

THE VALUE OF INFORMATION
IN THE DELEGATION PROBLEM*

by

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ABSTRACT

The value of information is examined in the delegation problem, i.e., in settings where a principal delegates limited decision-making authority to a better-informed agent. It is shown that for any pair of information systems there exist two delegation problems such that the agent's ranking of the systems is reversed for the two problems. It is shown that from the principal's point of view, an improvement in an information system of rank greater than two is valuable in all delegation problems if and only if it is of a special "success-enhancing" type. For information systems of rank two, a somewhat broader class of improvements is shown to be necessarily valuable to the principal.

1. Introduction

In many organizations there are individuals or groups whose primary function is to gather information and transmit it to a decision-maker. The latter then uses this information when choosing among the available courses of action. If the two parties have identical interests, their joint behavior mimics that of a single individual, and it is well known that having access to better information always has nonnegative value.¹ Here we examine the value of improved information to the two parties when their interests do not coincide.

As Holmstrom [1978] has pointed out, if the decision-maker described above is a Stackelberg leader, the situation is equivalent to one in which the better-informed subordinate is given the power to choose the action within a prescribed range. It is what Holmstrom calls the "delegation" problem, i.e., the problem of a principal delegating limited decision-making authority to a better-informed agent whose interests may be different from his own. The principal's problem is to choose the limitations to be placed on the agent.

Examples of this type of situation abound. An executive in a firm may delegate authority to a subordinate, a patient to a doctor, a client to a lawyer, etc. Moreover, double-maximum problems also fall into this class. In the optimal taxation problem (see Mirrlees [1971]) the tax authority allows each individual to choose how much labor to supply; in the nonuniform pricing problem (see Spence [1977]) the seller allows each consumer to choose how much to purchase; etc. In these cases, since the individual is better-informed (he knows his "type"), the principal has an incentive to allow him to choose from a "menu" of alternatives. The principal's problem is to design the "menu".

Little has been said about the value of improving the agent's information in delegation problems. Holmstrom examines this question using quadratic loss

functions for both parties and a signal variable that is uniformly or normally distributed. He shows that as the agent's information improves, the principal grants him more freedom and each party's expected loss decreases. Holmstrom also provides an example showing that when the loss functions are not quadratic, improving the agent's information can increase the principal's expected loss.

In a related analysis using arbitrary payoff functions, Green [1979] shows that if the signal is a binomial channel, i.e., if the signal variable can take only two values, then better information cannot decrease the expected payoff of the principal. If in addition only two actions are available, then better information cannot decrease the expected payoff of the agent. Green also gives an example involving three signal values, in which better information decreases the expected payoff of the principal.

Finally, in a comment on Green's paper Postlewaite [1979] provides four examples showing that when the payoff structures of the two parties differ, any pattern of effects is possible — better information can increase or decrease the expected value to either or both parties. In some cases the principal offers the same range of choice to the agent, but the agent's new information structure allows him to choose within this set in a way that is detrimental to the principal. Thus, the principal's expected payoff can fall as the agent's rises. In other cases, with a better informed agent the principal changes the set of alternatives offered to the agent, eliminating one or more of the old alternatives (and perhaps adding others). When this happens the principal's expected payoff may rise and the agent's fall, or the expected payoff of each may fall.

In this paper we continue the analysis of the value of better information in the delegation problem. As the foregoing discussion suggests, the value of

improved information depends on how sharply the interests of the principal and agent diverge, on how seriously their prior beliefs differ, and on the nature of the informational improvement. Two types of questions arise. How similar must the interests of the two parties be in order to make better information valuable to either or both of them? In particular, are there any restrictions on the payoff functions sufficient to insure that better information will be more valuable? The answer is 'no': any pattern of welfare changes can be produced from divergence in priors alone. In this way Postlewaite's examples can be generalized.² Alternatively, are there restrictions on pairs of information systems sufficient to imply a ranking between them, from the viewpoint of either player, independent of payoff structures? This is the question we address below.

First we examine the value of improved information from the agent's point of view. In Theorem 1 we show that for any pair of information systems there exist two delegation problems such that the agent's ranking of the two systems is reversed for the two problems. In particular, even if the information structure is strictly improved in the sense of statistical decision theory, the agent's welfare may go down.

In Theorem 2 we show that an improvement in information is necessarily valuable to the principal if it is of a type we call "success-enhancing." These can occur only if one of the observations provides "no information" in the sense that it induces no change in beliefs from the prior to the posterior. Such an observation, which corresponds to "dropping the test tube," leaves any prior belief unchanged. A success-enhancing improvement in information decreases the probability of "dropping the test tube" and increases the probabilities of all the remaining outcomes proportionately.

The converse of Theorem 2 is obviously true if the less informative

system has rank one (is completely uninformative), since in this case any improvement is success-enhancing and is also valuable to the principal. In Theorem 3 we prove that the converse of Theorem 2 also holds if the less informative system has rank greater than two. The converse fails if the less informative system has rank two, when a somewhat wider class of improvements is necessarily valuable to the principal. These are partially characterized in Theorem 4, and include any improvement in a binomial channel.

The rest of the paper is organized as follows: the formal model is described in section 2; the theorems are stated and proved in section 3; and the conclusions are discussed in section 4.

2. THE MODEL

Initially we will assume that the principal retains decision-making authority and that the agent merely communicates information. Later we will show that when the principal is a Stackelberg leader, this situation is equivalent to one in which the principal delegates limited decision-making authority to the agent.

Throughout we will denote by $M_{K \times N}$ the set of all $K \times N$ Markov matrices; by P_N the set of all $N \times N$ permutation matrices; by $\underline{1}$ the vector $(1, 1, \dots, 1)^T$; by $\underline{0}$ the vector $(0, 0, \dots, 0)^T$; and by e_i the vector $(0, \dots, 0, 1, 0, \dots, 0)^T$.

Let $A = \{a_1, \dots, a_k\}$ denote the set of actions from which the principal can choose (or among which he can randomize), and $\Theta = \{\theta_1, \dots, \theta_M\}$ the set of possible states of nature. The state of nature is unknown at the time an action must be taken. However, a signal about the state of nature is available to the agent. Specifically, $Y = \{y_1, \dots, y_N\}$ denotes the set of possible values that the signal can take on. Let λ_{ij} be the conditional probability that the signal takes on the value y_j if the state of nature is

θ_i , and let $\Lambda \equiv [\lambda_{ij}] \in M_{M \times N}$ be the (Markov) matrix of these conditional probabilities. An information structure is completely described by its likelihood matrix Λ .

The principal (agent) has prior probability distribution $\pi = (\pi_1, \dots, \pi_M)$ ($\pi' = (\pi'_1, \dots, \pi'_M)$) over the possible states of nature, where π_i (π'_i) is the prior probability assigned to θ_i . Let Π (Π') be the $M \times M$ diagonal matrix with the elements of π (π') on the diagonal.

Both parties are expected utility maximizers. The utility of the principal (agent) if action a_k is taken and state θ_i occurs is u_{ki} (u'_{ki}). Let $U \equiv [u_{ki}]$ and $U' \equiv [u'_{ki}]$ denote the $K \times M$ matrices of payoffs. The elements of these matrices are the von Neumann-Morgenstern utilities of the two players. To the extent that they are risk averse with respect to monetary payments, this has already been incorporated in these utilities.

One further comment should be made at this point concerning the possibility of the principal using monetary compensation to motivate the agent. Such transfers, which may depend on the mutually observable variables of the model, are the essence of the "principal-agent problem" as it has usually been treated. In our model, however, there is no explicit mention of these transfers.

Our results can be interpreted as applying in cases where direct contingent monetary transfers between the principal and agent are not possible. The literature to date notwithstanding, there are many such cases. An investment advisor may receive a management fee, but fees conditional on portfolio performance may be illegal or impractical. However, the advisor's reputation, and therefore his fee or level of operations with other clients, will be indirectly dependent upon the portfolio performance for the principal in question, and in this way his utility depends on the state as

well as his action. The problem of allocating investment among divisions of a firm provides another example. The state of nature in this case is the entire future course of events relevant to all the potential investments. While monetary transfers can be used to motivate managers, the relevant horizon for managers is likely to be shorter than the economic lifetime of the projects they initiate. Consequently, monetary transfers must be effected before the true state is observed, so that transfer schemes are severely restricted if not eliminated.

Our model works as follows. The agent observes the signal and then reports some observation to the principal. He need not be truthful. Because he is allowed to randomize among transmissions his reporting function :

$R \equiv [r_{jj}']$ can be any $N \times N$ Markov matrix, where r_{jj}' is the conditional probability that he reports y_j' if he observes y_j .

After receiving information from the agent, the principal must choose an action. We allow randomized decisions for the principal, so that his decision function $Z \equiv [z_{j,k}']$ can be any $N \times K$ Markov matrix, where $z_{j,k}'$ is the conditional probability that the principal chooses action a_k if the agent reports y_j' .

The situation we have modelled is a two-person nonzero-sum game. Because we imagine that the principal can commit himself to a decision function, the appropriate solution concept is the Stackelberg equilibrium.³ Formally, the strategies are $Z \in M_{N \times K}$ for the principal and, for the agent, a mapping assigning to each $Z \in M_{N \times K}$ a reporting function $R \in M_{N \times N}$. The interpretation is that when the principal has committed himself to Z , the optimal reporting function is R . Given $U, U', \Pi, \Pi', \Lambda$, and the pair (R, Z) , the principal's expected utility is $\text{tr } ZU\Pi AR$ and the agent's is $\text{tr } ZU'\Pi'AR$.⁴

The Stackelberg equilibrium can be found by first solving the agent's

problem for each Z, and then computing the maximum payoff for the principal over all Z's, with R varying so that it is always among the optimal responses to Z.

Given U', Π', Λ , and Z, the agent's set of optimal responses

$R(ZU'\Pi'\Lambda) \in M_{N \times N}$ is given by:

$$R(ZU'\Pi'\Lambda) \equiv \{R \mid R \in \operatorname{argmax}_{S \in M_{N \times N}} \operatorname{tr} SZU'\Pi'\Lambda\}.$$

Given any strategy Z_0 for the principal and any optimal response to it $R_0 \in R(Z_0U'\Pi'\Lambda)$, consider the strategy $Z_1 \equiv R_0Z_0$. If the principal adopts the strategy Z_1 , it is clear that accurate reporting is an optimal strategy for the agent, i.e., $I \in R(Z_1U'\Pi'\Lambda)$. Moreover, the strategy pairs (Z_0, R_0) and (Z_1, I) lead to identical outcomes for each party. Hence, without loss of generality, we can limit our attention to decision functions for the principal that induce accurate transmission of information by the agent. Let $Z(\cdot)$ denote this set.

$$Z(U'\Pi'\Lambda) \equiv \{Z \in M_{N \times K} \mid I \in R(ZU'\Pi'\Lambda)\}.$$

The set $Z(U'\Pi'\Lambda)$ will be called the set of incentive-compatible strategies.

Note that $Z \in Z(U'\Pi'\Lambda)$ if and only if the $N \times N$ matrix $ZU'\Pi'\Lambda$ is column dominant along the diagonal, i.e., if and only if for all $n = 1, \dots, N$:

$$\sum_{k,m} z_{nk} u'_{km} \pi'_m \lambda_{mn} > \sum_{k,m} z_{n'k} u'_{km} \pi'_m \lambda_{mn}, \quad n' = 1, \dots, N.$$

(We assume that the agent transmits the information correctly unless he has a positive incentive to do otherwise.) $Z(U'\Pi'\Lambda)$ is a closed, convex subset

of $M_{N \times K}$, containing all matrices of rank one.

The principal's problem is to choose an element of $Z(U' \Pi' \Lambda)$ that maximizes his own expected payoff. We denote his set of optimal strategies by $Z^*(U \Pi \Lambda, U' \Pi' \Lambda)$.

$$Z^*(U \Pi \Lambda, U' \Pi' \Lambda) \equiv \{Z | Z \in \operatorname{argmax}_{X \in Z(U' \Pi' \Lambda)} \operatorname{tr} X U \Pi \Lambda\}.$$

Now we can see why the Stackelberg solution of this game is equivalent to a situation where the principal delegates authority to the agent. Let Z be any incentive-compatible strategy for the principal, and let $z_{i \cdot}$ denote the i^{th} row of Z . Consider any other strategy \hat{Z} , with generic row $\hat{z}_{i \cdot}$, and suppose that for each $i \in \{1, \dots, N\}$, there exists $j \in \{1, \dots, N\}$, such that $z_{i \cdot} = \hat{z}_{j \cdot}$. That is, all the row vectors appearing in Z are represented in \hat{Z} . Then clearly there exists a strategy \hat{R} for the agent (composed of zeros and ones) such that $Z = \hat{R} \hat{Z}$.

In effect, the principal's only choice is about the composition of the individual rows of the matrix Z --the order of the rows is unimportant. These row vectors act as a "menu" from which the agent chooses. It is as if the principal selects a set of (possibly randomized) alternatives (the row vectors in Z) and delegates to the agent authority to choose (possibly in a randomized fashion) from among those alternatives after the latter observes the signal. (Here it is assumed that the agent acts in the principal's best interest if he himself is indifferent among two or more alternatives.) In the rest of the paper we will talk about the game in terms of the delegation problem.

3. When is Information Valuable?

Our goal is to develop partial orderings of information structures for

the two players in this game that parallel the criterion of informativeness developed by Blackwell [1951] for the single player case. Specifically, we wish to develop a partial ordering that says " Λ is better for the principal than Λ' " if for all utility matrices and priors U, U', Π, Π' , the principal's expected utility at the equilibrium of the game is higher when the agent's information structure is Λ than when it is Λ' . An analogous criterion will be used for the agent.

If the principal and agent have identical priors and utility matrices, their joint decisions mimic those of a single decision-maker. In this case, one information system is better than another for both players if and only if it is more informative in the sense of Blackwell. I.e., more informativeness in Blackwell's sense is clearly necessary for either player's welfare to be surely not decreased. Thus, the partial orderings for the two-player case can only add restrictions to Blackwell's criterion.

In Theorem 1 we prove that there is no condition sufficient to guarantee an increase in the equilibrium expected utility of the agent, for all utility matrices and priors. Better information can always hurt the agent.

In Theorem 2 we prove that the principal is necessarily better off if the improvement in information takes a special form that we call success-enhancing. These improvements correspond to decreasing the probability that the signal is totally uninformative, and increasing equiproportionately the probability of receiving each of the genuinely informative observations.

Obviously any improvement in a totally uninformative system is valuable to the principal. Since any such improvement is success-enhancing, the converse of Theorem 2 holds if the less informative system has rank one. In Theorem 3 we prove that the converse of Theorem 2 also holds if the less informative system has rank three or more. Information structures of rank two

are a somewhat special case. There are some improvements in rank two structures that are not success-enhancing, and yet never decrease the principal's equilibrium expected utility. A wider class of improvements that are necessarily valuable to the principal in the rank two case are characterized in Theorem 4.

Before proceeding, we need to define a new term. Note that different information structures may convey exactly the same information. For example, given $\Lambda \in \mathbf{M}_{M \times N}$, for any $P \in \mathbf{P}_N$, the information system $\Lambda' = \Lambda P$ conveys exactly the same information as Λ does. Moreover, two information systems may convey identical information even if they are of different dimension. For example, consider:

$$\Lambda = \begin{bmatrix} .2 & .8 \\ .6 & .4 \end{bmatrix} \qquad \Lambda' = \begin{bmatrix} .2 & .6 & .2 \\ .6 & .3 & .1 \end{bmatrix}$$

The information system Λ' is identical to Λ except that it "divides" the last signal value, randomly calling it y'_2 with probability .75 and calling it y'_3 with probability .25. This conveys no additional information.

We will sometimes find it useful to deal with information systems that do not contain these redundant signal values. Thus, we will say that an information structure $\Lambda_{M \times N}$ is concise if:

there does not exist $j, k \in \{1, \dots, N\}$, $j \neq k$, $\alpha > 0$;
 such that $\lambda_{ij} = \alpha \lambda_{ik}$, $i = 1, \dots, M$.

That is, Λ is concise if none of its columns are scalar multiples of each

other. Clearly, if Λ and Λ' are both concise, then they are equally informative if and only if $\Lambda' = \Lambda P$ for some $P \in P_N$. If Λ and Λ' are both concise, we will say that they are essentially identical if $\Lambda' = \Lambda P$ for some $P \in P_N$, and essentially different otherwise.

We are now ready to prove that better information can always decrease the expected utility of the agent. That is, for any two information structures Λ and Λ' , there exists a payoff structure $V = U\Pi$ for the principal and a pair of payoff structures $V' = U'\Pi'$ and $V'' = U''\Pi'$ for the agent such that:

- 1) if the agent's payoff structure is V'' his expected utility is higher under Λ than under Λ' ;
- 2) if the agent's payoff structure is V' his expected utility is higher under Λ' than under Λ .

Theorem 1: Let $\Lambda_{M \times N}$ and $\Lambda'_{M \times N}$ be given. Assume that each is concise and that they are essentially different. Then there exist matrices V, V', V'' such that:

$$\begin{aligned} \text{tr } Z^* V' \Lambda &> \text{tr } Z^* V' \Lambda' \\ \text{tr } Z^* V'' \Lambda &< \text{tr } Z^* V'' \Lambda' \end{aligned} \tag{1}$$

where

$$\begin{aligned} \{Z^*\} &= Z^*(V\Lambda, V''\Lambda) = Z^*(V\Lambda, V'\Lambda) \\ \{Z^*\} &= Z^*(V\Lambda', V''\Lambda') = Z^*(V\Lambda', V'\Lambda') \end{aligned}$$

Proof: First a proof for the case $M = 2$ will be given, then the extension to the case $M > 2$.

Assume $M = 2$. Since Λ and Λ' are both concise and are essentially different, either $\Lambda' \neq \Lambda B$ for all $B \in M_{N \times N}$, or $\Lambda \neq \Lambda' B$ for all $B \in M_{N' \times N}$.

Without loss of generality suppose the latter holds. Then as shown in the Appendix there exists an $N \times 2$ matrix \hat{V} and a constant c such that:

$$\text{tr } Z^\dagger \hat{V} \Lambda > c > \text{tr } Z^{\dagger\dagger} \hat{V} \Lambda' . \quad (2)$$

where

$$Z^\dagger \equiv \underset{Z \in M_{N \times N}}{\text{argmax}} \text{tr } Z \hat{V} \Lambda$$

$$Z^{\dagger\dagger} \equiv \underset{Z \in M_{N' \times N}}{\text{argmax}} \text{tr } Z \hat{V} \Lambda'$$

Define the $(N+1) \times 2$ matrices:

$$V \equiv \begin{bmatrix} \hat{V} \\ \hline c \quad c \end{bmatrix}, \quad V' = V + \begin{bmatrix} \alpha \underline{1} & \alpha \underline{1} \\ \hline 0 & 0 \end{bmatrix}, \quad V'' \equiv V + \begin{bmatrix} 0 & 0 \\ \hline \alpha & \alpha \end{bmatrix},$$

where c is the constant in (2), and $\alpha > 0$. For $\alpha > 0$ sufficiently large, obviously:

$$Z(V' \Lambda) = Z(V'' \Lambda) = \text{Co} \{ \{ \underline{1} \ 0 \ \dots \ 0 \}, \dots, \{ 0 \ \dots \ 0 \ \underline{1} \}, \{ Z^\dagger \mid 0 \} \}$$

$$Z(V' \Lambda') = Z(V'' \Lambda') = \text{Co} \{ \{ \underline{1} \ 0 \ \dots \ 0 \}, \dots, \{ 0 \ \dots \ 0 \ \underline{1} \}, \{ Z^{\dagger\dagger} \mid 0 \} \}$$

It is obvious that in either case all strategies except the next-to-last are, from the principal's point of view, dominated by the last. Since (2) holds, the optimal strategies are:

$$Z^*(V \Lambda, V' \Lambda) = Z^*(V \Lambda, V'' \Lambda) = \{ \{ Z^\dagger \mid 0 \} \} ,$$

$$Z^*(V \Lambda', V' \Lambda') = Z^*(V \Lambda', V'' \Lambda') = \{ \{ 0 \ \dots \ 0 \ \underline{1} \} \} .$$

For $\alpha > 0$ sufficiently large, clearly (1) holds.

Suppose $M > 2$. Since Λ and Λ' are essentially different, there exist two states--without loss of generality suppose they are θ_1 and θ_2 --such that

either $\Lambda_2' \neq \Lambda_2 B$ for all $B \in M_{N \times N'}$, or else $\Lambda_2 \neq \Lambda_2' B$ for all $B \in M_{N' \times N}$, where Λ_2 and Λ_2' are the submatrices consisting of the first two rows of Λ and Λ' respectively. Without loss of generality suppose the latter holds. The analysis above can be carried out for Λ_2, Λ_2' , yielding $(N+1) \times 2$ matrices V, V', V'' . The $(N+1) \times M$ matrices $[V|0 \dots 0], [V'|0 \dots 0], [V''|0 \dots 0]$, clearly satisfy (1). Q.E.D.

Since we can choose Π, Π', U, U', U'' so that $\Pi U = V, \Pi' U' = V'$ and $\Pi' U'' = V''$, Theorem 1 shows that from the agent's point of view, any two information systems that are essentially different are incomparable unless something is known about the payoff structures.

Next we show that from the principal's point of view a partial ordering of information structures is possible. However, because the conditions are quite restrictive, the ordering is very incomplete. The required conditions depend on the rank of the less informative information system Λ' . There are three cases to consider: $r(\Lambda') = 1, r(\Lambda') = 2$, and $r(\Lambda') > 3$. However, first we need the following definition.

We will say that an information matrix $\Lambda_{M \times N} = [\lambda_{.1} \dots \lambda_{.N}]$ is in standard form if:

$$\Lambda = [\alpha \underline{1} \mid (1-\alpha)\Gamma], \quad \alpha > 0, \Gamma \in M_{M \times (N-1)},$$

where Γ is in concise form and Λ is in concise form if $\alpha \neq 0$. Thus, a matrix in standard form is simply a matrix in concise form, with the additional requirement that a signal value which is completely uninformative is labeled as the first. If none of the signal values have this property a dummy signal value, which occurs with probability zero in every state, is labeled as the

first. If we think of the signal variable as the outcome of an experiment, then the outcome y_1 corresponds to "dropping the test tube," so that the experiment provides no information. If the experiment is successfully carried through, it provides the information described by Γ .

A success-enhancing improvement in information corresponds to decreasing the probability that the experiment fails, but does not change the information provided by successful experiments. Thus an improvement in information from Λ' to Λ is success-enhancing if, when Λ and Λ' are both in standard form, there exists $\alpha, \alpha', \Gamma, P$ with:

$$0 < \alpha < \alpha' < 1, \quad \Gamma \in M_{M \times (N-1)}, \quad P \in P_{N-1},$$

such that,

$$\Lambda = [\alpha \underline{1} \mid (1-\alpha)\Gamma]$$

$$\Lambda' = [\alpha' \underline{1} \mid (1-\alpha')\Gamma P].$$

Note that for success-enhancing improvements the garbling matrix $B \in M_{N \times N}$ satisfying $\Lambda' = \Lambda B$ has the form:

$$B = \left[\begin{array}{c|c} 1 & \underline{0}^T \\ \hline \sigma \underline{1} & (1-\sigma)P \end{array} \right], \quad (1-\sigma) = (1-\alpha')/(1-\alpha). \quad (3)$$

If an information matrix Λ' has rank one, i.e., if the distribution of signal values is independent of the true state, then it is completely uninformative. If it is in standard form, $\Lambda' = [\underline{1}]$. Obviously any improvement in such a system is success-enhancing. Moreover, any improvement in such an information system can only benefit the principal, since he always has the option of ignoring all information. In Theorem 2 we show that this conclusion can be extended to all success-enhancing improvements in information.

Theorem 2: If Λ is a success-enhancing improvement of Λ' , then:

$$\max_{Z \in Z(V'\Lambda)} \text{tr } ZVA > \max_{Z \in Z(V'\Lambda')} \text{tr } ZVA', \quad \forall V, V'. \quad (4)$$

Proof: Without loss of generality assume that $\Lambda' = \Lambda B$, where:

$$\Lambda = [\alpha \underline{1} \mid (1-\alpha)\Gamma], \quad 0 < \alpha < 1 ;$$

$$B = \begin{bmatrix} 1 & \vdots & 0^T \\ \vdots & \ddots & \vdots \\ \sigma \underline{1} & \vdots & (1-\sigma)I \end{bmatrix} = \sigma [\underline{1} \ 0 \dots 0] + (1-\sigma)I, \quad 0 < \sigma < 1.$$

Recall that $Z \in Z(V'\Lambda)$ if and only if $ZV'\Lambda$ is column dominant along the diagonal. Choose $Z' \in Z(V'\Lambda B)$, and let $X \equiv Z'V'$. It follows straightforwardly that:

$$Z'V'\Lambda B = XAB = [(\alpha + \sigma(1-\alpha))X\underline{1} \mid (1-\sigma)(1-\alpha)X\Gamma] ;$$

$$(BZ')V'\Lambda = BXA = \sigma \underline{1} w^T + (1-\sigma) [\alpha X\underline{1} \mid (1-\alpha)X\Gamma] ;$$

where w^T is the first row of XA . Each of these matrices is column dominant along the diagonal if and only if $[X\underline{1} \mid X\Gamma]$ is. Hence

$Z' \in Z(V'\Lambda B) \implies (BZ') \in Z(V'\Lambda)$, and (4) must hold. Q.E.D.

In Theorem 3 we prove that if $r(\Lambda) > 3$, the converse of Theorem 2 holds. The proof uses the following lemma, which establishes that the information structure Λ is more valuable to the principal than Λ' in all cases involving only two actions, only if the set of joint distributions of actions and states attainable under Λ includes all those attainable under Λ' .

Define V_M by:

$$V_M \equiv \{V_{2 \times M} \mid V = [v \ 0]^T, v \in R^M\}.$$

Lemma: Let $\Lambda \in M_{M \times N}$, $B \in M_{N \times N}$, be given. Then

$$\text{Max}_{Z \in Z(V' \Lambda)} \text{tr } ZV\Lambda > \text{Max}_{Z' \in Z(V' \Lambda B)} \text{tr } Z'V\Lambda B, \quad \forall V, V' \in V_M;$$

only if for all $V' \in V_M$:

$$Z' \in Z(V' \Lambda B) \implies \text{there exists } Z \in Z(V' \Lambda), \text{ such that } \Lambda Z = \Lambda B Z'. \quad (5)$$

Proof: Note that for any $V' \in V_M$ and any $\Lambda \in M_{M \times N}$, each element of $Z(V' \Lambda)$ has the form $[z \ \underline{1-z}]$, $z \in [0,1]^N$. For any N -vector $x = v^T \Lambda$, define the function z by:

$$z(x) \equiv \{z \in [0,1]^N \mid [z \ \underline{1-z}] \in Z([x \ 0]^T)\}.$$

Since $Z(X)$ is a convex set for any X , $z(x)$ is convex for any x .

Suppose (5) fails for $V' = [v' \ 0]^T$. Then there exists $z^\dagger \in z(v'^T \Lambda B)$ such that:

$$\Lambda(z - Bz^\dagger) \neq 0, \quad \forall z \in z(v'^T \Lambda).$$

Since $z(v'^T \Lambda)$ is convex, by the separating hyperplane theorem, there exists $v \in R^M$ such that:

$$v^T \Lambda(z - Bz^\dagger) < 0, \quad \forall z \in z(v'^T \Lambda).$$

Hence for $V = [v \ 0]^T$:

$$\text{Max}_{Z' \in Z(V' \Lambda B)} \text{tr } VABZ' > v^T \Lambda B z^\dagger > \text{Max}_{z \in Z(v^T \Lambda)} v^T \Lambda z = \text{Max}_{Z \in Z(V' \Lambda)} \text{tr } VAZ. \quad \text{Q.E.D.}$$

Theorem 3: Let $\Lambda_{M \times N}$, $\Lambda'_{M \times N}$, be given, with both in standard form. If (4) holds and $r(\Lambda') > 3$, then Λ is a success-enhancing improvement of Λ' .

Proof: Assume (4) holds. Then in particular it holds whenever $V' = V$, and it follows from Blackwell's Theorem that there exists $B \in M_{N \times N}$, such that $\Lambda' = \Lambda B$.

For any $v \in R^M$ and any $\Lambda \in M_{M \times N}$, define the function z^\dagger by:

$$z^\dagger(v^T \Lambda) \equiv \{z^\dagger \mid z^\dagger \in \text{argmax}_{z \in [0,1]^N} v^T \Lambda z\}.$$

Note that generically $z^\dagger(v^T \Lambda)$ is a singleton, and that in any case:

$$z(v^T \Lambda) = \text{co}(\{0, 1\}, z^\dagger(v^T \Lambda)),$$

where $z(v^T \Lambda)$ is defined as in the Lemma. Thus (5) holds for $V = [v \ 0]^T$ if and only if:

$$\begin{aligned} &\text{for each } z^{\dagger'} \in z^\dagger(v^T \Lambda'), \text{ there exists } z^\dagger \in z^\dagger(v^T \Lambda), \\ &\text{and coefficients } \alpha_1, \alpha_2 > 0, \text{ with } \alpha_1 + \alpha_2 < 1, \\ &\text{such that } \Lambda' z^{\dagger'} = \alpha_1 \Lambda z^\dagger + \alpha_2 \underline{1}. \end{aligned} \quad (6)$$

Since by hypothesis (4) holds, by Lemma 1, (6) holds for all $v \in R^M$.

Assume that Λ and Λ' are both in standard form, and let

$\lambda_{.j}$, $j = 1, \dots, N$, and $\lambda'_{.j}$, $j = 1, \dots, N'$, denote the column vectors of Λ and Λ' respectively. First we will show that each element in the set $\{\lambda_{.2}, \dots, \lambda_{.N}\}$ is a scalar multiple of an element in $\{\lambda'_{.2}, \dots, \lambda'_{.N'}\}$ and conversely. Suppose that for $\lambda_{.j^*}$, $j^* \in \{2, 3, \dots, N\}$,

$$\lambda_{.j^*} \neq \alpha \lambda'_{.j}, \quad \forall \alpha > 0, \quad \forall j \in \{2, \dots, N'\}.$$

Since $r(\Lambda) > r(\Lambda') > 3$, and since Λ, Λ' are both in standard form, by the Baire category theorem, there exists $v \in R^M$ such that:

$$v^T \lambda_{.j^*} = 0; \tag{7a}$$

$$v^T \lambda_{.j} \neq 0, \quad \text{for all } j \neq j^*, j \in \{2, \dots, N\}; \tag{7b}$$

$$v^T \lambda'_{.j} \neq 0, \quad \text{for all } j \in \{2, \dots, N'\}; \tag{7c}$$

$$v^T \lambda'_{.j} > 0, \quad \text{for some } j \in \{2, \dots, N'\}; \tag{7d}$$

$$v^T \lambda'_{.j} < 0, \quad \text{for some } j \in \{2, \dots, N'\}; \tag{7e}$$

and $z^\dagger(v^T \Lambda)$ and $z^\dagger(v^T \Lambda')$ are each singletons. Moreover, there exists $\epsilon \in R^M$, with $|\epsilon|$ sufficiently small, such that:

$$z^\dagger((v+\epsilon)^T \Lambda') = z^\dagger(v^T \Lambda');$$

$$z^\dagger((v+\epsilon)^T \Lambda) = z^\dagger(v^T \Lambda) + e_{j^*}.$$

Since $z_j^\dagger(v^T \Lambda') = 1$ if and only if $v^T \cdot \lambda'_{.j} > 0$, from (7d) and (7e) it follows that $\Lambda' z^\dagger(v^T \Lambda') \neq \beta \mathbf{1}$, $\forall \beta$. Hence (6) cannot hold for both v and $v+\epsilon$.

Repeating the argument with the roles of Λ and Λ' reversed establishes the converse.

Hence there exists a diagonal matrix D and a permutation matrix P such that $\Lambda' = \Lambda DP$. Without loss of generality assume that $P = I$, so that $z^\dagger(v^T \Lambda') = z^\dagger(v^T \Lambda)$, $\forall v \in R^M$. Since generically in v^T , $z^\dagger(v^T \Lambda)$ is a singleton, (6) requires that generically for $v \in R^M$:

$$\Lambda D z^\dagger(v^T \Lambda D) = \alpha_1 \Lambda z^\dagger(v^T \Lambda) + \alpha_2 \underline{1}. \quad (8)$$

Now consider a sequence of vectors $\{v_\nu\}_{\nu=0, \dots, N}$, constructed so that each set in the sequence $\{z^\dagger(v_\nu^T \Lambda)\}$ is a singleton, and:

$$z^\dagger(v_0^T \Lambda) = \{0\}$$

$$z^\dagger(v_\nu^T \Lambda) = z^\dagger(v_{\nu-1}^T \Lambda) + e_{k_\nu}, \quad \nu = 1, \dots, N,$$

where (k_1, \dots, k_N) is a permutation of $(1, \dots, N)$. Clearly (8) is satisfied for every element of $\{z^\dagger(v_\nu^T \Lambda)\}$ if and only if $d_i = d_j$, $i, j = 2, \dots, N$. Hence B has the form in (3). Q.E.D.

Theorems 2 and 3 taken together establish that for information structures of rank three or more, an improvement in information is necessarily valuable to the principal if and only if it is success-enhancing. The proof of Theorem 3 does not go through if $r(\Lambda') = 2$, and in fact, if $r(\Lambda') = 2$ a wider class of improvements are valuable to the principal. For example, Green [1979] has shown that any improvement in a binomial channel increases the expected utility of the principal. A class of improvements in information systems of

rank two that are necessarily beneficial for the principal and that include all improvements in a binomial channel, is characterized in Theorem 4.

Theorem 4: Let $\Lambda \in M_{M \times N}$ and $\Lambda' \in M_{M \times N'}$ be given, with $r(\Lambda) = 2$. Without loss of generality, assume that the columns of Λ are ordered so that:

$$\Lambda = [\lambda_{.1} \quad \lambda_{.N}] [\alpha \quad \beta]^T, \quad \alpha, \beta \in R_+^N; \quad (9a)$$

$$i < j \implies \beta_i \alpha_j < \beta_j \alpha_i, \quad i, j, = 1, \dots, N. \quad (9b)$$

Suppose that:

$$\Lambda_2 = \Lambda'D \quad (10)$$

for some partition $[\Lambda_1 \mid \Lambda_2 \mid \Lambda_3]$ of Λ and some diagonal matrix D with $0 < d_{ii} < 1$, $i = 1, \dots, N'$. Then Λ is better for the principal than Λ' , i.e., (4) holds.

Proof: Note that if Λ and Λ' satisfy (9) and (10), then there exists

$B = [B_1^T \mid I_2 \mid B_3^T]^T \in M_{N \times N'}$, such that $\Lambda' = \Lambda B$, where B is partitioned to conform to Λ . Define $\alpha', \beta' \in R_+^{N'}$ by $\Lambda' \equiv [\lambda_{.1} \quad \lambda_{.N}] [\alpha' \quad \beta']^T$.

Let V and V' be given, and choose $Z^* \in Z^*(V\Lambda', V'\Lambda')$. By definition Z^* satisfies:

$$0 < (z_{i.}^* - z_{j.}^*) V' \lambda'_{.i} = (z_{i.}^* - z_{j.}^*) V' [\lambda_{.1} \quad \lambda_{.N}] \begin{pmatrix} \alpha'_i \\ \beta'_i \end{pmatrix}, \quad i, j = 1, \dots, N'.$$

Hence it follows from (9b) that for $a, b \geq 0$:

$$\begin{aligned} (z_{i.}^* - z_{j.}^*) V' [\lambda_{.1} \quad \lambda_{.N}] \begin{pmatrix} a \\ b \end{pmatrix} > 0 & \quad \text{if } a\beta_i\beta_j < \alpha_i b\beta_j < \alpha_j b\beta_i, \\ & \quad \text{if } a\beta_i\beta_j > \alpha_i b\beta_j > \alpha_j b\beta_i. \end{aligned} \quad (11)$$

Define Z^* , partitioned to conform with Λ , by:

$$Z^* \equiv [E_1^T \mid I_2 \mid E_{N'}^T]^T Z^{*'} ,$$

where each row of E_1^T is the vector e_1^T . First note that (9) and (11) imply directly that Z^* is an incentive-compatible strategy when the agent's information structure is Λ , i.e., that $Z^* \in Z(V'\Lambda)$. Next, note that since Z^* is optimal for the principal under Λ' , it follows that:

$$(z_{1'}^* - z_{j'}^*)V[\lambda_{.1} \quad \lambda_{.N'}][\begin{matrix} a \\ b \end{matrix}] > 0, \quad \text{if } b\alpha_1' < \beta_1'a;$$

and

(12)

$$(z_{N'}^* - z_{j'}^*)V[\lambda_{.1} \quad \lambda_{.N'}][\begin{matrix} a \\ b \end{matrix}] > 0, \quad \text{if } b\alpha_{N'}' > \beta_{N'}'a ;$$

$$j = 1, \dots, N',$$

Using (9b) and (12), we see that:

$$\text{tr } VAZ^* - \text{tr } VA'Z^{*'} = \text{tr } VA[E_1^T - B_1^T \mid 0 \mid E_{N'}^T - B_3^T]^T Z^{*}'$$

$$= \text{tr } [E_1^T - B_1^T \mid 0 \mid E_{N'}^T - B_3^T]^T Z^{*}' V[\lambda_{.1} \quad \lambda_{.N'}][\alpha \quad \beta]^T > 0.$$

Hence the principal's expected utility is higher under Λ with the strategy Z^* than under Λ' with the strategy $Z^{*'}$, i.e., (4) holds. Q.E.D

Note that any improvement in a binomial channel satisfies the conditions of Theorem 4, with $\Lambda = [\lambda_{.1} \quad 0 \quad 0 \quad \lambda_{.4}]$.

4. Conclusions

We have shown that any improvement in information may decrease the agent's expected utility, and that, except for success-enhancing improvements and for cases where the worse information system has rank two, improvements may decrease the principal's expected utility. Moreover, we know that restrictions on utility payoffs alone cannot guarantee that arbitrary improvements in the agent's information system will increase either party's equilibrium expected utility (cf. footnote 2). Therefore, it seems that joint restrictions on the payoff functions, prior beliefs, and type of improvement in information are needed to insure that expected utilities do not fall.

The situation analyzed above does not include side-payments. Allowing such payments, contingent on the signal transmitted or on the action taken, can help to narrow the gap between the interests of the two parties. The extent to which they reduce the incentive problems in multi-party decision situations is an area for further investigation.

APPENDIX

For any $M \times N$ matrix $X = [x_{ij}]$, let $\rho(X)$ denote the row ravel of X , i.e., $\rho(X)$ is the MN -vector $(x_{11}, \dots, x_{1N}, x_{21}, \dots, x_{MN})$. Similarly, let $\hat{\rho}(X)$ denote the column ravel of X , i.e., $\hat{\rho}(X) = \rho(X^T)$.

Suppose that $\Lambda \neq \Lambda'B$, for all $B \in M_{N' \times N}$. Define T as the following set of MN -vectors:

$$T = \{t \in R^{MN} \mid t = \hat{\rho}(\Lambda'B - \Lambda), B \in M_{N' \times N}\}.$$

Clearly T is a closed, convex set and $0 \notin T$. Hence by the separating hyperplane theorem there exists $\hat{v} \in R^{MN}$ such that $\hat{v} \cdot t < 0$, for all $t \in T$. Define the $N \times M$ matrix \hat{V} by $\rho(\hat{V}) = \hat{v}$. By construction:

$$\text{tr } \hat{V}(\Lambda'B - \Lambda) = \rho(\hat{V}) \cdot \hat{\rho}(\Lambda'B - \Lambda) < 0, \quad \text{for all } B \in M_{N' \times N},$$

i.e.,

$$\text{Max}_{B \in M_{N' \times N}} \text{tr } \hat{V}\Lambda'B < \text{tr } \hat{V}\Lambda < \text{Max}_{B \in M_{N \times N}} \text{tr } \hat{V}AB.$$

FOOTNOTES

¹Throughout the paper the term "better information" will mean "better" under the criterion of statistical decision theory, i.e., in the sense of Blackwell [1951].

²Examples to this effect are available from the authors on request.

³If commitment is not possible, then Nash equilibrium is the appropriate concept. This has been studied by the authors in a separate paper [1980].

⁴If the strategy choices are Z and R the expected utilities of the two individuals can be written as:

$$\sum_m \pi_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u_{km}$$

and

$$\sum_m \pi'_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u'_{km}$$

for the principal and agent respectively. The interpretation of these expressions is straightforward. We simply sum up all the ways in which each action could be taken given each possible state, by multiplying the conditional probabilities of observations given states, transmissions given observations, and actions given transmissions, and weighting by the prior probabilities. A more compact way of writing these is as $\text{tr } \Pi \text{ARZU}$ and $\text{tr } \Pi' \text{ARZU}'$ respectively.

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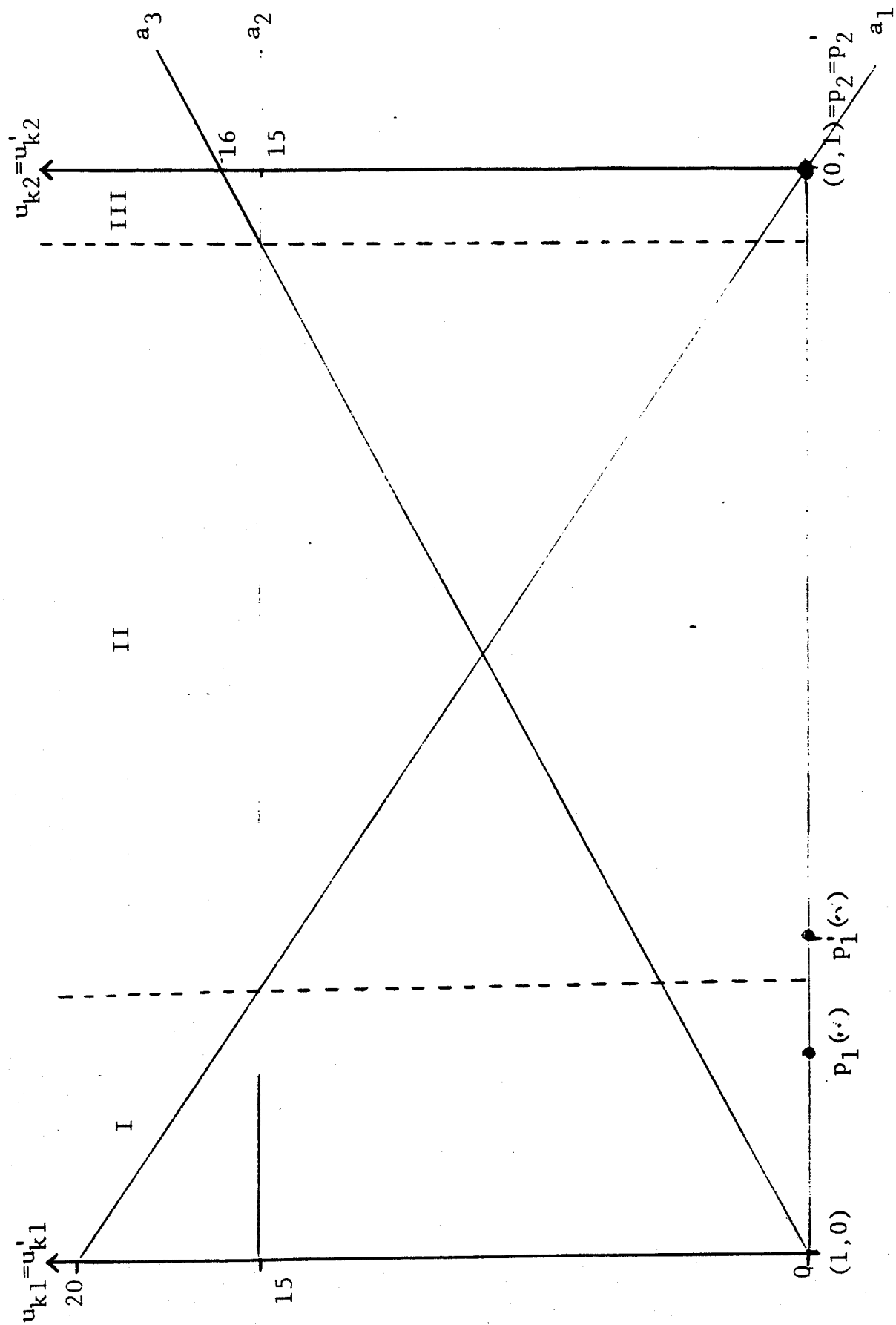


Figure 1

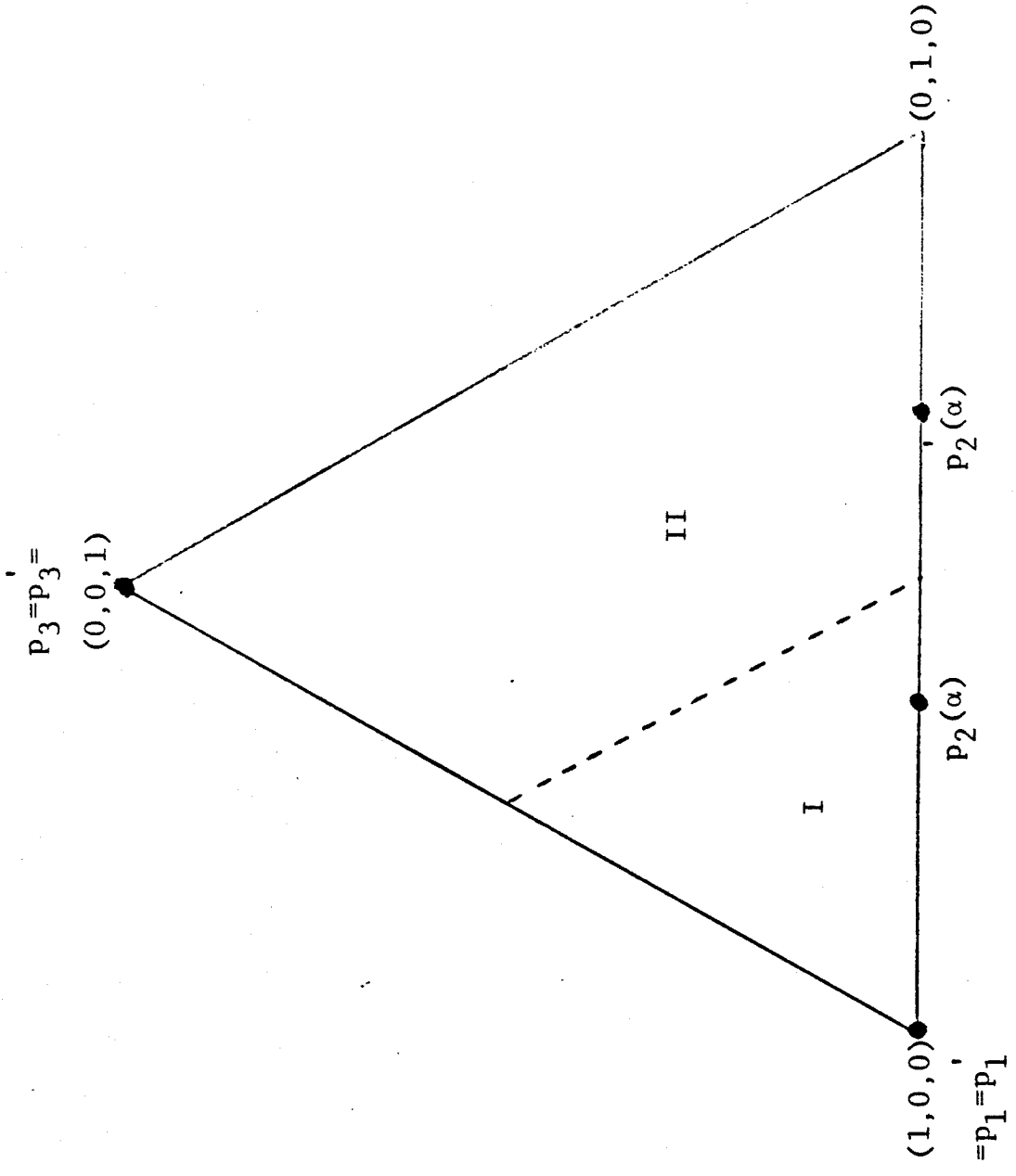
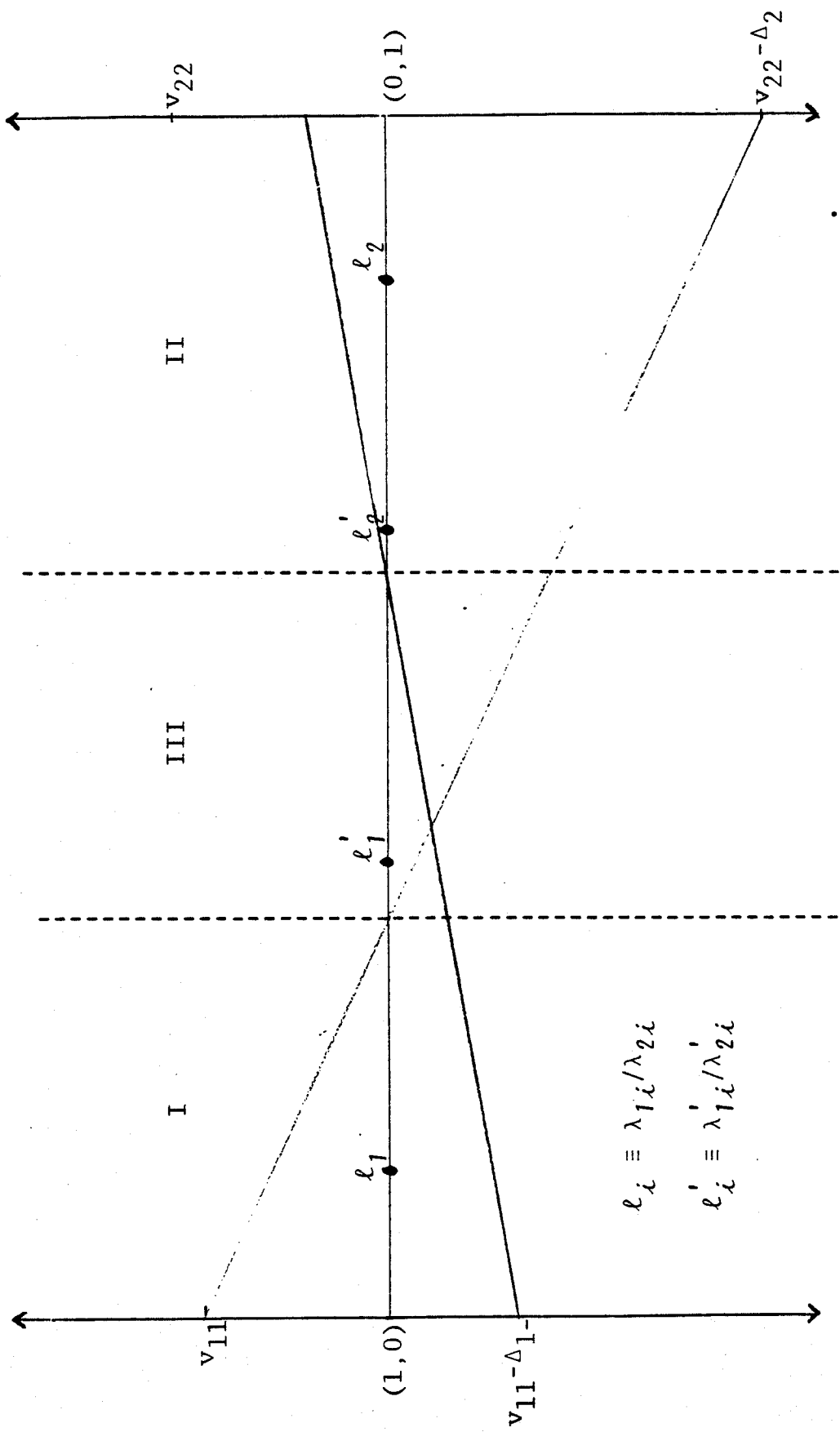


Figure 2

"



$$l_i \equiv \lambda_{1i} / \lambda_{2i}$$

$$l'_i \equiv \lambda'_{1i} / \lambda'_{2i}$$

Figure 3