1}\phi_\psi^2 - C_{\psi,n}\right)\psi_t^2 \\ +\left(C_{z\psi,n-1}\phi_z\phi_\psi - C_{z\psi,n}\right)z_t\psi_t + \left(B_{z,n-1}\phi_z - B_{z,n} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}\phi_\psi - B_{\psi,n} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{array}\right] \\ &\quad + \left[\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}\varepsilon_{X,t+1} + C_{z,n-1}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right] \end{aligned}$$

We assume that the unexpected stock return is assumed to be

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

Since the $\varepsilon$'s are conditionally jointly normal and mean zero we have $Cov_t\left(\varepsilon_{a,t+1}, \varepsilon_{b,t+1}^2\right) = 0$ and $Cov_t\left(\varepsilon_{a,t+1}, \varepsilon_{b,t+1}\varepsilon_{c,t+1}\right) = 0$ for all $a, b, c$. Furthermore, we impose that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$.Thus, the expression for the conditional covariance of stock returns with returns on a long-term real bond is

$$\begin{aligned}
Cov_t\left(r_{e,t+1}, r_{n,t+1}\right) &= \beta_{ex}\begin{pmatrix} \left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z} \\ +\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi} \end{pmatrix} \\
&+\beta_{eX}B_{x,n-1}\sigma_X^2 \\
&+\beta_{em}\begin{pmatrix} B_{x,n-1}\sigma_{Xm}+\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m} \\ +\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m} \end{pmatrix} \\
&+\left[\beta_{ex}\left(2C_{z,n-1}\sigma_{xz}\phi_z+C_{z\psi,n-1}\sigma_{x\psi}\phi_z\right)+\beta_{em}\left(2C_{z,n-1}\sigma_{zm}\phi_z+C_{z\psi,n-1}\sigma_{\psi m}\phi_z\right)\right]z_t \\
&+\left[\beta_{ex}\left(B_{x,n-1}\sigma_x^2+C_{z\psi,n-1}\sigma_{xz}\phi_\psi+2C_{\psi,n-1}\sigma_{x\psi}\phi_\psi\right)+\beta_{em}\left(B_{x,n-1}\sigma_{xm}+C_{z\psi,n-1}\sigma_{zm}\phi_\psi+2C_{\psi,n-1}\sigma_{\psi m}\phi_\psi\right)\right]\psi_t
\end{aligned}$$

**Stock-Nominal Bond Return Covariance** As we saw above, the holding period return on an $n$-period nominal bond is

$$\begin{aligned}
r^\$_{n,t+1}-r^\$_{1,t+1} &= p^\$_{n-1,t+1}-p^\$_{n,t}-r^\$_{1,t+1} \\
&= \begin{bmatrix} A^\$_{n-1}-A^\$_n+B^\$_{x,n-1}\mu_x\left(1-\phi_x\right)+B^\$_{z,n-1}\mu_z\left(1-\phi_z\right)+B^\$_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^\$_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C^{\$2}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ +C^\$_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)+\left(B^\$_{x,n-1}\phi_x-B^\$_{x,n}-1\right)x_t+\left(B^\$_{\xi,n-1}\phi_\xi-B^\$_{\xi,n}-1\right)\xi_t \\ +\left(C^\$_{z,n-1}\phi_z^2-C^\$_{z,n}\right)z_t^2+\left(C^\$_{\psi,n-1}\phi_\psi^2-C^\$_{\psi,n}\right)\psi_t^2+\left(\sigma_{m,\pi}+C^\$_{z\psi,n-1}\phi_z\phi_\psi-C^\$_{z\psi,n}\right)z_t\psi_t \\ +\left(B^\$_{z,n-1}\phi_z-B^\$_{z,n}+2C^\$_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C^\$_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B^\$_{\psi,n-1}\phi_\psi-B^\$_{\psi,n}+2C^\$_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C^\$_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{bmatrix} \\
&+\begin{bmatrix} B^\$_{x,n-1}\psi_t\varepsilon_{x,t+1}+B^\$_{x,n-1}\varepsilon_{X,t+1}+B^\$_{\lambda,n-1}\psi_t\varepsilon_{\lambda,t+1}+B^\$_{\lambda,n-1}\varepsilon_{\Lambda,t+1}+B^\$_{\xi,n-1}\psi_t\varepsilon_{\xi,t+1} \\ +C^\$_{z,n-1}\varepsilon^2_{z,t+1}+C^\$_{\psi,n-1}\varepsilon^2_{\psi,t+1}+C^\$_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B^\$_{z,n-1}+2C^\$_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^\$_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B^\$_{\psi,n-1}+2C^\$_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^\$_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{bmatrix}
\end{aligned}$$

We assume that the unexpected stock return is assumed to be

$$r_{e,t+1}-E_t r_{e,t+1}=\beta_{ex}\varepsilon_{x,t+1}+\beta_{eX}\varepsilon_{X,t+1}+\beta_{em}\varepsilon_{m,t+1}$$

Thus, the conditional covariance with the real return on short term nominal bond is

$$Cov_t\left(r_{e,t+1}, r^\$_{1,t+1}-\pi_{t+1}\right)=Cov\left(\beta_{ex}\varepsilon_{x,t+1}+\beta_{eX}\varepsilon_{X,t+1}+\beta_{em}\varepsilon_{m,t+1}, -\psi_t\varepsilon_{\pi,t+1}\right)=-\psi_t\left(\beta_{ex}\sigma_{x\pi}+\beta_{em}\sigma_{m\pi}\right)$$

since we impose the condition that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$.

Again, the $\varepsilon$'s are conditionally jointly normal and mean zero we have $Cov_t\left(\varepsilon_{a,t+1},\varepsilon_{b,t+1}^2\right)=0$ and $Cov_t\left(\varepsilon_{a,t+1},\varepsilon_{b,t+1}\varepsilon_{c,t+1}\right)=0$ for all $a,b,c$. Additionally, note that we impose $\sigma_{x,\Lambda}=0$ and that the only non-zero covariance of $\varepsilon_{\Lambda,t+1}$ is $\sigma_{\Lambda,m}$. Thus, the conditional covariance of stock returns with the returns on a long term nominal bond is

$$\begin{aligned}
Cov_t\left(r_{e,t+1},r_{n,t+1}^{\$}\right) = {} & \beta_{ex}\left(\begin{array}{c}\left(B_{z,n-1}^{\$}+2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z}\\ +\left(B_{\psi,n-1}^{\$}+2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi}\end{array}\right)\\
& +\beta_{eX}B_{x,n-1}^{\$}\sigma_X^2\\
& +\beta_{em}\left(\begin{array}{c}B_{x,n-1}^{\$}\sigma_{Xm}+B_{\lambda,n-1}^{\$}\sigma_{\Lambda m}+\left(B_{z,n-1}^{\$}+2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m}\\ +\left(B_{\psi,n-1}^{\$}+2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m}\end{array}\right)\\
& +\left[\beta_{ex}\left(2C_{z,n-1}^{\$}\sigma_{xz}\phi_z+C_{z\psi,n-1}^{\$}\sigma_{x\psi}\phi_z\right)+\beta_{em}\left(2C_{z,n-1}^{\$}\sigma_{zm}\phi_z+C_{z\psi,n-1}^{\$}\sigma_{\psi m}\phi_z\right)\right]z_t\\
& +\left[\begin{array}{c}\beta_{ex}\left(B_{x,n-1}^{\$}\sigma_x^2+B_{\lambda,n-1}^{\$}\sigma_{x,\lambda}+B_{\xi,n-1}^{\$}\sigma_{x,\xi}+C_{z\psi,n-1}^{\$}\sigma_{xz}\phi_\psi+2C_{\psi,n-1}^{\$}\sigma_{x\psi}\phi_\psi\right)\\ +\beta_{em}\left(B_{x,n-1}^{\$}\sigma_{xm}+B_{\lambda,n-1}^{\$}\sigma_{m,\lambda}+B_{\xi,n-1}^{\$}\sigma_{m,\xi}+C_{z\psi,n-1}^{\$}\sigma_{zm}\phi_\psi+2C_{\psi,n-1}^{\$}\sigma_{\psi m}\phi_\psi\right)\end{array}\right]\psi_t
\end{aligned}$$

**Volatility of Real Bond Returns** We have

$$r_{n,t+1} - E_t r_{n,t+1} = \left[\begin{array}{c} B_{x,n-1}\psi_t \varepsilon_{x,t+1} + B_{x,n-1}\varepsilon_{X,t+1} + C_{z,n-1}\varepsilon^2_{z,t+1} + C_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right)\varepsilon_{z,t+1} \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right]$$

so that

$$\begin{aligned} Var_t\left(r_{n,t+1}\right) \quad = \quad & \left[\begin{array}{c} B^2_{x,n-1}\sigma^2_X + 2C^2_{z,n-1}2\sigma^4_z + 2C^2_{\psi,n-1}\sigma^4_\psi + C^2_{z\psi,n-1}\left(\sigma^2_z\sigma^2_\psi + \sigma^2_{z\psi}\right) + \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma^2_z \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma^2_\psi \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \times \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi} \end{array}\right] \\ & + \left[\begin{array}{c} 4\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z\sigma^2_z \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z\sigma^2_\psi \\ +2\left[\begin{array}{c} 2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \end{array}\right]\phi_z\sigma_{z,\psi} \end{array}\right] z_t \\ & + \left[\begin{array}{c} 2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi\sigma^2_z \\ +4\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi\sigma^2_\psi \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B_{x,n-1}\sigma_{xz} \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B_{x,n-1}\sigma_{x\psi} \\ +2\left[\begin{array}{c} 2C_{\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +C_{z\psi,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \end{array}\right]\phi_\psi\sigma_{z,\psi} \end{array}\right]\psi_t \\ & + \left[4C^2_{z,n-1}\phi^2_z\sigma^2_z + C^2_{z\psi,n-1}\phi^2_z\sigma^2_\psi + 4C_{z,n-1}C_{z\psi,n-1}\phi^2_z\sigma_{z,\psi}\right]z^2_t \\ & + \left[B^2_{x,n-1}\sigma^2_x + C^2_{z\psi,n-1}\phi^2_\psi\sigma^2_z + 4C^2_{\psi,n-1}\phi^2_\psi\sigma^2_\psi + 2C_{z\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{xz} + 4C_{\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{x\psi} + 4C_{\psi,n-1}C_{z\psi,n-1}\phi^2_\psi\sigma_{z,\psi}\right]\psi^2_t \\ & + \left[\begin{array}{c} 4C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi\sigma^2_z + 4C_{\psi,n-1}\phi_\psi C_{z\psi,n-1}\phi_z\phi_\psi\sigma^2_\psi + 4C_{z,n-1}\phi_z B_{x,n-1}\sigma_{xz} \\ +2C_{z\psi,n-1}\phi_z B_{x,n-1}\sigma_{x\psi} + 2\left(4C_{z,n-1}C_{\psi,n-1} + C^2_{z\psi,n-1}\right)\sigma_{z\psi}\phi_\psi\phi_z \end{array}\right]z_t\psi_t \end{aligned}$$

**Volatility of Nominal Bond Returns** We have

$$r^{\$}_{n,t+1} - E_t r^{\$}_{n,t+1} = \left[ \begin{array}{c} B^{\$}_{x,n-1}\psi_t \varepsilon_{x,t+1} + B^{\$}_{x,n-1}\varepsilon_{X,t+1} + B^{\$}_{\lambda,n-1}\psi_t \varepsilon_{\lambda,t+1} + B^{\$}_{\lambda,n-1}\varepsilon_{\Lambda,t+1} + B^{\$}_{\xi,n-1}\psi_t \varepsilon_{\xi,t+1} \\ + C^{\$}_{z,n-1}\varepsilon^2_{z,t+1} + C^{\$}_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C^{\$}_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left( B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right)\varepsilon_{z,t+1} \\ + \left( B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array} \right]$$

so that

$$
\begin{aligned}
Var_t\left(r^{\$}_{n,t+1}\right) &= \left[\begin{array}{c}
B^{\$2}_{x,n-1}\sigma^2_X + B^{\$2}_{\lambda,n-1}\sigma^2_\Lambda + 2C^{\$2}_{z,n-1}\sigma^4_z + 2C^{\$2}_{\psi,n-1}\sigma^4_\psi + C^{\$2}_{z\psi,n-1}\left(\sigma^2_z\sigma^2_\psi + \sigma^2_{z\psi}\right) \\
+\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma^2_z \\
+\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma^2_\psi \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi}
\end{array}\right] \\
&+ \left[\begin{array}{c}
4\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\sigma^2_z\phi_z \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_\psi\phi_z \\
+2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_z \\
+2\left[\begin{array}{c}
2C^{\$}_{z,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_z
\end{array}\right] z_t \\
&+ \left[\begin{array}{c}
2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{x,n-1}\sigma_{xz} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{x,n-1}\sigma_{x\psi} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_z\phi_\psi \\
+4\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\sigma^2_\psi\phi_\psi \\
+2B^{\$2}_{\lambda,n-1}\sigma_{\lambda,\Lambda} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\lambda,n-1}\sigma_{z,\lambda} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda} \\
+2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\Lambda,\xi} + 4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_\psi \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\xi,n-1}\sigma_{\xi,z} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\xi,n-1}\sigma_{\psi,\xi} \\
+2\left[\begin{array}{c}
2C^{\$}_{\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_\psi
\end{array}\right]\psi_t
\end{aligned}
$$

$$+\left[4C^{\$2}_{z,n-1}\phi_z^2\sigma_z^2+C^{\$2}_{z\psi,n-1}\phi_z^2\sigma_\psi^2+4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_{z,\psi}\phi_z^2\right]z_t^2$$

$$+\left[\begin{array}{c} B^{\$2}_{x,n-1}\sigma_x^2+B^{\$2}_{\lambda,n-1}\sigma_\lambda^2+B^{\$2}_{\xi,n-1}\sigma_\xi^2+2B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\sigma_{x,\lambda}+2B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\sigma_{x,\xi} \\ +2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_\psi+4C^{\$}_{\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_\psi+C^{\$2}_{z\psi,n-1}\phi_\psi^2\sigma_z^2+4C^{\$2}_{\psi,n-1}\phi_\psi^2\sigma_\psi^2+2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi\lambda}+2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{z,\lambda}\phi_\psi \\ +4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda}\phi_\psi+2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_\psi+4C^{\$}_{\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_\psi+4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma_{z,\psi}\phi_\psi^2 \end{array}\right]\psi_t^2$$

$$+\left[\begin{array}{c} 4C^{\$}_{z,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_z+2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_z+4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_z^2\phi_z\phi_\psi \\ +4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma_\psi^2\phi_z\phi_\psi+4C^{\$}_{z,n-1}B^{\$}_{\lambda,n-1}\sigma_{z,\lambda}\phi_z+2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda}\phi_z \\ +4C^{\$}_{z,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_z+2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_z+2\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1}+C^{\$2}_{z\psi,n-1}\right)\sigma_{z\psi}\phi_\psi\phi_z \end{array}\right]z_t\psi_t$$