A TWO-PERSON GAME OF INFORMATION TRANSMISSION*

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ABSTRACT

We consider a statistical decision problem faced by a two player organization whose members may not agree on outcome evaluations and prior probabilities. One player is specialized in gathering information and transmitting it to the other, who takes the decision. This process is modeled as a game. Qualitative properties of the equilibria are analyzed. The impact of improving the quality of available information on the equilibrium welfares of the two individuals is studied. Better information generally may not improve welfare. We give conditions under which it will.

I. Introduction

When a decision is made by a group rather than an individual, the twin tasks of acquiring information on the one hand, and choosing a course of action on the other, are often delegated to separate sets of individuals. If all members of the group share common evaluations of the outcomes and have identical prior beliefs, then there is no conflict between the information-gatherers and the action-takers. Information will be accurately transmitted by the former and optimally utilized by the latter.

Here we study the situation that arises when interests do not coincide. When interests diverge, complete transmission may result in actions that are suboptimal from the information-gatherers' point of view. The situation is one of partial conflict. We model it as a game in which each of the two functions is executed by a single rational individual, neglecting conflicts among information-gatherers or among agents controlling different aspects of the group's action. We examine the Nash equilibria of the resulting two-person game. In particular, we look at the effect on the expected utilities of the two players of improvements in the available information.

The two individuals will be called the agent and the principal. Their joint decision problem is to choose one of the actions, a_k, from the set $A = \{a_1, \ldots, a_K\}$. The von Neumann - Morgenstern utility levels of the two participants depend upon the chosen action and the realization of the state of nature, θ_m, from the set $\Theta = \{\theta_1, \ldots, \theta_M\}$. These utilities can be represented by K × M matrices $U = (u_{km})$ and $U' = (u'_{km})$ for the principal and the agent respectively, where the elements are the utilities realized if a_k is chosen and θ_m occurs.

The agent receives an observation which is statistically related to the true state in Θ, and transmits the observation to the principal. He might not do so truthfully. There are N possible observations, y_n, in the set $Y = \{y_1, \ldots, y_N\}$. Allowing randomizations, his strategies are therefore representable by a Markov N × N matrix $R = (r_{nn'})$, where $r_{nn'}$ is the probability that $y_{n'}$ will be transmitted given that the actual observation is y_n.

The principal chooses the action $a_k \epsilon A$ given that the observation $y_{n'}$ has been transmitted to him. Again, allowing randomization, his strategy is a N × K Markov matrix $Z = (z_{n'k})$, where $z_{n'k}$ is the probability that a_k will be chosen given that $y_{n'}$ was transmitted.*

The statistical relationship between states and observations is called the information structure. It is represented by an M × N Markov matrix

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* Wherever possible we will try to follow the convention of labeling the typical actions and states with the running indices k and m, and the true and transmitted observations by n and n'.

$\Lambda = (\lambda_{mn})$, where λ_{mn} is the probability that y_n will be observed if the true state is θ_m. The interpretation of y_n depends on the prior beliefs of the individual in question. We allow different beliefs, $\pi = (\pi_m)$ and $\pi' = (\pi'_m)$ for the principal and agent respectively. The posterior probabilities of the states, given an observation, can be derived from π and Λ by Bayes rule. These posteriors are denoted $(p_1, \ldots, p_N)$, where $p_n \varepsilon \Delta^M$, the set of all M-dimensional probability vectors, is the posterior given y_n. The probability of receiving each y_n is also implied by π and Λ. Thus we have a distribution of the posterior which is simply the measure over Δ^M assigning the corresponding weight to each of the p_n.

If the strategy choices are Z and R the expected utilities for the principal and agent respectively are:

(1.1) $\quad \text{tr } U\hat{\pi}\Lambda RZ$

and

(1.2) $\quad \text{tr } U'\hat{\pi}'\Lambda RZ$

where the ^ over the vectors π and π' denotes the square matrices with these vectors on the diagonal and zeros elsewhere.*

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* In more detail, we have

$$\sum_m \pi_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u_{km}$$

and

$$\sum_m \pi'_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u'_{km}$$

The interpretation of these expressions is straightforward. We simply sum up all the ways in which each action could be taken given each possible state, by multiplying the conditional probabilities of observations given states, transmissions given observations, and actions given transmissions, and weighting by the prior probabilities.

In this paper we examine the Nash equilibria of this game. A pair of Markov matrices, (Z,R) is a <u>Nash equilibrium</u> if Z maximizes (1.1) and R maximizes (1.2).

The main results of the paper can be viewed in the tradition of comparative statics. We are interested in the consequences of changes in the information structure (Y,Λ) on the equilibria of the game. Specifically, it is well-known that a partial ordering of information structures according to the criterion of <u>more informativeness</u> can be given a precise mathematical characterization. This is an ordering based on single-person statistical decision theory. An information structure (Y,Λ) is said to be <u>more informative</u> than (Y',Λ') if for any U and any π, the decision problem under the former has at least as high a value as that under the latter. Using the notation developed above, this can be restated as

$$\max \operatorname{tr} U\hat{\pi}\Lambda Z \geq \max \operatorname{tr} U\hat{\pi}\Lambda' Z'$$

where the maximum in each case is taken over all Markov matrices Z (or Z') of the appropriate dimensionality. Blackwell (1951) has shown that the following condition is equivalent to more informativeness:

there exists a Markov matrix B
such that $\Lambda' = \Lambda B$.

We want to study the relation between this condition and conditions sufficient for the improvement of the welfare of one or both of the players in our two-person organization. Because of the compounding of game theoretic aspects with the usual decision theoretic issues, the welfare of the two players may not be monotonic with respect to the quality of the information structure. Several types of complication arise.

First, as in most games, there can be multiple equilibria. We have found it hard to analyze all of them. However, a natural classification of equilibria can be given, and one type, which we shall call <u>partition</u> <u>equilibria</u> have a rather regular behavior. Moreover we will give some arguments to the effect that these equilibria have desirable properties, and are hence "more likely" to be observed.

Second, as in the case of general equilibrium theory, the set of equilibria behaves lower hemi continuously with respect to changes in the parameters. Comparative static results therefore tend to be only "local." "Small" improvements in the information structure, suitably defined, are thus the objects of our investigation.

Third, and finally, the comparative static results turn out to be different for the two players. For the agent, any small improvement in the information structure will improve his equilibrium utility at a partition equilibrium. For the principal this may not be the case. His welfare can be guaranteed to be monotonic only when a very special kind of improvement in information is considered.

We define a <u>success-enhancing</u> improvement in information as one in which the probability that the observation is uninformative decreases, with a corresponding equiproportional increase in the probabilities of each of the other observations. If in the original information structure there is no such observation, that is if the posterior is unequal to the prior for every possible observation, then no success-enhancing improvements are possible. We show that small success-enhancing improvements in information necessarily improve the welfare of the principal at any partition equilibrium.

The remainder of the paper is organized as follows:

Section 2 covers the classification of types of equilibria and presents some genericity and stability-like arguments to bolster the case for considering partition equilibria.

Section 3 contains the main comparative static results mentioned above.

Section 4 contains several examples, primarily designed to illustrate directions in which our results cannot be extended.

II. Types of Equilibria and their Properties

Basic Classification

We begin by examining some general features of the set of equilibria. First we need the following definition.

We will call an information structure Λ' a partition of Λ if $\Lambda' = \Lambda PDP'$, where P and P' are permutation matrices and D is a block diagonal Markov matrix in which each block has rank one. When Λ' has this form, it is as if there is a partition of the signal space Y. Under Λ' if signal value y_k occurs, the partition element containing y_k is reported. Thus P rearranges signal values so that those in a common partition element are clustered together; each block along the diagonal of D corresponds to the report for one partition element; and P' rearranges the new signal values in any arbitrary way.

Of prime importance in our later analysis will be equilibria in which ΛR, the information the agent transmits, is a partition of Λ, the information he receives; these will be called partition equilibria. In addition there are two types of non-partition equilibria, distinguished by whether the principal uses pure or mixed strategies. It is useful to begin with an example that illustrates all three types.

Example 1

There are two states, two actions, and two observations: K = M = N = 2.

$$\Lambda = I$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$U' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\pi = \pi' = (1/2,\ 1/2)$$

One equilibrium of this game is the pair of strategies Z = I, R = I. Since the information transmitted by the agent is a partition (the complete refinement partition) of the space of observations, this is an example of a partition equilibrium.

Another equilibrium is one involving no transmission of information. This equilibrium is represented by any pair (Z,R) with:

$$Z = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix} , \qquad \alpha \in [0,1].$$

Note that for any value of α, $RZ = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Clearly $\Lambda' = \Lambda R$ is a partition of Λ (the partition consisting of one set, equal to the whole space), so that this is another example of a partition equilibrium.

Another type of equilibrium for this game is given by:

$$Z = I$$

$$R = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ 0 & 1 \end{pmatrix}$$

for $0 < \varepsilon \leq 1/2$.

Since $\Lambda' = \Lambda R$ is not a partition of Λ, this is not a partition equilibrium. Because the principal uses two distinct, nonrandomized actions, we call this a determinate action equilibrium. Note that in this case, randomization by the agent occurs only because he is indifferent between the two actions a_1 and a_2, given his information. Clearly this situation is non-generic.

The last type of equilibrium is given by:

$$Z = \begin{pmatrix} 1 & 0 \\ \delta & 1-\delta \end{pmatrix}$$

$$R = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

$$\delta \in (0,1) \ .$$

As in the previous case, $\Lambda' = \Lambda R$ is not a partition of Λ. However, in contrast to the previous case, the principal is playing a mixed strategy; we call this a random action equilibrium.

All equilibria of this form are non-robust to perturbations in U'. Whenever the agent evaluates the two actions differently after both observations, he will not be willing to transmit a randomized signal as long as Z has distinct rows. In this sense, these equilibria are non-robust for the same reason that applied to the previous type. However, in more general models the previous type, with non-randomized actions by the principal in all cases, is always non-robust, whereas this type, with both players randomizing, may behave continuously in the parameters. These differences are explored more fully below.

Returning to the equilibria with both players randomizing, we note that they are all unstable in the sense that the agent has many optimal responses to Z, namely all R of the form $R = \begin{pmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in [0,1]$ and the principal has many optimal responses to R, namely all Z of the form given above. But if the principal misperceives R even slightly, his optimal response is a non-randomized strategy (a Z matrix composed of zeros and ones), and the outcome would depart markedly from the equilibrium outcome.

Robustness of Partition Equilibria

In the rest of this section we will define partition, determinate action, and random action equilibria precisely, and argue that partition equilibria are robust in ways that the others are not. Specifically, we will show that determinate action equilibria are non-generic, and that random action equilibria are unstable against small perturbations in either player's strategy.

Formally we will say that an equilibrium pair (Z,R) is: a partition equilibrium if $\Lambda' = \Lambda R$ is a partition of Λ; a determinate action equilibrium if $\Lambda' = \Lambda R$ is not a partition of Λ, and each row of Z receiving positive weight under R has only a single positive element; a random action equilibrium if $\Lambda' = \Lambda R$ is not a partition of Λ, and some row of Z receiving positive weight under R has two or more nonzero entries.

Roughly speaking, we are presenting a "structural stability" argument to eliminate determinate action equilibria from consideration and a "dynamic stability" argument to eliminate the random action equilibria. Of course, as we do not present any real simultaneous adjustment process, we do not actually have any dynamics. This objection to random action equilibria is meant to be suggestive only, and not compelling. Nevertheless, partition equilibria will generically pass both of these tests.

Theorem 2.1

The set of all (U',Λ) for which there is any determinate action equilibrium is closed and null.

Proof

The existence of a determinate action equilibrium requires that for some observation the agent is indifferent among some of the actions in A. It therefore suffices to show that the set of U',Λ for which this indifference holds is closed and null. But this property is obvious because unless two rows of U' are identical (i.e. two actions are really the same) the set of all posterior probabilities under which there is more than one optimal action is of lower dimensions than Δ^M. Hence the set of all Λ matrices for which these posteriors arise is null. Closedness is obvious.

Theorem 2.2

Let (Z,R) be a random action equilibrium. Then generically in (U,U',Λ) there exists a sequence of Markov matrices (R_υ) converging to R, such that

i) each R_υ is optimal against Z

ii) the set of optimal responses by the principal to each of the R_υ is bounded away from Z.

Proof

Since (Z,R) is a random action equilibrium there exists $y_n \in Y$ and n_1, n_2 such that $r_{nn_1} > 0$, $r_{nn_2} > 0$, $n_1 \neq n_2$; the row vectors $z_{n_1\cdot}$ and $z_{n_2\cdot}$ are distinct; and the row vector $z_{n_1\cdot}$ has at least two positive entries. Since R is an optimal response to Z, it is also an optimal response to increase r_{nn_1} to $r_{nn_1} + \nu$ and decrease r_{nn_2} to $r_{nn_2} - \nu$. Generically, in Λ this will alter the principal's posterior beliefs given y_{n_1} as a received transmission.

Generically in U, this will destroy the equivalence between the expected utilities of the actions represented in the mixture in the n_1^{st} row of Z. Letting $\nu \to 0$ establishes the theorem as stated.

Theorem 2.3

Generically in (U,U',Λ), if (Z,R) is a partition equilibrium, then,

i) every row of Z for which the corresponding column of R has a positive entry is uniquely determined in the optimal response to R.

ii) R is the unique optimal response to Z, and it is itself a partition.

Proof

Obvious.

The main comparative static results of this paper apply to the generic instance of partition equilibria with the properties stated in Theorem 2.3. To delineate this class of equilibria more sharply, we give the following definition.

A partition equilibrium (Z,R) is called an essential equilibrium if the following two conditions hold:

i) if Z' is an optimal response to R then RZ' = RZ;

ii) if R' is an optimal response to Z the R'Z = RZ.

The idea of essential equilibria is that the strategies of each player are "essentially" unique, that is, choosing a different strategy from the optimal set does not alter the statistical relationship between the observations and the action taken. An essential equilibrium remains an equilibrium when either player chooses a different element in his set of optimal responses. Essential equilibria also possess a kind of "stability" in that they are robust to small deviations from optimal responses.

The distinction between partition equilibria in general and essential equilibria can be seen in the following example, to which we will refer again in Section 3.

Example 2

$$K = M = N = 2$$

$$U = U' = I$$

$$\pi = (.4, .6)$$

$$\pi' = (.6, .4)$$

$$\Lambda = \begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix}$$

Consider $Z = R = I$. It is straightforward to verify that this is a partition equilibrium. Posterior probabilistic beliefs are given as follows: For the principal,

$$p_1^P = (\frac{1}{2}, \frac{1}{2}) = (\text{prob } (\theta_1|y_1), \text{ prob } (\theta_2|y_1))$$

$$p_2^P = (\frac{4}{13}, \frac{9}{13}) = (\text{prob } (\theta_1|y_2), \text{ prob } (\theta_2|y_2))$$

and for the agent, symmetrically,

$$p_1^A = (\frac{9}{13}, \frac{4}{13})$$

$$p_2^A = (\frac{1}{2}, \frac{1}{2})$$

Therefore the agent is indifferent between both actions when he receives the observation y_2, and the principal is indifferent when the agent transmits y_1. The choices of $Z = R = I$ are mutually fortuitous; and neither of the requirements in the definition of essential equilibrium holds. The non-genericity of partition equilibria similar to this is responsible for their peculiar comparative static properties, as we will see below.

III. Improvements in the Information Structure

In this section we present the main comparative static results of this paper. We ask the question: When can one be sure, independent of a knowledge of the preferences and beliefs of the two individuals, that one information structure is better than another in the sense of providing a higher level of expected utility in equilibrium? The answer depends on whose welfare is being considered. Broadly speaking we find that the agent benefits from any improvement in the information structure. The principal, however, can well be hurt. Only for one very special, though interesting, type of improvement in the agent's information can we be sure that the principal benefits.

One further qualification is important to emphasize. As in most models of games or economic equilibria, and as we have seen in the examples of Section 2, we often will have multiple equilibria. Because these can be regarded as fixed-points of a suitable mapping, they will be continuous in the parameters of the problems for almost all parameter values. However, at some critical points the set of equilibria changes radically. Non-essential partition equilibria, for example, are likely to display this behavior.

For this reason our results are "local" in nature. We define a concept of "small" changes in the information structure. The comparative static results described above apply to changes in information that are sufficiently small, at an equilibrium that moves continuously in this change.

The concept of small changes, or mathematically, a topology on the space of information structures, is given by the following definition of convergence. It is natural to say that a sequence of information structures (Λ_υ) converges to Λ_o if for any decision maker with utility matrix U and prior π, the values of these decision problems $V(U,\pi,\Lambda_\upsilon) = \max_{D\epsilon M} \text{tr } U\pi\hat{\Lambda}_o D$ converge to $V(U,\pi,\Lambda_o)$. It is obvious that this is equivalent to the weak convergence of the distribution of posteriors for any strictly positive prior. Information structures representing a <u>small improvement</u> in information from Λ_o are those in a neighborhood of Λ_o which are also more informative in the sense of Blackwell.

It is important to point out that the dimensionality of the likelihood matrices, that is to say the number of possible observations, is not held constant. We are able to compare information structures in which the qualitative nature of the signals are quite different.

We will now show that essential equilibria have an invariance property that is responsible for the comparative static results that we will obtain.

<u>Theorem 3.1</u>

Let Λ_o be an information structure for which (Z_0,R_0) is an essential equilibrium. Let (Λ_υ) be a sequence of information structures converging to Λ_o. There exists a sequence $((Z_\upsilon,R_\upsilon))$ of equilibria, corresponding to (Λ_υ), with the property that $(\Lambda_\upsilon R_\upsilon)$ converges to $\Lambda_o R_o$.

<u>Proof</u>:

The proof is by construction of the tail for a sequence (Z_ν, R_ν) having required properties.

For each $a_k \epsilon A$, let $B_k \subseteq \Delta^M$ be the set of posterior beliefs for which the principal strictly prefers a_k to the other actions in A. Note that these sets are disjoint. Since (Z_o, R_o) is essential, the principal's posterior given any signal from $\Lambda_o R_o$ lies in the interior of one of the sets B_k.

Let $A_o \subseteq A$ be the subset of actions receiving positive weight under Z_o, and for each $a_k \epsilon A_o$, let $C_k \subseteq \Delta^M$ be the set of posterior beliefs for which the agent strictly prefers a_k to the other actions in A_o. Since (Z_o,R_o) is essential, the agent's posterior given any signal from Λ_o lies in the interior of one of the sets C_k.

Let $\hat{Z}$ be any matrix of appropriate dimension whose distinct rows are precisely the distinct rows of Z_o receiving positive weight under R_o. Select a subset J of the rows of $\hat{Z}$ containing all the distinct rows of $\hat{Z}$ and no duplicates. Let R_ν be any response to $\hat{Z}$ that is optimal for the agent, given the information structure Λ_ν and subject to the constraint that only signals in J are transmitted. By construction R_ν is also an (unconstrained) optimal response for the agent, given Λ_ν and $Z_\nu = \hat{Z}$.

Moreover, since (Λ_ν) converges to Λ_o, as $\nu \to \infty$, with probability approaching one the agent's posteriors given the signals in Λ_ν lie in the interiors of the same sets C_k as they do under Λ_o. Hence $\Lambda_\nu R_\nu$ converges to $\Lambda_o R_o$.

Finally, note that by construction the number of distinct signals transmitted under $\Lambda_\nu R_\nu$ is the same as the number transmitted under $\Lambda_o R_o$. Hence for ν sufficiently large the principal's posterior under any signal from $\Lambda_\nu R_\nu$ lies in the interior of the set B_k corresponding to the action selected by $\Lambda_\nu R_\nu \hat{Z}$. Hence $Z_\nu = \hat{Z}$ is an optimal response for the principal, and the sequence (Z_ν,R_ν) has the required properties. Q.E.D.

Theorem 3.2

Let Λ_o be an information structure for which (Z_o,R_o) is an essential equilibrium. Let (Λ_ν) be a sequence of information structures, each element of which is more informative than Λ_o (in the sense of Blackwell), and such that (Λ_ν) converges to Λ_o. Let (Z_ν,R_ν) be the sequence of equilibria whose existence is established in Theorem 3.1. Then, for ν sufficiently large, the agent is better off under Λ_ν with the equilibrium (Z_ν, R_ν) than under Λ_ν with the equilibrium (Z_o, R_o).

Proof:

As the rows of Z_υ are precisely the distinct rows of Z_o, by construction, the agent is facing a fixed decision problem and hence his welfare cannot diminish under any improvement in the sense of Blackwell.

To see the role of the hypothesis that the equilibrium is essential, it is useful to consider a sequence of information structures converging to that given in example 2.

Example 3

As in example 2, K = M = 2. However for all υ there will be three possible observations, with the likelihood matrices

$$\Lambda_\upsilon = \begin{pmatrix} .6-\varepsilon_\upsilon & 2\varepsilon_\upsilon & .4-\varepsilon_\upsilon \\ .4-\varepsilon_\upsilon & 2\varepsilon_\upsilon & .6-\varepsilon_\upsilon \end{pmatrix}$$

where (ε_υ) converges to zero.

Clearly (Λ_υ) converges to the information structure of example 2,

$$\Lambda_o = \begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix} \quad .$$

Moreover each Λ_υ is an improvement of Λ_o in the sense of Blackwell.

It is demonstrated in the appendix that the only equilibrium for Λ_υ with the utilities and priors of example 2 is the equilibrium of no communication. The principal always chooses a_2 and the resulting expected utilities are .6 for the principal and .4 for the agent.

In the limit, however, as example 2 shows, there is an inessential partition equilibrium of complete communication. This gives rise to the payoff $\frac{8}{13}$ for each player, which is above the .4 achievable by the agent when he has superior information.

We now consider the principal's welfare. Further conditions are required to guarantee that better information raises the principal's equilibrium expected utility. Again, a modification of Example 3 will be useful in gaining insights to the results.

Example 4

We modify Example 3 so that the priors are slightly less definitive, i.e. $\pi = (.45, .55), \pi' = (.55, .45)$. The other data of the example are unchanged and are repeated here simply for the readers' convenience.

$$U = U' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda_o = \begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix}$$

$$\Lambda_\upsilon = \begin{pmatrix} .6-\varepsilon_\upsilon & 2\varepsilon_\upsilon & .4-\varepsilon_\upsilon \\ .4-\varepsilon_\upsilon & 2\varepsilon_\upsilon & .6-\varepsilon_\upsilon \end{pmatrix}$$

As above, we call the signals under Λ_o, y_1 and y_2, and those under Λ_υ, $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ to avoid confusion.

Under Λ_o it is straightforward to verify that $Z = R = I$ is an essential partition equilibrium. It results in an expected utility of .6 to each player.

For ε_ν sufficiently small, the game with information structure Λ_ν has an essentially unique partition equilibrium (other than the no-transmission equilibrium). The agent transmits the partition $\{y_1,y_2\}$, $\{y_3\}$, and in response the principal chooses a_1 and a_2 respectively. Thus action a_2 is taken when y_2 occurs, as the agent prefers. The expected utilities at this equilibrium are $.6 - \varepsilon_\nu$ and $.6 + \varepsilon_\nu$ for the principal and agent respectively. Although the principal would rather have the agent transmit the partition $\{y_1\}$, $\{y_2,y_3\}$, there is nothing he can do to enforce this. Complete communication is not an equilibrium because the principal would choose a_2 after y_2, making complete communication irrational for the agent. In this example the better information structure entails a positive probability of a signal that causes the two players to disagree. The principal loses because at equilibrium the information is used by the agent in a way opposite to what the principal would like.

Our positive comparative static results rely on a condition that we call success-enhancing.

The motivation for examining success enhancing improvements is that there are many situations where one hopes to receive an informative observation but in fact nothing happens. Either the experiment has "failed", or, within the relevant time interval the outcome is not yet known. Since many improvements in information reduce the failure rate or cut the average delay time without affecting the quality of the experimental procedure itself, these results are of considerable interest.

The formal definition of a success-enhancing change is as follows:

We will say that Λ is in standard form if

$$\Lambda = \left[\begin{array}{c|c} \alpha & \\ \vdots & (1-\alpha)\ \Gamma \\ \alpha & \end{array}\right]$$

where $\alpha \geqq 0$ and Γ is an $M \times (N-1)$ Markov matrix, and where Γ has no two of its columns proportional.

The first signal of an information structure in standard form represents the totally uninformative observation: "dropping the last tube." A success-enhancing improvement lowers the probability of this observation and raises all others proportionately.

We will say that Λ is a <u>success-enhancing improvement</u> of Λ' if

$$\Lambda = \left[\begin{array}{c|c} \alpha & \\ \vdots & (1-\alpha)\ \Gamma \\ \alpha & \end{array}\right]$$

$$\Lambda' = \left[\begin{array}{c|c} \alpha' & \\ \vdots & (1-\alpha')\Gamma P \\ \alpha' & \end{array}\right]$$

$1 \geqq \alpha' > \alpha \geqq 0$, and P is a permutation matrix.

Note that for success-enhancing improvements the garbling matrix takes the form

$$B = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \alpha & & & \\ \vdots & & (\frac{1-\alpha'}{1-\alpha})P & \\ \alpha & & & \end{array}\right]$$

Finally, a <u>small</u> <u>success-enhancing</u> <u>improvement</u> is a change in information structures having both properties -- that is an improvement as described above for which α' and α are close.

Whenever Λ is a success-enhancing improvement of Λ' we can reorder the columns of Λ' so that the permutation matrix P referred to in the above definition is the identity. For the rest of this section we will suppose that this is the case, as this in no way changes the structure of the game. Moreover, not only is $\Lambda' = \Lambda B$, but also, with this normalization, we can write

$$\Lambda' = \Lambda D$$

where D is the <u>diagonal</u> matrix whose entries are given by

$$d_{11} = \frac{\alpha'}{\alpha}$$

(3.1)
$$d_{nn} = \frac{1-\alpha'}{1-\alpha} \text{ for } n \geq 2$$

We will make use of this representation in the proof of the main theorem, which follows:

<u>Theorem 3.3</u>

Let (Z,R) be an essential equilibrium for Λ_o and let Λ be a small success-enhancing improvement of Λ_o. Then (Z,R) remains an equilibrium for Λ and the principal's expected utility cannot decrease.

<u>Proof</u>:

That (Z,R) remains an equilibrium follows from an argument parallel to that used in the proof of Theorem 3.1. The assertion that the principal's expected utility cannot decrease will be proven using the special structure of success-enhancing improvements. We will express Λ_o and Λ in standard form and note that $\Lambda_o = \Lambda D$ where the elements of D are given by (3.1). The principal's expected utility under Λ is tr $RZ\hat{U}\pi\Lambda$ and under Λ_o it is tr $RZ\hat{U}\pi\Lambda D$. Thus the gain in going from Λ_o to Λ is

(3.2)
$$\text{tr } RZ\hat{U}\pi\Lambda(I-D)$$

This quantity will be proven to be necessarily non-negative.

Using (3.1) we see that

$$(I-D) = \left((1-d_{22})\, I - (d_{11}-d_{22})\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}\right)$$

$$= \left(\left(\frac{\alpha'-\alpha}{1-\alpha}\right) I - \left(\frac{\alpha'-\alpha}{\alpha(1-\alpha)}\right) \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix} \right)$$

Substituting into (3.2) we have

$$(3.3) \quad \left(\frac{\alpha'-\alpha}{1-\alpha}\right) \{\mathrm{tr}\; RZU\hat{\pi}\Lambda - \frac{1}{\alpha}\,\mathrm{tr}\; RZU\hat{\pi}\Lambda \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix} \}$$

As Λ is in standard form its first column is the constant α and we have,

$$(3.4) \quad \Lambda \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix} = \begin{pmatrix} \alpha & \\ \vdots & 0 \\ \alpha & \end{pmatrix}$$

Substituting (3.4) into (3.3) we obtain

$$\left(\frac{\alpha'-\alpha}{1-\alpha}\right) \{\mathrm{tr}\; RZU\hat{\pi}\Lambda - \mathrm{tr}\; RZU\hat{\pi} \begin{pmatrix} 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix} \}.$$

Using the commutativity of matrix multiplication under the trace, this becomes,

$$(3.5) \quad \left(\frac{\alpha'-\alpha}{1-\alpha}\right)\{\mathrm{tr}\; U\hat{\pi}\Lambda RZ - \mathrm{tr}\; U\hat{\pi} \begin{pmatrix} 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix} RZ\}.$$

The two terms in the brackets in (3.5) have straightforward interpretations. The first is the value of the principal's problem in the equilibrium with the better information structure. The second is the expected utility he would obtain if he used the action matrix Z in a decision problem with information structure $\begin{pmatrix} 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix} R$. But this information structure is totally uninformative, as it is just the first row of R repeated. Therefore the value of the second term in brackets is at most that of the optimal value for this problem, where Z is a matrix of identical rows and the action for which $U\pi$ has its maximal component is always chosen. Moreover, the value of this problem is at most the value of the problem with the information structure ΛR, the first term in brackets, as anything is more informative than no information. Thus the entire bracketed expression is non-negative and, as $\alpha'-\alpha$ is also non-negative, so is (3.5).

We note that (3.5) will be strictly positive whenever the principal is not indifferent, in equilibrium, between the choice of Z and a rank one matrix composed of a repeated row. Q.E.D.

IV. Comments and further examples

In this section we gather a few comments showing why the results above cannot be strengthened, and addressing some conjectures about the qualitative nature of the Nash equilibria.

1. Success-enhancing improvements in information have the property that the set of posterior beliefs that can arise after seeing the observation remains fixed. One might imagine that this property alone is responsible for the beneficial nature of the change.

An improvement in information from Λ' to Λ can be called posterior-preserving improvement if $\Lambda' = \Lambda D$, where D is a diagonal matrix, and $\Lambda' = \Lambda B$, where B is a Markov matrix. Note that a non-informative signal may not exist. If (Z,R) is an equilibrium for the information structure Λ', then we know that R is among the agent's best responses to Z under Λ. The following example shows, however, that the principal's welfare may decrease if he plays Z, and moreover that there may be no possibility for him to achieve the former level of utility:

$$\Lambda = \begin{pmatrix} .8 & 0 & 0 & .2 \\ 0 & .9 & 0 & .1 \\ 0 & 0 & .8 & .2 \end{pmatrix}$$

$$\Lambda_o = \begin{pmatrix} 6 & 0 & 0 & .4 \\ 0 & .8 & 0 & .2 \\ 0 & 0 & .6 & .4 \end{pmatrix}$$

It is easy to see that Λ is a posterior-preserving improvement of Λ_o. The first three signals in either case are perfect predictors of the state, while the fourth carries some information but does not limit the set of states that are possible.

Let the utilities and priors be such that

$$U\hat{\pi} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U'\hat{\pi}' = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

The finest partition equilibria convey $\{y_1,y_2\}$, $\{y_3,y_4\}$. For example the strategies

$$Z = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

constitute an equilibrium.

This results in an expected utility for the principal of .1 under Λ but .2 under Λ_o. More information has therefore been harmful.

Notice that even though Λ is strictly more informative than Λ_o, ΛR is non-comparable to $\Lambda_o R$. For this particular utility function it is worse.

2. Even if the improvement in information is success enhancing, a discrete change from Λ' to Λ may be such that (Z,R) is no longer an equilibrium. R remains a best response to Z, but Z may not be best against R.

An example, available from the authors on request, shows that all the equilibria under Λ may be inferior for the principal to a given partition equilibrium for Λ'. Theorem 3.3 relies on the changes being small enough that (Z,R) persists as an equilibrium.

3. We investigated the conjecture that the common refinement of the partitions implicit in two partition equilibria always corresponds to another partition equilibrium. A counterexample to this conjecture is also available on request.

4. Recent work by Crawford and Sobel (1981) shows that some further generality can be attained in our results if a smaller set of decision problems is considered. Specifically, they show that improved information always benefits the principal if utilities are concave in actions and states and if both the space of states and the signal space are one dimensional. Moreover, in that situation there are no equilibria other than partition equilibria.

5. Finally, it should be emphasized that the main results of this paper are crucially dependent on the finiteness of the set of possible actions. These results are local in nature, as noted in 3. above. The structure of our model is such that within a neighborhood of a given information structure, partition equilibria are locally constant with respect to success-enhancing or posterior-preserving improvements. This enables us to evaluate welfare changes by examining the effect of the improved information in a fixed equilibrium.

If there were a continuum of actions the neighborhood of local constancy might vanish. Changing information would induce locally continuous shifts in the equilibria. Welfare effects would then depend upon the nature of these shifts, as well as on the difference in the quality of information.

Andrew Postlewaite (1980) has provided us with an example of a game with a unique partition equilibrium in which the principal's welfare responds negatively to a success-enhancing improvement in the agent's information for this reason.

V. Conclusion

We have examined a simple two-person game designed to represent the separation of functions in an organization. It has been argued that, although this game may have multiple equilibria, there is one type of equilibrium of particular note. In analyzing the comparative statics of individual welfare with respect to improvements in information, we have concentrated on this form of equilibrium.

In general, improvements in information may be harmful for one or even both of the players. We therefore tried to find restrictions on the nature of the improvement in information that imply that it is surely beneficial.

For large shifts in the information structure, nothing can be said, in general. Locally, an arbitrary improvement in the information structure will generically benefit the agent, but the principal may be hurt. To guarantee that neither player can be hurt by a small improvement in information, we needed to assume that the shift is "success enhancing" -- that is, that it represents a decrease in the probability of receiving an uninformative observation, and, correspondingly, proportional increases in the probabilities of receiving all other observations.

There are many possible extensions of this model. We can mention only two of them here.

Our analysis concentrated on restricting the information structure. An alternative would be to look for restrictions on utilities and priors. In this regard the paper by Crawford and Sobel (1981) cited above is relevant.

Our model is related to, but distinct from, the "principal-agent" problem that has been widely discussed in the literature. There, the "agent" plays the role of information gatherer and decision-maker. The "principal" is present only to help offset risks by making contingent payments of a transferable resource. We have no such resource, the essential feature of our model resting in the separation of the informational and decision-making functions within the organization. The possibility of effecting such conditional payments would add an entirely new dimension to the analysis. The principal might, for example, set up a payment schedule that would coax a more accurate transmission out of the agent. Paralleling the "principal-agent" literature, it is probably best to model this as the Stackelberg, rather than Nash, equilibrium of a game in which the principal is the "leader." We begin the study of this solution concept in a companion paper, Green and Stokey (1981), retaining the structure presented here in all other respects -- including the absence of transferable utility.

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Appendix

We now show that the equilibrium of "no communication" is the unique equilibrium in example 3 for the particular value of $\varepsilon_\upsilon = .2$. It will be clear in the construction that the same result holds for all $\varepsilon_\upsilon > 0$. We proceed by enumerating all the possibilities. This is necessary because a computation based on an algorithm for finding fixed-points might not find all equilibria, and because the example itself is non-generic (see the discussion in Section 2 above), so that we cannot be sure that such methods would be successful.

$$U = U' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\pi} = \begin{pmatrix} .4 & 0 \\ 0 & .6 \end{pmatrix} \qquad \hat{\pi}' = \begin{pmatrix} .6 & 0 \\ 0 & .4 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} .4 & .4 & .2 \\ .2 & .4 & .4 \end{pmatrix}$$

The posteriors that would be held by the principal if he had the agent's information can be summarized in the following matrix, where the n, mth element is the posterior probability of θ_m given y_n.

$$P_{princ.} = (\text{prob } (\theta_m \mid y_n)) = \begin{pmatrix} 4/7 & 3/7 \\ .4 & .6 \\ 1/4 & 3/4 \end{pmatrix}$$

For the agent

$$P_{agent} = \begin{pmatrix} 3/4 & 1/4 \\ .6 & .4 \\ 3/7 & 4/7 \end{pmatrix}$$

From the form of the utilities we see that they would agree about the action to be taken given y_1 or y_3, but would disagree given y_2.

Let us write R and Z, incorporating their Markovian structure, as

$$R = \begin{pmatrix} r_{11} & r_{12} & 1-r_{11}-r_{12} \\ r_{21} & r_{22} & 1-r_{21}-r_{22} \\ r_{31} & r_{32} & 1-r_{31}-r_{32} \end{pmatrix}$$

$$Z = \begin{pmatrix} z_1 & 1-z_1 \\ z_2 & 1-z_2 \\ z_3 & 1-z_3 \end{pmatrix}$$

The non-negativity constraints will be incorporated later. With this notation the objective function of the principal can be written out explicitly in terms of the elements of the R and Z matrices as:

(1) $$\text{tr } U\hat{\pi}\Lambda RZ = z_1(.04\, r_{11} - .08\, r_{21} - .16\, r_{31})$$
$$+ z_2(.04\, r_{12} - .08\, r_{22} - .16\, r_{32})$$
$$+ z_3(-.20 - .04\, r_{11} - .04\, r_{12} + .08\, r_{21} + .08\, r_{22} + .16 r_{31} + .16 r_{32})$$
$$+ .60\ .$$

For the agent we have,

(2) $$\text{tr } U'\hat{\pi}'\Lambda RZ = r_{11}(.16\, z_1 - .16\, z_3) + r_{12}\,(.16\, z_2 - .16\, z_3)$$
$$+ r_{21}\,(.08\, z_1 - .08\, z_3) + r_{22}\,(.08\, z_2 - .08\, z_3)$$
$$+ r_{31}\,(-.04\, z_1 + .04\, z_3) + r_{32}\,(-.04\, z_2 + .04\, z_3)$$
$$+ (.20\, z_3 + .40)\ .$$

The Pareto efficient expected utilities can be computed from these formulas by setting R = I and varying Z. One obtains that $z_1 = 1$ and $z_3 = 0$, for efficiency, and that the Pareto Frontier is

$$(EU_{agent}, EU_{princ.}) = (.56 + .08\, z_2,\ .64 - .08\, z_2)$$

parameterized by z_2.

Thus at the two extremes these utilities are:

$z_2 = 1$ (.64, .56)

$z_2 = 0$ (.56, .64) .

To search for equilibrium strategies we use the following procedure.

We first categorize Z into 9 classes based on inequality relationships among z_1, z_2 and z_3. We test each class to see if there can be any equilibrium there by taking (z_1, z_2, z_3) in that class, finding the optimal strategies for the agent from (2) and then, for every possible optimal strategy by the agent, testing whether (z_1,z_2,z_3) could in turn be optimal against it, using (1).

Because the complete computation involves so many cases, we will illustrate the method only. The interested reader will be able to write down the others straightforwardly.

The cases are:

A. $z_1 > z_3$

$z_2 > z_3$

B. $z_1 > z_3$

$z_2 = z_3$

C. $z_1 > z_3$

$z_2 < z_3$

D. $z_1 = z_3$

$z_2 > z_3$

E. $z_1 = z_3$

$z_2 = z_3$

F. $z_1 = z_3$

$z_2 < z_3$

G. $z_1 < z_3$

$z_2 > z_3$

H. $z_1 < z_3$

$z_2 = z_3$

I. $z_1 < z_3$

$z_2 > z_3$

They are obviously mutually exclusive and exhaustive. This particular breakdown is chosen because, from (2), one can see that the optimal responses by the agent depend precisely on these inequality relations. Take case A. From (2) we have that

$$r_{11} = 1,\ r_{12} = 0,\ r_{21} = 1,\ r_{22} = 0,\ r_{31} = 0 \text{ and } r_{32} = 0 \ .$$

This follows since case A entails for example, that $z_1 - z_3 > z_2 - z_3$, so the term $r_{11}(.16\ z_1 - .16\ z_2)$ dominates $r_{12}\ (.16\ z_2 - .16 z_3)$. Substituting these values into (1) we have that the principal's objective is

$$-.04\ z_1 - .08\ z_3 + .60 \ .$$

Therefore $z_1 = z_3 = 0$ is optimal. The optimal value of z_2 could be anything, but the conditions of case A are surely violated. Thus there can be no equilibria in case A.

Some further flavor of this method can be seen by examining case D. We have $r_{11} = 0$, $r_{12} = 1$, $r_{21} = 0$, $r_{22} = 1$, $r_{32} = 0$, but r_{31} remains indeterminate. Any value between zero and one could equally well be played. Substituting into (1), we have,

$$-.16\ r_{31}\ z_1 - .08\ z_2 + (.16\ r_{31} - .16)\ z_3 + .60$$

Clearly $z_2 = 0$. The optimal values of z_1 and z_3 depend on r_{31}. There are three possibilities

$\underline{r_{31} = 0}$

z_1 is indeterminate, but $z_3 = 0$. Hence $z_2 = z_3$, violating the conditions of case D.

$\underline{1 > r_{31} > 0}$

$z_1 = z_3 = 0$; again the conditions fail.

$\underline{r_{31} = 1}$

$z_1 = 0$, but now z_3 is indeterminate. However we now must have $z_1 \overset{<}{=} z_3$, again contradicting the hypotheses.

Finally, we show that in case E there is an equilibrium -- the "no communication" equilibrium in which the principal sets $z_1 = z_2 = z_3 = 0$, because his prior favors θ_2.

From (2) we see that there are no constraints at all on the elements of R. This is natural. If the principal's action will always be the same, the agent obtains the same utility with any transmission.

If $z_1 = z_2 = z_3$ is to be established, we must have either that

(4) $$.04\, r_{11} - .08\, r_{21} - .16\, r_{31} \geqq 0$$

(5) $$.04\, r_{12} - .08\, r_{22} - .16\, r_{32} \geqq 0$$

(6) $$-.20 - .04\, r_{11} - .04\, r_{12} + .08\, r_{21} + .08 r_{22} + .16 r_{31} + .16 r_{32} \geqq 0$$

or that,

(7) $$.04\, r_{11} - .08\, r_{21} - .16\, r_{31} \leqq 0$$

(8) $$.04\, r_{21} - .08\, r_{22} - .16\, r_{32} \leqq 0$$

(9) $$-.20 - .04\, r_{11} - .04\, r_{12} + .08\, r_{21} + .08 r_{22} + .16 r_{31} + .16 r_{32} \leqq 0$$

In the former case we obtain a contradiction by substituting (4) and (5) into (6).

The second set of inequalities have many solutions, for example they hold when all the r_{nn}, are zeros. At all these equilibria $z_1 = z_2 = z_3 = 0$ and the expected utilities are .40 and .60 for the agent and principal respectively. Note that this is strictly below the Pareto frontier, since (.56, .64) is attainable.

The material in this appendix confirms the assertions made in connection with example 3 in the text.