

# Minimal Settlement Assets in Economies with Interconnected Financial Obligations

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## Abstract

A model is developed where firms belonging to a group are obliged to make payments to one another by using a liquid asset. The paper studies the exogenous endowments of this asset that are necessary to assure that all obligations are met. Conditions are presented under which the degree to which firms are interconnected (so that each creditor has more debtors and each debtor has more creditors) increases the number of firms that must be endowed with the liquid asset. Interconnectedness then makes payment defaults more likely. By acquiring too many payment obligations, firms may also become too interconnected. (JEL G20, D53, D85)

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Many financial institutions are expected to make and receive payments from each other. The funds that these firms receive from other financial institutions can often be used to make subsequent payments on the same day. This means that complete settlement of a day's obligations is possible even if firms do not start out with an endowment of settlement funds that equals their individual obligations and even if they do not borrow additional settlement funds during the day. But, what properties must this endowment have if one wants to make sure that all obligations are fulfilled? This paper takes up this question and pays particular attention to how this required endowment varies as firms become more interconnected in the sense of having to make payments to a larger collection of other firms.

The setup discussed in this paper is meant to resemble a stripped down version of a large value payments systems (LVPS). Firms belonging to actual LVPS's make payments to each other, and the daily volume of payments that they handle is enormous relative to the cash that they start out with at the beginning of the day. As discussed in Bech, Preisig and Soramäki (2008), most pre-1990 LVPS systems were based on netting (so that firms only had to pay their net obligation at the end of the day). Participants' fears of the difficulties caused by bank failures in this system led to their replacement by "Real-Time Gross Settlement" (RTGS) systems. In RTGS systems, payments are cleared continuously and a receiving bank treats an obligation as settled only after a sending bank gives up funds in its possession. In many real-world interbank systems of this sort, central banks play a crucial role by providing ample intraday credit to banks wishing to make payments. In order to reduce this provision of liquidity, several LVPS's have now modified their RTGS algorithms so that they are able to net some obligations during the course of the day.<sup>1</sup>

The focus of this paper is essentially a "pure" RTGS system without the provision of central bank intraday loans. As discussed by Martin (2005), such a system was used in Switzerland between 1987 and 1999 and led to considerable delays. According to Nield

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<sup>1</sup>See Martin and McAndrews (2008) for a model showing why this is not necessarily an improvement.

(2006), similar problems arose in New Zealand during 1999-2006, when banks held few overnight funds and the central bank provided only limited intraday credit. These problems waned when banks were encouraged to hold more overnight reserves.

The analysis of Bech and Soramäki (2001) clarifies an important source of these difficulties. As they show, pure RTGS systems are paralyzed by what they term “gridlock” when bank  $A$  lacks  $X$  dollars that it wishes to pay to bank  $B$ , who lacks  $X$  dollars that it wishes to pay to bank  $C$ , who in turn lacks  $X$  dollars that it wishes to pay to bank  $A$ . When I focus on settlement that can be reused without limit, I effectively analyze the amount of funds that the system needs to avoid this kind of gridlock. This is an analytical counterpart to the simulations carried out in Bech and Soramäki (2001).

While stylized, the model presented in this paper may also give insights into the way debts would be settled among financial firms if the government were not involved.<sup>2</sup> It may thus provide a useful benchmark against which one can compare financial systems with government intervention in the settlement process. A remarkable aspect of settlements systems is that the magnitude of their operations has grown tremendously in recent years. Between 1996 and 2008, Fedwire transfers grew by an average of 9.7% per year even as nominal GDP grew by an average of 5.2% per year.

Portfolio decisions by financial firms may explain some of this growth. To see this, imagine that firm  $A$  borrows 100\$ from  $B$ , which borrows them from  $C$ , which in turn borrows them from  $A$ . The result is that required interest and principal payments must flow from  $A$  to  $B$  to  $C$  and back to  $A$ . Flows identical to these can be induced by coupling a loan from  $C$  to  $B$  with transactions involving credit default swaps. Suppose, in particular, that  $A$  “buys protection” from  $B$  on security  $X$ , so that  $A$  commits itself to make payments to  $B$  as long as  $X$  does not default. Suppose further that  $A$  subsequently offsets this position by selling

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<sup>2</sup>See Martin and McAndrews (2010) for a general discussion of the desirability of intraday loans provided by central banks.

protection on  $X$  to  $C$ . As long as  $X$  does not default,  $A$  is then a conduit of funds from  $C$  to  $B$ .  $B$ 's repayments on its loan from  $C$  then complete the circle.

In “real” models of financial intermediation such as Allen and Gale (2000), repayments are most easily thought of as taking the form of transfers of consumption goods. When it comes to payments that flow from  $A$  to  $B$  to  $C$  and back to  $A$ , however, transfers of physical consumption goods would be wasteful. Financial entities avoid this waste in practice by making payments using claims on monetary assets. This is partly for legal reasons, as legal tender laws privilege such assets as a means to settle obligations. I thus consider a financial system that has an exogenous endowment of assets that can be used to make payments.

At the risk of some terminological confusion, I call this the financial system's endowment of “liquidity.”<sup>3</sup> An alternative would be to call this the system's endowment of money.<sup>4</sup> I choose the broader term liquidity because this recognizes that, in many institutional contexts, certain assets can be traded cheaply and rapidly with outsiders so as to obtain immediate settlement funds.<sup>5</sup> An important implication of this is that changes in the institutional context can reduce the system's effective endowment of settlement funds even if the balance sheets of financial institutions contain an unchanged level of monetary assets. This would be true, for example, if outsiders become less willing to accept the assets of financial institutions as collateral for immediate loans.<sup>6</sup> It follows that the settlement problems that are emphasized

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<sup>3</sup>Martin and McAndrews (2008) also use the term liquidity to denote assets that are useful for settlement. As noted in footnotes 4 and 7, the term has been used for a variety of different, if related phenomena in the academic literature.

<sup>4</sup>Let  $q_{mi}$  represent the amount of money (currency or bank funds) one must pay to buy asset  $i$  while  $q_{im}$  is the quantity of asset  $i$  one must pay to buy one unit of money immediately. By one measure, asset  $i$  is liquid if its bid-ask spread in terms of money  $q_{mi} - 1/q_{im}$  is low. Since  $q_{mm} = 1$ , money is very liquid by this measure. Since money is used to buy goods and assets, the same would be true if the bid-ask spread were measured in terms of the number of goods that have to be given up to buy (indirectly) an asset minus the number of goods one could buy (indirectly) by selling an asset.

<sup>5</sup>This was true, for example, of government securities in the New Zealand environment described in Nield (2006). At the time, the central bank was willing to make intraday loans collateralized by these securities.

<sup>6</sup>The collapse in the repo market described in Gorton and Metrick (2009) might be an example of this. While the repo market is not a market for intraday funds, repo funds can be available for use on the same day that the contract is agreed upon.

in this paper are more likely to be relevant when liquidity problems of the various kinds discussed by Tirole (2010) become more severe.<sup>7</sup>

I study a group of firms whose contracts require them to make payments to other members of the group. To focus on settlement problems, I abstract from intertemporal lending and solvency by supposing that the obligations of each firm within a period equal the payments that the firm is expected to receive.<sup>8</sup> In this idealized environment, the canceling of all obligations in each period can be accomplished by “netting” the obligations of each firm or by providing short term loans, each of which would be repaid as soon as the borrowing firm receives payment from other firms. These solutions may not be available when financial market participants fear insolvency. Even with full rationality by all participants, fear of insolvency may exist even in states of nature where firms are not actually insolvent. This fear then leads to the settlement problems I discuss.

This solvency ensures that a trivial amount of liquidity is sufficient to clear all obligations if the sequence of individual payments is chosen by an omniscient planner and if there is no bound on the number of times a unit of liquidity can be reused. In this case, the extent to which financial firms are interconnected does not matter. This conclusion no longer holds when the sequence of payments is chosen in a less sophisticated manner, as might be expected if this sequence is chosen by a series of financial institutions, each of which knows little about the full set of obligations that need to be settled. In that case, there is a “worst case scenario” in which the degree of interconnection increases the minimum liquidity that is needed to ensure that all obligations are settled. In this paper, a financial system is treated

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<sup>7</sup>While recognizing that there are ambiguities involved, he attempts to distinguish between deteriorations in funding liquidity (difficulties in increasing the level of unsecured liabilities) and in market liquidity (difficulties in converting certain assets into cash).

<sup>8</sup>This sets this paper apart from the literature that studies whether interconnections increase the likelihood that a shock to the balance sheet of one firm will lead to systemic failure. The literature on this issue is extensive (e.g Allen and Gale 2000, Freixas *et al.* 2000, Eisenberg and Noe 2001, Cifuentes *et al.* 2005, Nier *et al.* 2007). The current paper is perhaps most closely related to the last three, because they also use graphs to model financial interconnections.

as being more interconnected if, for given sets of gross individual debts, each firm has more (net) creditors. Interconnections then give payees more choice regarding whom to pay. This choice is subject to an externality which, to my knowledge, has not been noted before. This is that a firm's choice of payee affects whether the given amount of liquidity will permit all obligations to clear (as when a planner is in charge) or not.

If there is a bound on the number of times that liquidity can be reused, even an omniscient planner guiding payments encounters more difficulties in achieving full settlement when each debtor has more creditors. When each debtor has just one creditor, it is sufficient for a relatively small number of firms to be endowed with liquidity. As creditors are added to each debtor while keeping total debt the same, it becomes necessary to endow more debtors with liquidity to ensure that all obligations are settled.<sup>9</sup>

These results establish that changes in the pattern of firms' obligations have implications for the settlement process as a whole. If these obligations were all exogenous, as in the case of banks who make transfers on behalf of their non-financial customers, there would be little to add. As mentioned above, however, many of the payments that financial firms must make to one another are the result of earlier commitments that they themselves took on. This raises the question of whether the process of taking on these commitments is inefficient in light of its implications for the settlement process.

To make this concrete, consider the example of how broker-dealers respond to the "termination" offers made by TriOptima. TriOptima has software that finds trades in Credit

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<sup>9</sup>There is an interesting connection between this paper and the more abstract treatment of monetary exchange in Ostroy and Starr (1974). They consider a situation where a set of agents has a vector of endowments and must make a vector of net trades to achieve a Walrasian allocation. Ostroy and Starr (1974) show that having each agent barter once with every other agent is not generally enough to achieve this allocation, as long as the exchanges that agents have in each bilateral encounter do not rely on information about the history of other agents' trades. They prove that, by contrast, a single round of bilateral meetings is sufficient if each agent's endowment of a "monetary" good is large enough to cover the cost of all his purchases. They leave open the question of whether smaller monetary endowments can also accomplish this. If one interprets my setting as one where firms wish to "repurchase" the coupons comprising their current obligations, the results indicate that somewhat less liquidity is needed even under fairly adverse conditions.

Default Swaps that can be eliminated through netting (such as when  $A$  buys protection on a particular entity from  $B$ , who buys it from  $C$  who buys it from  $A$ ). After TriOptima finds sets of trades can be mutually terminated, dealers still get to decide which of these candidate trades to terminate. Now consider a set of trades whose elimination would increase the likelihood that all obligations could be settled with the liquidity available to the financial system. The question is whether dealers would properly internalize this benefit in their decision to terminate trades.<sup>10</sup> In the model's version of this decision, they do not, even when they are aware of the beneficial effect of these terminations on future settlements. The reason is that some of these benefits accrue to dealers other than the ones that terminate their trades.

To establish this, a model is needed where firms derive some benefit from establishing and maintaining interconnected financial links with one another. Unfortunately, existing models of financial interconnections do not appear to provide a solid rationale for this. In variants of the Allen and Gale (2000) model, for example, interconnections result from the desire of financial institutions to insure each other against idiosyncratic shocks. As they show, a simple way of implementing this insurance is through demandable deposits, which imply that only banks that have suffered negative shocks receive any payments.<sup>11</sup> This leads me to introduce a simple model in which firms have exogenous reasons, *i.e.*, "tastes," to wish to take different positions in financial markets. The results will hopefully extend to other reasons for maintaining interconnections, including differences in beliefs about the performance of different assets or different interests concerning the accounting consequences of unwinding trades.

The paper proceeds as follows. Section 1 contrasts the standard model of intermediaries

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<sup>10</sup>While the TriOptima system leads to many trade terminations, not all of its suggestions are accepted by dealers.

<sup>11</sup>The same is true in Freixas *et al.* (2000), who use credit lines instead. Kahn and Santos (2009) consider the case where firms consolidate to provide mutual support. This presumably also implies that funds flow only to banks that need them. In Wagner (2010), banks are concerned with idiosyncratic shocks to the value of their assets. Wagner (2010) lets them address this by trading assets with one another which, in principle, does not require transfers of funds across financial institutions.

where these channel funds from ultimate borrowers to ultimate lenders (see Diamond (1984) for a classic example) with a setting where there are required payment “cycles” among intermediaries. A simple cycle would be a situation where  $A$  is expected to make a payment to  $B$ , who is expected to make an equal payment to  $C$ , who is expected to make an equal payment to  $A$ .<sup>12</sup>

Section 2 considers a more interconnected system where each firm is expected to make an equal-sized payment to  $K$  other financial firms. The graph of the resulting obligations is shown to be Eulerian, a property that extends to most situations where each firm’s obligations equal its claims and where firms in a group only owe funds to other members of the group. Obligations that can be described by an Eulerian graph have the property that, when there is no limit on the number of times a unit of liquidity can be reused within a period, a central planner guiding payments can clear all debts with an arbitrarily small dose of liquidity. After showing this, I turn to decentralized outcomes. Section 3 endogenizes the debt structure of Section 2. Section 4 considers a setting where there is an exogenous limit on the number of times that a unit of liquidity can be used to settle obligations within a period. Section 5 offers some concluding remarks.

## 1 Setting the stage: Vertical lending versus debt cycles

A simple, and standard, view of financial intermediaries is that these channel funds from ultimate lenders to ultimate borrowers. As ultimate borrowers repay their obligations, intermediaries are able to repay their obligations to ultimate lenders as well. If contracts are simple and intermediaries are solvent in the sense of having claims on borrowers that equal

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<sup>12</sup>An interesting model that appears to contain a chain of this sort is presented in Kahn, McAndrews and Roberds (2003). They array firms in a circle and let each firm produce a final good using as input an endowment owned by the firm that precedes it. The result is that each firm ends up with an obligation to the firm that precedes it, and the set of these obligations looks like a cycle of obligations. In this model, however, each borrower has enough final output to pay his creditor. This means that, in the absence of the voluntary defaults that are the focus of Kahn, McAndrews and Roberds’ (2003) analysis, the cyclical nature of loans has no effect on repayments.

their liabilities to lenders, the capacity of all ultimate borrowers to repay their debts assures that all intermediaries are able to settle their own obligations as well. To see this, start with a trivial example where a lender has a claim of  $z$  against an intermediary, who in turn has a claim of  $z$  against a borrower. When the borrower repays the  $z$  that he owes, the intermediary is able to fulfill his obligation as well.

This result readily extends to more general situations where claims are “vertical,” so that intermediaries only channel repayments from ultimate borrowers to ultimate lenders. To see this, suppose that there are  $N$  firms indexed by  $i$ . Let  $d^{ij}$  denote the amount that firm  $i$  is expected to pay firm  $j$  and let all these obligations be multiples of  $z$ .

It is helpful to notice that one can represent these obligations with a *directed graph*  $\mathcal{G}$  where the vertices represent firms and where there is an “edge” going from vertex  $i$  to vertex  $j$  whenever  $i$  owes  $z$  to  $j$ . Debts that are multiples of  $z$  are represented by multiple identical edges. A certain amount of graph-theoretic nomenclature proves useful. The *in-degree* of a vertex is the number of edges that end at this vertex while its *out-degree* is the number of edges that originate from it. Further, a directed graph is connected if, for every pair of vertices  $i$  and  $j$ , one can reach  $i$  from  $j$  by traveling from the origin to the destination of a series of edges. When traveling in this way, a *cycle* denotes a set of edges that constitute a path from one vertex back to the same vertex.

A generalized “vertical” system of obligations can be represented by a graph that contains no cycles, and Figure 1 shows an example of such a graph. An acyclic graph must have some sources (the  $B_i$ ’s) and some sinks (the  $L_i$ ’s), where the former are vertices with an in-degree of zero and the latter have an out-degree of zero. In the current context, the sources are net borrowers and sinks are net lenders. Intermediaries are represented by vertices that are neither sources nor sinks, and Figure 1 displays three of them. It is apparent that intermediaries can have obligations to each other in such a graph. To avoid cycles, however, an intermediary that receives funds from a second cannot have debts towards intermediaries

that have obligations towards this second intermediary.

I suppose that funds paid by one firm to another can be used by the receiver to make further payments “within the period.” The idea is that obligations are due on a particular calendar day while money can be reused multiple times within the day. As the following proposition indicates, the result is that all obligations are met when firms are solvent and there are no cycles.

**Proposition 1.** *Let the graph describing the obligations of firms be acyclic. Then all debts are settled if firms are solvent.*

*Proof.* Solvency ensures that all sources are able to meet their obligations while also implying that any vertex that is neither a source or a sink has an in-degree equal to its out-degree. Let an edge from  $i$  to  $j$  be removed whenever  $i$  makes a payment of  $z$  to  $j$  so that full repayment of all obligations takes place when all edges are removed.

Start with the payment of  $z$  from a source to one of its creditors and remove the corresponding edge. Now let this particular creditor make a payment to one of his own creditors and remove the corresponding edge and continue making payments and removing edges until this particular payment reaches an ultimate lender. Next remove the edges associated with another payment from a source and, when this payment reaches an ultimate lender, continue in the same fashion until all sources have made all their required payments. It should be apparent that the graph contains no further edges. The reason is that any remaining edge would either have to originate from a source (which is impossible) or from a firm that still has an extant obligation that is traceable to a source (which is equally impossible).  $\square$

Once financial firms move beyond vertical lending and trade claims against each other, cycles are likely to arise. A simple cycle is generated if, for example, firm 1 owes  $z$  to firm 2, who owes  $z$  to firm 3, who owes  $z$  to firm 1. These obligations leave all three firms “solvent” but unable to settle their debts without any outside source of liquidity. In this particular

case, the needed liquidity might be obtained by inserting one of them into a vertical lending relationship. This is depicted in panel (a) of Figure 2, which combines the cycle I just described with a debt of  $z$  from  $B$  to 1 and a corresponding debt of  $z$  from 1 to  $L$ . Now, when  $B$  repays his debt, firm 1 can first repay firm 2, which repays firm 3, which then makes  $z$  available to firm 1 so that it can repay  $L$ . Thus, all debts can be settled by the simple device of giving firm 1 access to liquidity from outside the system consisting of  $\{1, 2, 3\}$ .

While this device can be effective, it is not infallible. Its success requires that firm 1 repay firm 2 before it repays  $L$ . In this simple case, it may seem obvious that this is in firm 1's interest. However, consider the simple variant depicted in panel (b) of Figure 2. Here, firm 1 does not owe funds to an identifiable ultimate lender  $L$  but to firm 4 who in turn owes funds to  $L$ . If firm 1 does not know the creditors of firm 2 and 4, he may sometimes pay firm 4 before he pays firm 2. It might then be necessary to give firm 1 additional sources of liquidity to guarantee that all debts are settled. This example demonstrates that, when there exist horizontal debt ties, full repayment by ultimate borrowers is no longer sufficient to ensure the settlement of all debts.

One potential way to proceed at this point would be to consider more general debt patterns that include both vertical relationships and cycles. Because the analysis becomes intractable quickly, I follow a simpler route and study how much "exogenous" liquidity is needed to settle obligations consisting exclusively of cycles. One source of exogenous liquidity is the holding of either monetary assets or assets that remain readily convertible into money even in a liquidity crisis. A second source consist of payments by ultimate lenders to financial intermediaries. These serve to pay off other financial firms as long as these funds are not funneled back in the direction of ultimate borrowers. The analysis that follows can thus be understood as one where some firms do have claims on outsiders and where this source of liquidity is used as much as possible because firms do not repay their obligations to agents outside the financial system until they have repaid all their debts to financial firms. One

conclusion from this analysis is that, as financial firms become more interconnected, ensuring that one firm has access to outside liquidity may no longer be sufficient for settling all debts.

## 2 A settlement model with long payments chains

Consider an economy populated by  $N$  financial institutions (or firms) indexed by  $i \in [0, 1, \dots, N - 1]$  and let these firms be arrayed in a circle so that firm  $N - 1$  is followed by firm 0. These firms have obligations that are due within the period. In particular, each firm is expected to pay  $z$  dollars to the firms whose index is  $i + j$  with  $j \leq K$  where the addition is taken modulo  $N$ .<sup>13</sup> I denote the graph of their obligations by  $C_N^K$ . Figures 3 and 4 present examples with  $K = 1$  and  $K = 2$  respectively. Because the in-degrees and out-degrees of each vertex equal  $K$ , all these firms are solvent.

Following Ostroy and Starr (1974), one can think of each firm as wanting to achieve an allocation where it re-acquires the  $K$  coupons of its own debt while it sells back to the issuers the  $K$  coupons that it holds. As in their model, this vector of net trades is consistent with budget balance for each firm. As they argue, these net trades are easy to consummate with a single round of bilateral encounters between each pair of possible firms if each firm starts out endowed with a value of money greater than or equal to  $zK$ . Each firm then buys a coupon it has issued whenever it encounters a firm that holds one. The end result is that all coupons are purchased by their issuers.

In the Ostroy and Starr (1974) analysis, the sequence of encounters is not necessarily related to the transactions agents wish to carry out. Here I consider a somewhat more directed sequence of pairwise exchanges that is based on the idea that firms incur a penalty if they do not pay their debts on time. Thus, firms that simultaneously have money and debts would like to use their liquidity to make payments to at least one creditor. In other

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<sup>13</sup>To ensure that firms do in fact have net obligations with one another (so that bilateral netting is insufficient to eliminate these obligations),  $2K \leq N - 1$ . In the case of maximum interconnectedness,  $2K = N + 1$  so that each firm is a creditor to half the firms other than itself and a debtor to the other half.

words, the structure of the problem suggests that firms know which firms they owe funds to, and that they wish to send funds to these firms if they can. This still leaves a firm with multiple creditors with the choice of whom to pay with the particular unit of liquidity at its disposal.

How firms deal with this choice is an empirical question that deserves attention. Since firms are reluctant to reveal the identity of their creditors, this choice is unlikely to be based on knowledge of the entire system of outstanding obligations. My focus is on the extreme opposite case, where a firm knows the list of firms to whom it owes funds but knows (or remembers) nothing else about them.<sup>14</sup> As a result, they are indifferent regarding whom they pay. Moreover, I do not allow creditors to communicate with debtors before the latter choose whom to pay, so creditors are unable to affect this choice.<sup>15</sup> Lastly, I suppose that firms use whatever cash they have to pay off one of their obligations in full before seeking to pay any fraction of another obligation.

In practice, the process of consummating, verifying and recording payments does take some time, so there may be a finite upper bound  $R$  to the number of payments that can be made within the period using a single unit of liquidity. Below, I consider the case where the upper bound  $R$  is binding. I start with the simpler case where each payment is processed so rapidly that  $R$  is effectively infinite.

To model the consequences of the finiteness of the supply of settlement assets, I endow financial firms with units of these assets in sequence. As soon as one firm receives some liquidity, it uses it to make payments, and the recipients of these payments make payments in turn. No further liquid assets are injected into the system until the existing units of

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<sup>14</sup>An interesting open question is the extent to which the liquidity needed to settle all debts is reduced if firms have more information regarding the debts of their creditors. This information might lead firms to give priority to creditors with more obligations and thereby lead to a more efficient settlement mechanism.

<sup>15</sup>This assumption may appear restrictive. It is important to note, however, that creditors do not want to have a reputation for accepting less than the amount that is due to them. They may thus seek ways to commit themselves not offer debtors inducements to obtain payment ahead of other creditors.

liquidity can no longer be used to settle existing obligations. At that point, another firm may receive a new endowment. After these asset introductions cease, and when the existing liquidity can no longer be used to satisfy obligations, the settlement period ends. A firm  $i$  that still has open obligations at this point must pay a default cost  $c$ . This section focuses on the number of (sequential) distributions of liquidity from outside the financial system that are needed to settle all the debts and thereby avoid these costs.

Because firms do not favor one creditor over another, there are many possible paths of equilibrium payments. This leads me to study a best case scenario as well as a scenario that uses the available liquidity in the least effective possible way. The best case scenario turns out to be remarkably good: the minimum amount of liquidity that is needed to settle all debts is arbitrarily small and it is enough that one firm be endowed with this minuscule amount of liquidity.

To demonstrate this, it is worth recalling that an *Eulerian cycle* is a cycle that traverses every edge of a graph once, while going in the direction of the edges. A graph is Eulerian if it has an Eulerian path. One elementary result in graph theory is that a graph is Eulerian if it is connected and each vertex has an in-degree that equals its out-degree. A second result that is relevant for this paper is that an Eulerian graph can be decomposed into *edge-disjoint* cycles that do not have any edges in common. In the case of  $C_N^K$ , both the in-degree and the out-degree of each vertex equal  $K$ . Moreover, the graph is connected since one can always reach vertex  $j$  from vertex  $i$  by traveling to  $i + 1$ ,  $i + 2$  and so on until one reaches  $j$  (by passing vertex 0 if  $j < i$ ). This graph is thus Eulerian.

**Proposition 2.** *Let firm  $i$  be endowed with an arbitrarily small amount of liquidity  $w$ . Using just this liquidity, a path of payments can be found such that all debts in  $C_N^K$  are settled within the period.*

*Proof.* Let  $E^i = \{i, j, k, \dots, i\}$  be an Eulerian cycle originating at  $i$ . Suppose first that

$w < z$ . Let  $i$  give  $w$  to  $j$  to settle part of his debt with him, let  $j$  use these funds to pay part of his debt to  $k$ , and so on along the Eulerian path until these funds reach  $i$ . At this point, every firm owes  $z - w$  to its  $K$  creditors. If this exceeds  $w$ ,  $i$  once again pays  $w$  to  $j$  and so on along the Eulerian path. When enough Eulerian cycles of payments have been completed that everyone's outstanding debt  $\tilde{z}$  is less than  $w$ ,  $i$  pays  $\tilde{z}$  to  $j$  who passes it on to  $k$ , and so on, until all debts are settled. This last case also applies when  $w \geq z$ .  $\square$

While the particular graph considered in this proposition is special, it is clear from the proof that the result applies to any pattern of debts that can be represented by Eulerian graph. As long as all financial firms are connected to one another, the graph of their obligations has this property whenever each firm has total obligations to other financial firms that equal its total claims on such firms. It is also worth noting that the Eulerian path(s) that accomplish this are straightforward to compute for a central planner with full information regarding everyone's debts.

In a decentralized system with the limits on information that I have imposed, much more liquidity may be needed. Before showing this formally, it is worth noting an important property of liquidity endowments in the graph  $C_N^K$ . This is that a liquidity endowment to firm  $i$  stops settling obligations only when  $i$  has fulfilled all his own obligations. To see this give an endowment of  $z$  to  $i$ . Since firms all firms have obligations that equal their claims on other firms, a firm  $j \neq i$  that receives this endowment still has a further obligation and thus makes an additional payment. Thus, any unit of liquidity with which  $i$  is endowed continues to be used for payments until it is back in the hands of  $i$  himself. As long as firm  $i$  still has obligations, it makes further payments and this implies that payments continue until  $i$  has settled all his obligations and is in possession of his initial endowment.

Firm  $i$  is part of  $K$  distinct cycles in graph  $C_N^K$ , so  $i$ 's endowment must travel through  $K$  cycles before it stops being useful for payments. I focus on a "worst case scenario" where

endowments settle as few obligations as possible. This requires that the  $K$  cycles traveled by  $i$ 's endowment be as short as possible. This is captured in Assumption A, which also covers subsequent endowments. In particular, let  $G_t$  denote the graph that is left after  $t$  firms have each been given an endowment of  $z$  and made all the payments that this endowment facilitates, with  $G_0 = C_N^K$ . Then,

**Assumption A.** If firm  $i$  receives an endowment when the graph of obligations is  $G_t$ , the path followed by his first payment follows one of the shortest cycles in  $G_t$  that includes  $i$ . If at any vertex  $j$  of this cycle (including the origin  $i$ ) there is more than one shortest path back to  $i$ , the one that is chosen is the one that maximizes  $z$  where the edge  $\{j, j + z\}$  is included in the cycle. If  $i$  still has outstanding debts after earlier payments return to him, he makes new payments. These are chosen in a like manner.

The inefficient use of liquidity in Assumption A is counteracted to some extent by Assumption B, which concerns how the sequential endowments are distributed. Let  $d_t^j$  represent the total obligations of firm  $j$  at  $t$ , with  $d_0^j = zK$ . Suppose that, at stage  $t$ , there still exists a firm  $i$  such that  $d_t^i > 0$ . Then,

**Assumption B.** If a firm  $i$  receives an endowment after  $t$  firms have received theirs (and made all possible payments),  $d_t^i \geq d_t^k$  for all  $k$  between 0 and  $N - 1$ .

The purpose of Assumption B is to ensure that liquid endowments go to the firms that need them the most (because they have the largest debts). Without this assumption, it is easy to waste massive amounts of liquidity by giving it to firms that have already settled all their obligations in the past, and this is not reasonable. Moreover, this assumption would be somewhat natural if one viewed outside liquidity as being costly, since firms with large outstanding obligations would have a greater incentive to acquire this liquidity.

**Proposition 3.** *Under Assumptions A and B, the minimum number of firms that must be provided with liquidity to settle all obligations in  $G_0 = C_N^K$  is  $K$ .*

*Proof.* Start by giving  $z$  to firm  $i$ . The  $K$  shortest cycles starting at  $i$  on  $C_N^K$  start at  $i + j$ ,  $1 \leq j \leq K$ , then go to  $i + j + K$ ,  $i + j + 2K$  and so on, until they reach  $\{i - K, \dots, i - 1\}$ , at which point they return to  $i$  (where all these numbers are modulo  $N$ ). These  $K$  cycles are edge-disjoint and the full set of them touches each vertex once. Once the edges that are part of these cycles are removed, one can remove vertex  $i$  as well since  $i$  is left with no debts or claims. This leaves the graph  $G_1$  which is given by  $C_{N-1}^{K-1}$ . In this graph, each vertex has  $d_1^j = K - 1$ .

Assumption B implies that one of these remaining firms receives the next unit of endowment. By the argument above,  $G_t$  is thus  $C_{N-t}^{K-t}$  for all  $t \leq K - 1$ . After  $K - 1$  firms have been given an endowment, the graph is  $C_{N-K+1}^1$ . Denoting the  $K$ 'th firm that receives an endowment by  $i$ , this firm pays  $i + 1$ , who pays  $i + 2$  and so on until all the debts are cleared.  $\square$

This proposition shows both that giving  $K$  separate firms an endowment is enough to clear all debts and that, under assumptions A and B, giving endowments to fewer firms leaves some firms unable to settle their obligations. Indeed, if only  $K - 1$  firms are given an endowment, only  $K - 1$  firms clear their debts and the remaining  $N + 1 - K$  firms are unable to do so. This shows that an increase in the interconnectedness of firms increases the liquidity that is needed to settle all debts under assumptions A and B.

Even though these assumptions imply that more liquidity is needed than in the best case scenario where payments follow Eulerian paths, the required liquidity  $zK$  is still smaller than the sum of all obligations, which equals  $zKN$ . This is worth noting because Ostroy and Starr (1974) only show that it is sufficient for each agent to have as much money as his total purchases, which here corresponds to giving each of  $N$  firms an endowment of  $zK$ . The reduced liquidity required in my setting is not the result of assuming that each firm has more bilateral exchanges with other firms. Ostroy and Starr (1974) suppose that, in one

“round,” each agent encounters every other agent once. Both in the Eulerian path and in the paths contemplated in Proposition 3, each firm pays each of its creditors only once so that the number of pairwise meetings is actually smaller. It thus appears that Ostroy and Starr (1974) could have found a tighter bound on the amount of money needed to achieve their desired outcome. Still, the random sequence of meetings envisaged by Ostroy and Starr (1974) may require more liquidity than the sequence I consider here, where meetings are initiated when a debtor is able to make a payment to a creditor.

As the following proposition demonstrates,  $zK$  units of liquidity can settle an even larger volume of total debt when liquidity can be reused without bound.

**Proposition 4.** *If debts have the pattern embodied in  $C_N^K$ , giving endowments of  $z$  to  $K$  firms under assumptions A and B clears all debts even if each firm’s bilateral obligation  $z_o$  exceeds  $z$ .*

*Proof.* Let  $q$  equal  $z_o/z$  when this ratio is rounded down and let  $z_r = z_o - qz$ . Start by instituting a cycle of payments that would clear all debts if  $z_o = z$ . Then repeat this path of payments an additional  $q$  times, where the first  $q$  such cycles transfer  $z$  each and the last transfers  $z_r$ . □

This proposition also implies that  $z$ , and thus the total size of debts, does not affect the minimum amount of liquidity needed. Only the interconnectedness of debts  $K$  matters when  $R$  is sufficiently large.

To gain intuition for the model and its behavior, consider the simple case shown on Figure 3 where  $N = 6$  and  $K = 1$ . Suppose that only firm 0 starts out with liquidity equal to  $z$ , perhaps because it is the only one involved in a vertical lending chain like the one depicted in panel (a) of Figure 2. Within the financial system, this firm has no one to pass this liquidity to other than firm 1, who passes it on to firm 2 and so on until firm 5 returns it to 0. In the process, all debts are settled. Figure 3 makes it clear that this result does not depend on

$N$  being equal to 6: no firm has a choice as to whom to pay when  $K = 1$ , so all payments complete a full circle before returning to firm 0.

This can be contrasted with Figure 4 where the left panel shows  $C_6^2$ . The middle panel shows an Eulerian cycle. In this cycle, 0 makes a second payment after the funds it has advanced first get returned to him. Suppose that the first payment is given by the dashed arrows so that the vertices it reaches, in order, are  $\{0, 2, 4, 5, 1, 3, 5, 0\}$ . The second payment then follows the solid arrows so that its path is  $\{0, 1, 2, 3, 4, 0\}$ . When all these payments have been made, all obligations have been settled. For this particular Eulerian path to be completed, it is crucial that 5 first pass to 1 and only later pass to 0. The right panel shows a less happy outcome where the first of 0's payments follows  $\{0, 2, 4, 0\}$  so that 4 passes immediately to 0, while the second payment follows  $\{0, 1, 3, 5, 0\}$ , so that 5 passes to 0 at his first available opportunity. The multiplicity of choices faced by each firm in the case  $K = 2$  makes it easier to construct paths of payments such that giving liquidity to just one firm is insufficient to settle all debts. As  $K$  is increased further, this multiplicity can be exploited so that even more firms must be given liquidity.

So far, this section has only considered the fully symmetric graph of obligations  $C_N^K$ . To study whether firms make optimal decisions when they acquire claims and debts, however, one must study what happens when one firm has fewer assets and liabilities. It is, of course, impossible to reduce *only one* firm's obligations since eliminating  $i$ 's obligation to  $j$  means that  $j$  is unable to pay off as many obligations as before. If  $j$  responds by reducing his obligations to  $k$ , firm  $k$  must further reduce his own debts. This logic implies that, starting with the graph  $C_N^K$ , at least one cycle must be removed for  $i$  to have one fewer obligation while ensuring that all firms still have the same number of claims as they do debts. Consider then, the graph  $\mathcal{G}_i = C_N^K - C_i$  where  $C_i$  is a cycle that passes through  $i$ .

Intuition would suggest that, since there are fewer debts to settle with  $\mathcal{G}_i$  than with  $C_N^K$ , complete settlement of all debts can be accomplished by providing fewer firms with liquidity

in the former case. This can be seen graphically for a special with  $N = 6$  and  $K = 2$  in Figure 5. In this Figure, one cycle has been removed from  $C_6^2$ , namely the cycle given by  $\{1, 3, 5, 1\}$ . Inspection of the Figure shows that giving a liquid endowment to any of the firms with two debts (0, 2 or 4) is enough to clear all debts because these firms first make a payment that travels along the dashed arrows and then make a second payment that travels along the solid ones.

Using numerical methods, it is readily shown that the basic conclusion from this example extends to other values of  $K$  and  $N$ . I have considered a range of values for these parameters and constructed  $\mathcal{G}_i$  by subtracting a shortest cycle from  $C_N^K$ . In other words, I subtracted a cycle such that all but one of its edges went from a vertex with index  $i$  to a vertex with index  $i + K$ , while the remaining edge went from a vertex with index  $i$  to a vertex with index  $i + r$  where  $r$  is the remainder in the division of  $N$  by  $K$ . I then assigned endowments using Assumption B. For each such endowment, Assumption A then uniquely determines the obligations it settles. Assumption B, on the other hand, does not uniquely determine which firm receives an endowment from among all the firms that have the maximum total debt. In the case of  $C_N^K$ , this ambiguity was not important because all firms were symmetrically placed after an endowment had been used as much as possible for payments. In the case where one cycle is removed from  $C_N^K$ , however, firms are not as symmetric.

To see this, the left panel of Figure 6 shows  $C_8^3$  with the cycle  $\{0, 3, 6\}$  removed. The next panel shows what happens after firm 1 is given an endowment of liquidity while the last panel shows what happens after an endowment is given to firm 2. These two graphs are not identical because the original graph is not fully symmetric. In this case, one additional endowment clears all obligations regardless of which firm is given the first endowment. This is consistent with all my numerical experiments for  $K \leq 11$ . For these values of  $K$ ,  $K - 1$  endowments were necessary and sufficient to settle all claims (after 1 cycle was removed) regardless of which firms received endowments. For  $K > 11$ , however, there were cases

where less than  $K - 1$  endowments were needed. While I did not study this dependence exhaustively, many possible allocations were considered and, in all cases, fewer than  $K$  firms had to be given liquidity to settle all obligations.

### **3 A model of multilateral claims acquisition**

Many required payments in LVPS are exogenous in the sense of being caused by firms outside the financial system. The massive growth in these systems appears to be driven partially by a growth in payments from one financial firm to another. There is then the question whether, if the government were absent from the intraday market, financial firms would have a sufficient incentive to avoid straining their liquidity endowments when they acquire obligations against one another. If financial firms simply ignore these consequences altogether, their behavior is clearly inefficient since we have seen that the pattern of obligations they acquire can affect the ease with which these obligations can be settled. I thus take on a more complex question here. This is whether firms would inefficiently take on obligations even if they correctly forecast that this would increase the likelihood of settlement difficulties down the road.

One question that arises at this point is whether the relevant firms for this issue consist exclusively of those that are directly active in LVPS or whether it also includes those that use correspondent banks to make their intraday payments. Leaving aside the fact that this paper considers only symmetrical firms, its analysis also applies to firms that do not have direct access to the payments system under two conditions. These are first, that these “outside” firms not be allowed to have intraday overdrafts and second, that they be able to make payments with the funds that arrive into their account during the day. Even if such arrangements are not ubiquitous when central banks provide ample intraday credit, they might become common if this credit were curtailed. The analysis would then potentially apply to the large number of financial firms whose “arbitrage” strategies involve owning

claims and acquiring liabilities whose cash flows are similar to one another.

Leaving these firms aside, many of the firms that are directly involved in LVPs also acquire obligations with one another. While these firms also process payments for others, their own portfolio decisions can still affect the settlement process. Moreover, as discussed below, there is a remarkable similarity between the decision at the center of the analysis of this section, and decisions that broker-dealer firms must make after receiving suggestions on trade terminations from TriOptima.

The particular model I consider is somewhat unusual both in the way that it creates demands for securities and in the centralized mechanism that it postulates for determining who holds claims on whom. It tries to capture two fairly conventional forces, however. The first is that firms differ in the claims that they wish to hold. The second is that financial intermediaries have an incentive to maximize the volume of intermediation.

As discussed in the introduction, cycles of required payments among a set of firms have not been rationalized with existing models. Here I consider a simple model based on “tastes” where such cycles arise endogenously. Firm  $i$  is assumed to derive utility  $u(j)$  from holding a claim of  $z$  on firm  $i - j$ . Claims smaller than  $z$  yield no utility, and neither does utility rise if the size of claims is increased above  $z$ .<sup>16</sup> I let  $u(j)$  be decreasing in the index  $j$  so that firms have an intrinsic preference for holding the claims of firms that are close to them when going in the direction where the firm index falls. There is an extensive literature demonstrating that people and firms’ portfolios contain relatively large proportions of claims on “local” creditors, and the model is partially faithful to this effect by giving firms a preference for claims whose indexes are close to their own.

Explicit modeling of a decentralized system where individuals have something to gain by arranging trades by third parties is also beyond the scope of this paper, even though the issue

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<sup>16</sup>Supposing that preferences are concave in the amount held of any particular asset ensures that exposures do not become too large.

occupies a central role in the financial services industry. I postulate instead a centralized mechanism whose aim is to maximize financial transactions on the basis of messages sent by participating firms. The message sent by firm  $i$  consists of the integer  $\ell_i$ . This integer is interpreted as the number of firms that  $i$  is willing to lend to if it has the resources to do so. Because  $i$  is known to have a preference to lend to local firms, the message is taken to mean that  $i$  is willing to lend resources to all firms whose index is  $i - j$  where  $1 \leq j \leq \ell_i$  and the subtraction  $i - j$  is modulo  $N$ .

On the basis of these messages, the mechanism determines the matrix  $X$  whose element  $X_{ji}$  is equal to 1 if firm  $i$  lends  $z$  to  $j$  and equals zero otherwise. The  $i$ th column thus indicates the firms to whom  $i$  lends funds, while the  $j$ th row indicates all the firms that lend resources to  $j$ . Letting  $\iota$  represent a vector of  $N$  ones, the requirement that each firm's total loans be equal to its total obligations can be written as

$$X\iota = X'\iota \quad (1)$$

where  $X'$  is the transpose of  $X$ . Thus, the sum of the elements of a row is equal to the sum of the elements of the corresponding column. With  $X_{ij}$  only being able to take the values of zero and one, the centralized mechanism maximizes the total value of claims

$$\iota'X\iota \quad (2)$$

subject to (1) and

$$\forall i, j \quad X_{ji}I_{ji} = 0 \quad \text{where} \begin{cases} I_{ji} = 0 & \text{if } \ell_i \geq i - j \\ I_{ji} = 1 & \text{otherwise} \end{cases} \quad (3)$$

This constraint ensures that firm  $i$  does not hold a claim on a firm that is further than  $\ell_i$  away from it.

Letting  $\ell$  denote the full set of messages, the solution to this optimization problem is the matrix  $X^*(\ell)$ . The matrix  $X^*$  is the *adjoining matrix* of a directed graph, since it has zeros

on the diagonal while some of its off-diagonal elements equal one. Since  $X_{ij}^*$  is equal to one when  $i$  owes funds to  $j$ , and since this debt contract requires  $i$  to pass  $z$  units of liquidity to  $j$ ,  $X^*$  is in fact the adjoining matrix for a settlements graph.

From the perspective of firms  $i$ , it is useful to decompose  $\ell$  into the message sent by  $i$  himself,  $\ell_i$  and the messages sent by all other firms  $\ell^i$ . Firm  $i$  then chooses  $\ell_i$  to maximize his own utility, which is given by

$$U_i = \sum_{j=i+1}^{i+N-1} u(i-j)X_{ji} - P_i(X^*)c$$

where  $P_i(X^*)$  is the probability that firm  $i$  will be unable to settle one of its obligations given the debts represented by the matrix  $X^*$ . The maximization of this utility requires individuals to have beliefs about the effect of  $\ell_i$  on  $i$ 's assets and liabilities as well as on  $i$ 's probability of being unable to meet his obligations. I require that these beliefs be rational in the sense that firm  $i$  has to know both the equilibrium value of  $P_i(X^*)$  as well as how this probability changes when  $\ell_i$  changes. Notice that this rational assumption does not require firms to know the full network of obligation, nor their position in this network. Asking for such knowledge would seem unreasonable in more realistic settings.

I focus on a symmetric Nash equilibrium. At such an equilibrium, all firms send a message  $\ell_i = \bar{\ell}$  and the maximization of  $\iota'X\iota$  leads the pattern of obligation to be equal to  $C_N^{\bar{\ell}}$ . This reproduces the debts considered in the previous section. If endowments of liquidity are expected to be given sequentially to  $\bar{\ell}$  firms chosen according to Assumption B and if settlements proceed according to Assumption A,  $P_i = 0$ . Under assumptions A and B, these default probabilities are higher if the number of firms that receive liquidity has a positive probability of being smaller than  $\bar{\ell}$ .

For a symmetric equilibrium to exist, no firm must want to unilaterally deviate from sending a message of  $\bar{\ell}$ . When a single firm deviates by setting  $\ell_i$  above  $\bar{\ell}$ ,  $X^*$  is unaffected. Since the mechanism limits the loans of all other firms to  $\bar{\ell}$ , firm  $i$  does not have the resources

to increase the number of his loans beyond this. The ineffectiveness of a message that is above that of all other firms implies that firms cannot gain or lose from sending messages that are above the consensus message  $\bar{\ell}$ . This indifference could justify assuming that firms send messages of  $\bar{\ell}$  whenever they believe that other firms do so, even if all firms preferred to make loans to more firms. This could then rationalize equilibria with arbitrarily small (and even zero) loans. Such equilibria are unattractive because they are inconsistent with the efforts of brokers to increase the volume of financial transactions when this is profitable. In addition, sending a message of  $\bar{\ell}$  when firms prefer to make more loans is a weakly dominated strategy. I thus center my attention on symmetric equilibria that are robust to the deletion of weakly dominated strategies.

Ignoring integer constraints, firms are then indifferent with respect to reductions in  $\ell_i$ . A reduction in  $\ell_i$  below  $\bar{\ell}$  affects equilibrium lending because it prevents the centralized mechanism from giving firm  $i$  claims on  $\bar{\ell}$  firms. Indeed, (1) requires a reduction also in the number of firms that lend to  $i$  and in the loans of at least some of the firms to whom  $i$  would have lent if  $\ell_i$  had been set equal to  $\bar{\ell}$ . As discussed in the introduction, it is precisely such a simultaneous reduction in liabilities along a cycle that TriOptima proposes when it suggests Credit Derivative Swap trades that are candidates for termination.

Consider then, a deviation where  $\ell_i = \bar{\ell} - 1$ . Because  $i$  can end up with at most  $\ell_i$  claims and obligations, the resulting  $X^*$  must feature at least one less cycle passing through  $i$  than the graph  $C_N^{\bar{\ell}}$ . Since the mechanism seeks to maximize the number of edges remaining in  $X^*$ , it removes a shortest cycle. As discussed in the previous section, this implies that endowing  $\bar{\ell} - 1$  firms with liquidity is then sufficient to settle all debts under assumptions A and B.

The aim of the current section is only to demonstrate that the acquisition of claims need not be optimal. I thus proceed to construct a special case where private and social interests diverge, with the hope that it provides some intuition that is more generally valid. Suppose that assumptions A and B hold, that it is certain that at least  $\bar{K} - 1$  firms will receive

endowments of liquidity and that there is a probability  $\mu$  that  $\bar{K}$  firms will do so. I now consider a sufficient condition for an equilibrium to exist such that all firms set  $\ell_i$  equal to  $\bar{K}$ .

At such an equilibrium, all debts are settled with probability  $(1 - \mu)$ . With the remaining probability,  $N - \bar{K} + 1$  firms are left with one unpaid debt. The remaining  $\bar{K} - 1$  firms settle all their debts because they are the lucky recipients of a liquidity endowment. So, the combination of not knowing how many units of liquidity will be available and not knowing which firms will receive liquidity in the case where only  $\bar{K} - 1$  units are available leads firms to have an expected default cost of  $\mu c(N - \bar{K} + 1)/N$ .

A firm that deviates from the proposed equilibrium by setting  $\ell_i = (\bar{\ell} - 1)$  avoids these default costs since it is certain to be able to fulfill all its obligations. Since this deviation costs the firm  $u(\bar{K})$ , it is indifferent with respect to this deviation if

$$u(\bar{K}) = \frac{\mu c(N - \bar{K} + 1)}{N} \quad (4)$$

I now study the social consequences of having firm  $i$  reduce  $\ell_i$  from  $\bar{K}$  to  $\bar{K} - 1$ . For a certain number of firms, this reduces the number of their debtors and creditors by one. Given that the mechanism maximizes total debts, the number of firms thus affected is  $N/\bar{K}$  rounded up to the nearest integer. These firms all lose  $u(\bar{K}) - \mu c(N - \bar{K} + 1)/N$  so that they neither gain or lose anything. For the rest of the firms, there is a net gain of  $\mu c(N - \bar{K} + 1)/N$  since their debts are now settled for sure. To obtain the total social gain, one multiplies this individual gain by the number of these indirectly affected firms, which is  $(N - N/\bar{K})$  rounded down. The reason these social gains exist is that Assumption A implies that liquidity is not used in its most socially efficient manner. As a result, reducing a few firms' liquidity requirements allows many other firms to take advantage of the liquidity that is thus freed up.

One possible interpretation of the social benefits of having a firm reduce its interconnec-

tions is that each firm’s financial transactions create a “congestion externality” in that it makes a demand on scarce (and unpriced) liquidity. This congestion externality is unusual, however, in that small reductions in a firm’s interconnections can have discrete benefits for a great many other firms. The reason is that even a small reduction in interconnections can eliminate some liquidity payments paths that are quite inefficient, and thereby allow the existing liquidity to settle many *more* debts. This can be seen in Figure 5 where the elimination of the cycle of obligations  $\{1, 3, 5, 1\}$  implies that, under Assumptions A and B, all debts are settled with just one unit of liquidity. By contrast, when the  $\{1, 3, 5, 1\}$  cycle remains present, one unit of liquidity is not enough.

## 4 Short payments chains

There are several reasons to be interested in situations in which the number of payments that can be settled by a unit of liquidity is limited. One might suppose, for example, that the processing of each payment takes a discrete amount of time  $\tau$  while the length of the trading day is itself limited and equal to  $T$ . It is then impossible to use a unit of liquidity for more than  $T/\tau$  payments on a given calendar day and this may affect the amount of liquidity that one needs to settle the debts that come due on that day. As one firm is paying a second during a particular time interval, a third firm might be able to learn that it will receive the resulting funds later on. This third firm may thus be both able and willing to make a nearly simultaneous payment to a fourth firm using funds raised through a “daylight” loan. This parallel processing of payments may allow a unit of liquidity to be used more than  $T/\tau$  in a given day.<sup>17</sup>

Nonetheless, there may well be limitations on the process of making payments in advance of receiving liquidity. One of these is that, when a bank’s daylight loan is repaid, the bank

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<sup>17</sup>If these loans are costly, firms would prefer to pay with cash that they have already received, and this might dampen the use of this borrowing. See Angelini (1998) for a model where priced intraday credit leads firms to postpone their payments until they have cash on hand.

receives the liquid asset. This liquidity can only be used to settle more debts if the bank lends it anew. If the bank fails to do so, only the original cascade of payments using the system's actual liquidity continues unabated.

This section thus takes up the case where the maximum number of times that a unit of liquidity can be used,  $R$ , is smaller than  $N - 1 + K$  so that the paths of payments considered in Section 2 are infeasible.<sup>18</sup> One immediate consequence of this is that the total liquidity that is needed to settle all debts now depends on the volume of debt in addition to depending on the number of interconnections among firms. To see this, imagine a pattern of liquidity endowments that settles all debts when each firm owes  $z$  to each of his creditors. If each bilateral debt is of size  $\lambda z$  (so that the total debt is multiplied by  $\lambda$ ), it can be settled by the same sequence of endowments, as long as each endowment is multiplied by  $\lambda$  as well. Conversely, if one multiplies every bilateral debt by a sufficiently large  $\lambda$ , the original distribution of liquidity endowments will be insufficient to settle all debts.

A somewhat more surprising consequence of short payment chains is that the *minimum* number of liquidity endowments that is needed to settle all obligations can now depend on the interrelatedness of obligations. This can happen even if a social planner gets to determine the payment paths, so that interrelatedness poses problems even leaving aside the externality from choosing whom to pay (though this externality can still increase the number of firms that must be given liquidity in worst case scenarios). This new problem arises only when individual liquidity endowments are lumpy, so that some firms have large endowments of liquid assets while others have no such endowments. What happens, then, is that interrelatedness makes it more difficult to channel large quantities of liquidity to firms without endowments.

To see this, I start with a case where endowment distributions are so small that  $N$  of

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<sup>18</sup>That the solution considered in Proposition 2 involves  $N - 1 + K$  payments when the first receives  $z$  units of liquidity can be seen by noting that this unit of liquidity ensures that  $N - 1$  firms receive one payment while the original recipient of liquidity receives  $K$  payments.

them are needed even if  $K = 1$ . I then show that  $N$  distributions can also lead all obligations to be settled when  $K$  is larger, though this requires that payment paths be chosen with great care. This irrelevance of  $K$  when payment paths are chosen by an outside planner harks back to the irrelevance of  $K$  when long payment chains were forced to move along Eulerian paths. I then show that this irrelevance stops being true when endowment distributions are larger.

Suppose that each firm has a total debt  $d$ , where  $d$  is independent of  $K$ , so that total required payments equal  $dN$ . If each endowment can be used  $R$  times and the size of each endowment is  $e$ , the minimum number of endowments that must be distributed equals  $dN/eR$ . If  $e = d/R$ , this minimum equals  $N$  and this value of  $e$  lets a planner achieve full settlement with minimum liquidity for many values of  $K$ .

**Proposition 5.** *If  $e = d/R$  and payment paths are chosen appropriately,  $N$  distributions are sufficient to cover all debts for any  $K$  if either  $R/K$  or  $K/R$  are integers.*

*Proof.* First let  $R/K$  be an integer  $r$ . The obligation  $z$  of firm  $i$  to firm  $i + 1$  is then equal to  $d/K = rd/R$ . One therefore needs to use  $r$  different distributions to cover a single obligation. To specify the  $j$ th payment made with a particular distribution let  $j$  be written as  $j - 1 = mK + f$  where  $m$  equals  $(j - 1)/K$  rounded down and  $f$  is the remainder from this division. Then, the  $j$ th payment made with the endowment given to  $i$  goes from  $O_{ij} = i + m \sum_{v=1}^K v + \sum_{v=1}^f v$  to  $O_{ij} + f + 1$ . This means that, for  $j < K + 1$ , the first payment travels a distance of 1, the second a distance of 2, and so on until the  $K$ th payment travels a distance of  $K$ . The  $K + 1$ st payment then again travels a distance of 1, followed by a payment that travels a distance of 2 and so on.

The edges corresponding to these payments cover the full edge set of the graph. To see this, recall that, for  $j \leq K$  firm  $i$  is required to make  $r$  payments of  $d/R$  to firm  $i + j$ . The procedure implies that the closest source of these payments is the endowment given to the

firm that is separated from  $i$  by  $\tilde{O}_j = 1 + 2 + \dots + (j - 1)$ . If  $r > 1$ , there are  $r - 1$  additional sources which are given by firms that are separated from  $i$  by  $\tilde{O}_j + mK(K + 1)/2$  where  $m$  takes values between 1 and  $r - 1$ . The firm thus has enough resources to make all its required payments.

Now consider the case where  $K/R$  is an integer  $r'$  greater than 1. The obligation of firm  $i$  to firm  $i + 1$  can now be written as  $d/K = d/(r'R)$  so that the firm is able to fully cover  $r'$  of its required payments with its endowment of  $d/R$ . Let these payments be made to firms with indices  $i + v$  where  $v$  goes between 1 and  $r'$ . Further, let firm  $i + v$  make a payment to firm  $i + 2v + r'$ , with subsequent payments going to firms with indices  $i + kv + k(k - 1)r'/2$  with  $k$  taking on values between 1 (the original recipient) and  $R$ . Notice that the last of these payments is made by the firm with index  $i + (R - 1)Rr'/2$  and made to a firm with index  $i + (R + 1)Rr'/2$ , *i.e.*, a firm that is  $Rr' = K$  firms away from the firm with the endowment. Thus, each endowment induces  $K$  payments, one of each possible length. Moreover, if the payment of length  $j$  made by firm  $i$  is induced by the endowment given to firm  $k$ , the endowment of firm  $k + 1$  leads to a payment of length  $j$  made by firm  $i + 1$ . Thus, the  $N$  endowments lead all payments of length  $j$  to be made.  $\square$

This demonstrates that the minimum of  $N$  distributions is achievable for both small and large values of  $K$  when the distributions are small enough that it is necessary to make  $N$  of them. In the case of  $K = 1$ , firms have no choice as to whom they pay. As a result, distributing  $d/R$  to each firm ensures that all firms make their required payments. In the case where  $K > 1$ , however, firms do face such a choice and the result of Proposition 5 hinges crucially on making these payment choices in a way that uses the endowments efficiently. To see this, I consider a simple example where the obligations are given by  $C_6^2$ , which is given in the left panel of Figure 4. Proposition 5 shows that 6 distributions of  $d/2$  can be sufficient to settle all debts when  $R = 2$ . I now show that, for a different set of payment choices, 6

distributions of  $d/2$  are not sufficient.

Start by giving endowments to firms with index  $i$  equal to 0, 2, and 4. Let each of these firms make a payment of  $d/2$  to firm  $i + 1$ , which then makes a payment to firm  $i + 3$ . The graph of remaining obligations is now given by Figure 7. It is apparent by inspection of this Figure that it is now impossible to make 3 further distributions of  $d/2$  that settle all obligations. After giving endowments of  $d/2$  to firms 1 and 2, for example, there remains an obligation from 0 to 2 and an obligation of 5 to 1, and these cannot both be settled by distributing one endowment. Thus, the externality of payment choice remains present in this example with shorter payment chains.

A more surprising aspect of short payment chains is that they create problems for settlements in interconnected financial systems even when a benevolent planner chooses who makes payments to whom. These problems only arise, however, when endowments are lumpier than in Proposition 5. This is demonstrated in the following proposition.

**Proposition 6.** *Let each of  $N$  firms have a total debt equal to  $d$  and be owed  $d$  by others. Suppose that  $N$  is divisible by  $R > 1$  and that liquidity endowments are equal to  $d$ . Then, the minimum number of liquidity endowments needed to clear all debts when these are given by  $C_N^1$  equals  $N/R$ . When debts are given by  $C_N^K$  with  $K \geq R$ , the number needed is strictly larger.*

*Proof.* See Appendix. □

The proposition suggests that settling all obligations is more complex when  $K \geq R$ . When  $K = 1$ , it suffices to space the recipients of exogenous liquidity so that their indices differ by  $R$ . By contrast, when  $K \geq R$  so that each firm makes payments to a variety of firms, each endowment leads many firms to have their debts reduced slightly. It then becomes impossible to space endowments so that none of their recipients have already reduced their

debts with funds received from other firms. Thus, endowments are not fully used for payments and more firms need to be endowed with funds to settle all debts.

Proposition 6 covers the case where  $K \geq R$  so that interconnectedness is large relative to the length of payment chains. This leaves open the question of what happens when  $1 < K < R$ . As the following examples indicate, it seems difficult to say much about this situation in general. When  $R = 4$  and  $K = 3$ , there exist values of  $N$  such that  $N/R$  endowments suffice (the example below shows this to be true for  $N = 12$  but the result ought to apply whenever  $N$  is divisible by  $R$ ). By contrast, when  $R = 4$  and  $K = 2$ , there do exist values of  $N$  that are multiples of  $R$  such that  $N/R$  is not sufficient. This suggests that the capacity to settle all claims with  $N/R$  endowments is not monotonic in  $K$  when  $K < R$ .

**Proposition 7.** *When the graph of obligations is given by  $C_{12}^3$  and  $R = 4$ , it is possible to clear all debts by giving 3 endowments of  $d$ .*

*Proof.* See Appendix. □

**Proposition 8.** *When the graph of obligations is given by  $C_{12}^2$  and  $R = 4$ , it is impossible to clear all debts with 3 endowments of  $d$ .*

*Proof.* See Appendix. □

## 5 Concluding Remarks

The paper shows that limits on the amount of liquid assets that are available to the financial system for the payment of debts can lead to more defaults when each firm has more creditors and debtors. One obvious weakness of the paper is that the graphs of required payments that it considers are extremely simple relative to those in actual markets. The benefit of focusing on such this setting is that it is possible to demonstrate the importance of a relatively simple

but understudied variable, namely the number of creditors per firm. To focus attention on the importance of this number, I let it be the same for all firms so that the vertices in the graph of financial obligations all have the same degree. In fact, graphs of social phenomena tend to display considerable variability in the degrees of their vertices and, as demonstrated by Soramäki et al (2007) this is true also of the network of payment flows transferred over Fedwire. It seems likely that this variability in degree characterizes also the number of firms that financial firms owe funds to (and the number of firms that are their creditors).

According to McAndrews and Rajan (2000) financial trades are an important component of Fedwire's massive volume. This suggests that, as in the model of Section 3, the commitments that financial firms make to one another may affect the amount of liquidity that the system requires to settle all its obligations. A key question that is raised by the heterogeneity of the firms involved, however, is whose interconnections matter more for the capacity of the system to settle all its obligations. Do the claims that hedge funds acquire with each other matter at all? Even in the case of banks that belong to LVPS, to what extent does the randomness of their exogenous obligations impair their ability to forecast the effect of their own obligations on the settlement process? To answer these questions, one needs to extend the model presented here. As discussed in section 3, the effect of financial firms that are not directly plugged into LVPS is likely to depend on the terms under which their correspondent banks give them intraday credit. It is thus worth exploring the sensitivity of settlement outcomes to the provision of this credit.

The analysis in this paper suggests that it would be extremely useful to learn what this particular network of obligations among financial firms looks like. With such a network in hand, one could use Assumptions A and B to obtain an estimate of the liquidity needed by the system on a daily basis to clear all the obligations that financial firms have with each other. One could also try to understand whether granting intraday credit to firms that have many interconnections is sufficient or whether it is necessary for some smaller firms to have

direct access to liquidity as well.

Data on the claims that financial firms have on other financial firms should also encourage further modeling of the reasons firms have for acquiring these claims. It would be interesting to study, for example, the extent to which these holdings can be rationalized by models based on the idea that financial firms seek to protect themselves from idiosyncratic shocks. The existence of financial firms that appear to use their entire capital just to hold risky claims against other financial firms suggests that these portfolios may be based on differences in beliefs, and models of this type deserve further exploration as well. The model based on “tastes” that I have presented to explain the existence of cycles of claims might also benefit from being extended. While this model is only a “reduced form,” it would be interesting to learn the extent to which a simple model of this type can help explain actual cross-holding of claims by financial firms.

The model also highlights the importance of another variable in determining the amount of liquidity that is needed to settle all claims, and this is the number of times that a unit of liquidity can be reused for payments during the day. The measurement of this variable is nontrivial, though progress can presumably be made by tracking payments in real time. One important determinant of this turnover is the extent to which firms are better off making payments as soon as they have liquidity available (which is the assumption made in this paper) rather than waiting until later. A model that tackles this issue is presented in Bech and Garratt (2003) and it would be useful to incorporate this analysis into the determination of the amount of liquidity needed to settle the obligations of financial firms.

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## Appendix: proofs of some propositions

### Proof of proposition 6

In the case of  $C_N^1$ , it suffices to give endowments to firm with indices given by  $iR$  with  $i = 0, 1, 2, \dots, (N/R - 1)$  to clear all debts. Each recipient of an endowment  $i$  uses the funds to pay off his entire obligation to  $i + 1$ . Payments from one firm to the next continue until firm  $i + R$  receives a payment from the endowment given to firm  $i$ . Firm  $i + R$  also receives an endowment and thus all obligations are cleared.

For the provision of  $d$  units of liquidity to  $N/R$  firms to be sufficient to settle all debts (whose total value is  $dN$ ), each liquidity endowment of  $d$  must on average settle debts with a value of  $dR$ . Since  $dR$  is the maximum amount of debt that an endowment can settle, every liquidity endowment must settle this amount of debt, and this is precisely what takes place in the case of  $C_N^1$  that I just discussed. I now show that this is impossible for  $C_N^K$  when  $K > R > 1$ . Since each firm owes  $d/K$  to  $K$  firms, settling debts with a value of  $dR$  requires that  $RK$  firms make payments of  $d/K$  to a creditor.

The first firm  $i$  that receives a liquidity endowment has a debt of  $d$  outstanding and makes payments of  $d/K$  to  $K$  firms with indices  $i + j$  where  $j$  is between 1 and  $K$ . If  $R > 1$ , each of these firms must make further payments so that each endowment distribution leads at least  $1 + K$  contiguous firms to make payments. If any of these  $1 + K$  firms receives a subsequent endowment, this endowment cannot be used to make  $K$  payments, which implies that fewer than  $RK$  firms use this endowment to make payments.

If a later endowment is given to a firm with index  $j < i$  such that  $j + K > i$ , one of the payees of firm  $j$  must be firm  $i$ . Since firm  $i$  has already discharged his obligations, it makes no further payments, so that the endowment given to firm  $j$  clears fewer than  $RK$  debts of  $d/K$ . It follows that a necessary condition for an endowment to clear  $RK$  debts is that no firm with an index between  $i - K - 1$  and  $i$  be given an endowment. This logic implies that, for any firm  $k$  that receives an endowment, no firm with an index between  $k - K - 1$  and  $k$  can receive an endowment if all endowments are to clear  $RK$  obligations. This means no more than one firm out of every  $1 + K$  can be given an endowment if they are each to pay

off debts of  $Rd$ . If  $K \geq R$ , the number of firms whose endowments clear this many debts is thus strictly smaller than  $N/R$  so more firms must be given an endowment to clear all debts.

### **Proof of proposition 7**

Each firm  $i$  that receives an endowment makes three payments. The first of these goes to  $i+1$ , who uses it to pay  $i+3$  who uses it to pay  $i+5$  who uses it to pay  $i+8$ . The second of these goes to  $i+2$ , who uses it to pay  $i+5$  who uses it to pay  $i+6$  who uses it to pay  $i+8$ . The last goes to  $i+3$ , who uses it to pay  $i+6$  who uses it to pay  $i+7$  who uses it to pay  $i+8$ . Let endowments be distributed to firms 0, 4 and 8. Denote by 1, 2 and 3 the three payments made by 0, by 4, 5 and 6 the three payments made by 4, and by 7, 8 and 9 the three payments made by 8.

Table 1 shows which payments are received by each firm, except that for firms 1, 4 and 8, it specifies both the payment that they receive (under column  $R$ ) and that they pay out (under column  $E$ ). The table demonstrates that each firm  $j$  receives payments from  $j-1$ ,  $j-2$  and  $j-3$  while it makes payments to firms  $j+1$ ,  $j+2$ , and  $j+3$ . This construction may seem arbitrary but notice that, aside from all ending in  $i+8$ , the three paths have the property that, in total, they involve four payments each of lengths 1, 2 and 3.

### **Proof of proposition 8**

Suppose that 3 endowments are given to firms  $i$ ,  $j$ , and  $k$ . If any path of payments starting at  $i$ ,  $j$ , or  $k$  ends at  $m \neq i, j, k$ ,  $m$  has at least one unfulfilled obligation (since it neither has an endowment nor is able to use one of the payments it receives to make a further payment). Thus, the paths of payments originating at  $i$ ,  $j$  and  $k$  must terminate at  $i$ ,  $j$ , or  $k$  for all obligations to be fulfilled. Because  $R = 4$  and  $K = 2$ , the maximum length of a payment chain is 8 and the minimum is 4. Since the path from  $i$  to  $i$  has length 12, this implies that chains of payments originating in  $i$  cannot end at  $i$ , so they must end at either  $j$  or  $k$ .

If the 3 firms receiving endowments are equidistant so that, for example  $j = i + 4$  and  $k = j + 4$ , some obligations are unfulfilled. The reason is that there is then only one path originating in  $i$  and terminating in  $j$  and the only path from  $i$  to  $k$  passes through  $j$  as

well (because the distance between  $i$  and  $k$  is 8, the path from  $i$  to  $k$  involves 4 segments of length 2). Thus, equidistant endowments imply either that both of  $i$ 's endowments end at  $j$  (meaning that one only makes two payments) or that  $j$  only originates a single payment from his own endowment.

This implies that, to fulfill all obligations, one distance between firms receiving endowments, say that between  $i$  and  $j$  must be smaller than 4. This implies that no payment originating in  $i$  terminates in  $j$ . Since the distance between  $j$  and  $i$  is at least 9 (when  $j - i = 3$ ), no payment originating in  $j$  ends in  $i$  either. Thus, to fulfill all obligations, all payments originating in both  $i$  and  $j$  have to terminate in  $k$ , which is impossible.

Table 1

A set of payments and endowments that clears  $C_{12}^3$  with  $R = 4$ 

	Firm $j$														
	0		1	2	3	4		5	6	7	8		9	10	11
	$R$	$E$				$R$	$E$				$R$	$E$			
Received from $j - 1$	6	1	1	8	9	9	4	4	2	3	3	7	7	5	6
Received from $j - 2$	5	2	7	2	1	8	5	1	5	4	2	8	4	8	7
Received from $j - 3$	4	3	8	9	3	7	6	2	3	6	1	9	5	6	9

Figure 1: An example of vertical obligations

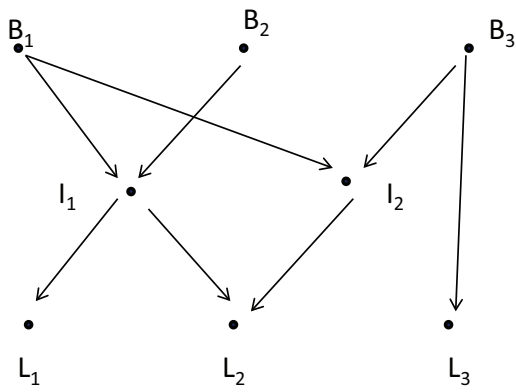


Figure 2: An example combining vertical lending with a cycle

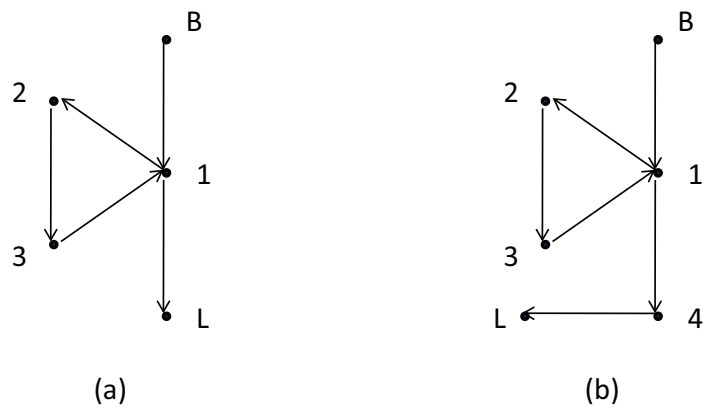


Figure 3: An example with  $K = 1$

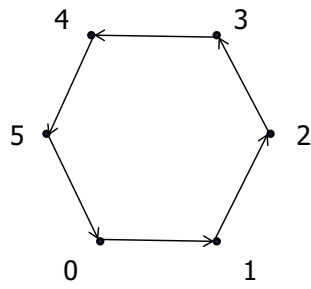


Figure 4: An example with  $K = 2$

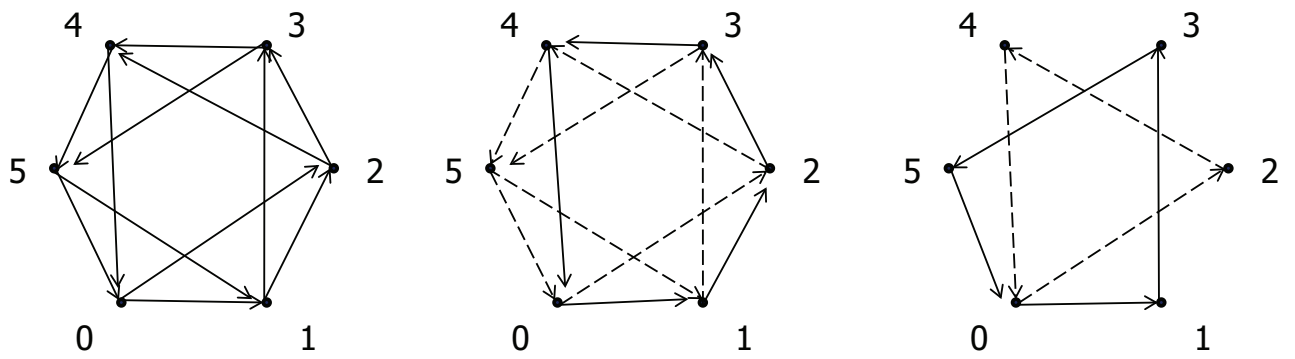


Figure 5: An example with  $K = 2$  and a missing cycle

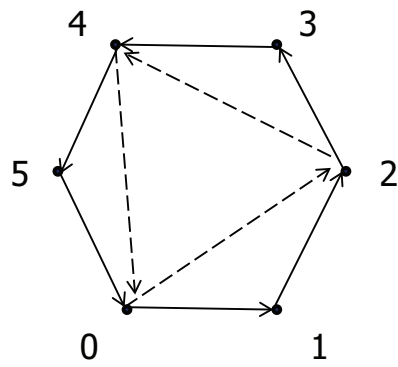


Figure 6: An example with  $K = 3$  and a missing cycle

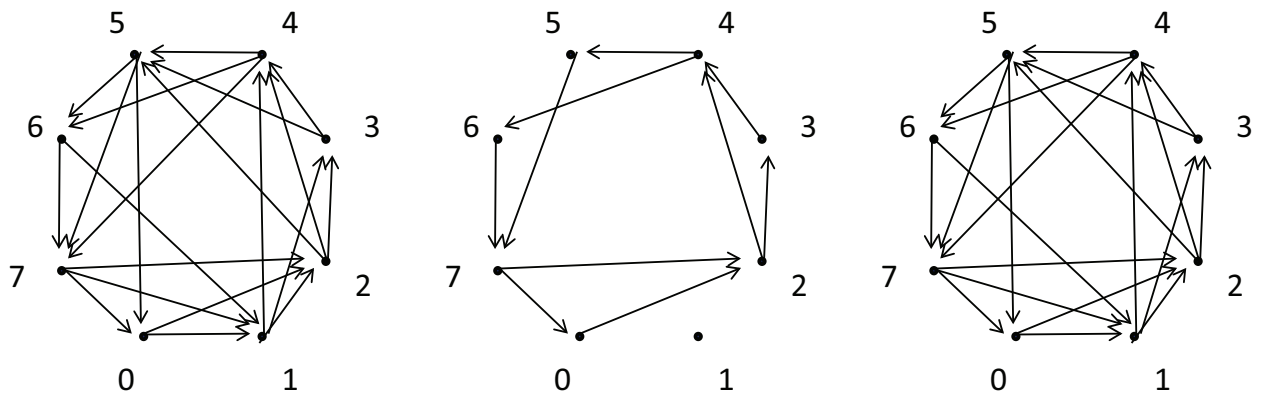


Figure 7: Remaining obligations of  $C_6^2$  when  $R = 2$

