Monotonicity and Implementability*

Itai Ashlagi† Mark Braverman‡ Avinatan Hassidim§
Dov Monderer¶

Oct 2009

Abstract

Finding which social goals can be achieved using monetary payments is a major task in the field of mechanism design. This paper provides a useful tool in this research agenda. Consider a model with finite number of alternatives, and with agents with private values and quasi-linear utility functions. A domain of valuations for an agent is a monotonicity domain if every finite-valued monotone randomized allocation rule defined on it is implementable, in the sense that there exists a randomized truth-telling direct mechanism that implements this allocation rule. We fully characterize the set of all monotonicity domains.

1 Introduction

Finding which social goals can be achieved using monetary payments is a major task in the field of mechanism design. This paper provides a useful tool in this research agenda. Consider a model with finite set of alternatives, and with agents with private values and quasi-linear utility functions. An allocation rule is said to be *implementable* if there exists a truth-telling direct mechanism that implements it. Rochet [22] (see also [23]) showed that *cycle monotonicity* is a necessary and sufficient condition for any allocation rule to be implementable. However cycle monotonicity is a very difficult condition to verify. A

*We would like to thank André Berger, Shahar Dobzinski, Ron Lavi, Rudolf Müller, Seyyed Hossein Naeemi, and Maria Polukarov for very helpful discussions. Dov Monderer thanks the Israeli Science Foundation (ISF), and the Fund for the Promotion of Research at the Technion.

†Harvard Business School.

‡Microsoft Research, New England.

§MIT.

¶Technion-Israel Institute of Technology.

much weaker necessary condition, *monotonicity*[1], has been shown to be also sufficient in certain domains of valuations and for certain families of allocation rules. Roughly speaking, monotonicity is a condition on every pair of valuations, and cycle monotonicity is a condition on every finite sequence of valuations.[2] Myerson [19] showed that in a single dimension domain, monotonicity is sufficient for any allocation rule to be implementable. Bickchandani et. al [4] showed a number of (convex) domains, most notably R_+^k, in which monotonicity is sufficient for a *deterministic* allocation rule to be implementable. Gui et. al [6] noticed that by a result of Roberts [21], the full domain $D = R^A$ is such a domain, and it was proved in addition that so is every cube. Saks and Yu [24] extended this result for any convex domain.[3] In their paper, Saks and Yu [24] asked in which domains every *deterministic* monotone allocation rule is implementable.

In this paper we study *finite-valued* (finite range) *randomized* allocation rules.[45] We give a full characterization of *monotonicity domains*, which are domains for which every finite-valued randomized monotone allocation rule is implementable. In particular, we answer the question posed by Saks and Yu for randomized allocation rules.

The Characterization:

Let A be the set of alternatives, and let H^A be the hyperplane which is orthogonal to the vector $(1, \ldots, 1) \in R^A$.

A domain $D \subseteq R^A$ is a monotonicity domain if and only if its projection to H^A, $\Pi(D)$, is either of dimension one or the closure of $\Pi(D)$ is convex.

Our characterization is given in three steps, which are formalized in Section 2. The first step has been proved by Monderer in [16]. Monderer proved that every domain with a convex closure is a monotonicity domain, which is a generalization for Saks and Yu's result. He also proved that the closure of a *proper monotonicity domain* with dimension 2 is convex, where proper monotonicity domains are defined similar to monotonicity domains with the exception that an allocation rule can output sub-probability vectors (instead of only probability vectors). This paper provides the full characterization together with all proofs.

Characterizations of implementability are important for several reasons. One can use the knowledge that a given domain D is a monotonicity domain as a tool for achieving

[1]What we call monotone is called weakly monotone by Bikhchandani et. al [4] and others. However, the term "monotone" is well-known in the convex analysis literature (e.g., [23]) and we therefore use it.

[2]For a good background on monotonicity and cycle monotonicity see Bikhchandani et. al [4] and Vohra [25]. See also Jehiel and Moldovanu [10], Jehiel et. al [11], Krishna and Maenner [12] for characterizations of Bayesian incentive compatible mechanisms.

[3]Berger et. al [3] extend the Saks and Yu's result to convex valuation functions.

[4]Note that very deterministic allocation rule is finite-valued in our setting.

[5]Archer and Kleinberg [1] and Bickchandani et. al [4] each provide an example for a monotone rule on a convex domain, with infinitely many outcomes which is not cyclic monotone.

the difficult task of finding a revenue-optimal truth-telling mechanism on D. As mentioned above, Myerson [19] showed that any domain of dimension 1 is a monotonicity domain. In fact this was this was a key result in his optimal single-item auction finding in the Bayesian setup. Since monotonicity becomes a sufficient condition in these domains, one can focus only on a "candidate" allocation rule rather than a "candidate" mechanism (allocation rule and payments) in order to verify wether there is truth-revealing mechanism implementing the allocation rule. For example, it is immediate to verify that every affine maximizer allocation rule is monotone and therefore it is implementable in a monotonicity domain.

In monotonicity domains it is easier to look for concrete characterizations of implementable allocation rules, such as the one by Roberts [21], where he proved that every implementable pure allocation rule is an affine maximizer. Indeed, Roberts proved the theorem using a condition (positive association of differences), which is very similar in spirit to monotonicity; See [13, 14] for details.[6]

Our characterization can be very useful in finding efficiency bounds; Consider some desired social choice function. How bad (lower bound) and how good (upper bound) can the ratio between the output of the desired social choice function and the output of an implementable allocation rule be, i.e. when one insists on incentive compatible mechanisms. See Nisan and Ronen [20] who initialized this line of research in the setup of allocating tasks. Ashlagi et. al [2] use the monotonicity condition to verify wether a task allocation mechanism is truth-revealing.

Another task is to characterize domains in which the revenue equivalence principle holds. Revenue equivalence holds in a certain domain of valuations if every implementable allocation rule on this domain determines the payment scheme uniquely up to an additive constant (see e.g. [19]). When agents are risk neutral, revenue equivalence holds in a certain domain if and only if utility equivalence holds in this domain, where the utility equivalence principle asserts that the utility function is uniquely determined up to an additive constant. Archer and Kleinberg [1] proved that utility equivalence holds in every polygonally connected domain. Holmstrom [9] proved this for differentiable path-connected domains. For the newest results in this area also in the Bayesian setting, and for a survey of previous results see [8, 18].

The first step in our characterization, Theorem 2, implies that convex domains are monotonicity domains. Although convex domains might be more natural, non-convex domains are also important to study as these rise in a variety of settings. Perhaps the most common settings are those with discrete valuations. A setting in which the domain of valuations is not convex and also not discrete, is a combinatorial auction with single minded bidders [15].

[6] Also see [5] and [17] for a different type of monotonicity, Maskin monotonicity, that is used to characterize incentive compatibility in non quasi-linear settings.

2 Model And Results

Let A be a finite set of alternatives. R^A is the set of all valuation functions on A, that is the set of all real valued functions defined on A. The value of a for an agent with valuation v is thus v_a. It is convenient to represent each alternative a by its associated unit vector $e^a \in R^A$, where $e^a_a = 1$ and $e^a_b = 0$ for every $b \neq a$. Let $Z(A)$ be the set of all probability vectors $z \in R^A$ respectively:

$$Z(A) = \{z \in R^A | \ z_a \geq 0 \ \forall a, \ \sum_{a \in A} z_a = 1\}.$$

Let $D \subseteq R^A$, and let $f : D \to Z(A)$. We think of D as the set of all possible valuations of a given agent with a quasi-linear utility function, and f is interpreted as a *randomized allocation rule* in some direct mechanism (D, f, c), where $c : D \to R$. If an agent with valuation v declares w, alternative a is chosen with probability $f_a(w)$, and therefore her utility is $\langle v, f(w) \rangle = \sum_{a \in A} v_a f_a(w)$ minus $c(w)$. A randomized allocation rule f is *finite-valued* if its range $\{f(v) | v \in D\}$ is a finite set.

We say that a randomized allocation rule f is *implementable* if there exists a function $c : D \to R$ such that truth telling is a dominant strategy in the direct mechanism (D, f, c). That is,

$$\langle v, f(v) \rangle - c(v) \geq \langle v, f(w) \rangle - c(w) \quad \forall v, w \in D. \tag{1}$$

If f is implementable, then a simple manipulation on (1) shows that:

$$\langle f(v) - f(w), v - w \rangle \geq 0 \quad \text{for every } v, w \in D. \tag{2}$$

A randomized allocation rule satisfying (2) is called *monotone*. By further using (1) it was noticed by Rochet [22] that every implementable randomized allocation rule f satisfies a stronger monotonicity property. f is called *cyclically monotone* if for every $k \geq 2$, for every k vectors in D (not necessarily distinct), $v_1, v_2, \ldots, v_k$ the following holds:

$$\sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i) \rangle \geq 0, \tag{3}$$

where v_{k+1} is defined to be v_1. By taking $k = 2$ in (3) it can be easily seen that every cyclically monotone randomized allocation rule is monotone. The following characterization of implementability is known:

Theorem 1 (Rochet) *A randomized allocation rule is implementable if and only if it is cyclically monotone.*

We say that a domain of valuation functions is a *monotonicity domain* if every finite-valued monotone randomized allocation rule defined on it is implementable. It is well known that every domain of dimension at most one is a monotonicity domain.

For our characterization we need a similar structure to a monotonicity domain. Let $\bar{Z}(A)$ be the set of all sub-probability vectors $z \in R^A$:

$$\bar{Z}(A) = \{z \in R^A | \ z_a \geq 0 \ \forall a, \ \sum_{a \in A} z_a \leq 1\}.$$

We say that a domain of valuation functions is a *proper monotonicity domain* if every finite-valued monotone function $f : D \to \bar{Z}(A)$ is implementable.[7]

Our characterization follows from the following three theorems we prove:[8]

Theorem 2 *Every domain with a convex closure is a proper monotonicity domain.*[9].

Theorem 9 *The closure of every proper monotonicity domain of dimension at least 2 is convex.*

Theorem 16 *A domain D is a monotonicity domain if and only if its projection to the hyperplane $H^A = \{v \in R^A : \sum_{a \in A} v_a = 0\}$ is a proper monotonicity domain.*

As a corollary every domain in R^2, and e.g. the unit sphere in R^3 are monotonicity domains. The main tool in proving Theorem 2 is Theorem 7, where we show that if f is a monotone randomized allocation rule on a convex set D, which is the union of two closed convex sets, and f is cyclically monotone on each of the two convex sets, then f is cyclically monotone on D.[10]

In Section 3 we prove Theorem 2. In Sections 4 and 5 we prove Theorems 9 and 16 respectively.

An important comment is that in the context of the problems discussed in this paper, there is no loss of generality in dealing with one agent. This follows from the fact that in a multi-agent model, a randomized allocation rule is implementable if and only if for each agent, for each fixed vector of valuations of all other agent, the resulting one-agent randomized allocation rule is implementable. For a detailed discussion see, e.g., [4]

3 Domains with Convex Closure

In this section we prove:

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[7]The monotonicity, cyclically monotonicity definitions are similar for functions of the form $f : D \to \bar{Z}$. Moreover, Rochet's theorem holds for such functions.

[8]We actually prove more than stated in our characterization.

[9]Theorem 2 implies Saks and Yu's result [24], but in our opinion its proof is significantly simpler than their proof. With some effort, Theorem 2 can be directly derived from their result.

[10]Theorem 7 uses the well-known Rockafellar' s characterization of cyclically monotone functions, which is described at Theorem 6.

Theorem 2 *Every domain with a convex closure is a proper monotonicity domain.*

3.1 Preparations

Before we proceed with the proof of Theorem 2 we need some preparations. Let $f : D \to \bar{Z}(A)$ be monotone and finite-valued, where D is an arbitrary set. Let $y^1, \ldots, y^m \in R^A$ be the distinct values of f. That is, for every $v \in D$ there exists $1 \le j \le m$ such that $f(v) = y^j$, and every y^j is attained at some valuation. If $m > 1$, for $j \ne k$ define:

$$\delta(j,k) = \delta_{D,f}(j,k) = \inf_{v \in D, f(v)=y^j} \langle v, y^j - y^k \rangle. \tag{4}$$

If $w \in D$ satisfies $f(w) = y^k$ then by monotonicity, $\langle v, y^j - y^k \rangle \ge \langle w, y^j - y^k \rangle$. Therefore $\delta(j,k) > -\infty$. Furthermore:

$$\delta(j,k) \ge \sup_{w \in D, f(w)=y^k} \langle v, y^j - y^k \rangle = -\delta(k,j).$$

Hence,

$$\delta(j,k) + \delta(k,j) \ge 0, \quad \forall j \ne k. \tag{5}$$

As (3) can be written as

$$\sum_{i=1}^{k} \langle v_i, f(v_i) - f(v_{i-1}) \rangle \ge 0, \tag{6}$$

where v_0 is defined to be v_k, the following useful lemma has been noted by many authors (see e.g. [7, 24]):

Lemma 3 *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone.*

a. f is cyclically monotone if and only if for every sequence $j_1, j_2, \ldots, j_k$, $k \ge 2$, such that $j_s \ne j_{s+1}$ for $1 \le s < k$ the following holds:

$$\sum_{i=1}^{k} \delta(j_i, j_{i+1}) \ge 0, \tag{7}$$

where j_{k+1} is defined to be j_1.

b. If in addition to the monotonicity $\delta(j,k) + \delta(k,j) = 0$ for every $j \ne k$, then f is cyclically monotone if and only if the inequalities (7) are satisfied as equalities.

For every j let:

$$D_j = \{v \in D | \ \langle v, y^j - y^k \rangle \ge \delta(j,k) \quad \forall k, \ k \ne j\}.$$

Obviously, $f(v) = y^j$ implies $v \in D_j$. Hence, $D = \cup_{j=1}^m D_j$.

The following sufficient condition will be useful:

Lemma 4 *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone. If $\cap_{j=1}^m D_j \neq \emptyset$ then f is cyclically monotone.*

Proof: Let $v \in D$ be in the intersection. Hence $\langle v, y^j - y^k \rangle \geq \delta(j,k)$ for all $j \neq k$. We claim that

$$\langle v, y^j - y^k \rangle = \delta(j,k) \quad \text{for all } j \neq k. \tag{8}$$

Indeed, $v \in D_j$ implies $\langle v, y^j - y^k \rangle \geq \delta(j,k)$, and $v \in D_k$ implies $\langle v, y^k - y^j \rangle \geq \delta(k,j)$. Therefore, from (5) we obtain (8) . By plugging (8) in (7) it follows that (7) is satisfied with equality for every sequence of indices, and hence f is cyclically monotone.■

We next show that in order to prove that a set is a proper monotonicity domain it suffices to prove that its closure is a proper monotonicity domain. For a domain D we denote its closure by $cl(D)$.

Lemma 5 *If $cl(D)$ is a proper monotonicity domain so is D.*

Proof: Suppose $cl(D)$ is a proper monotonicity domain, and let $f : D \to \bar{Z}(A)$ be a finite-valued monotone function on D. Extend f to $cl(D)$ as follows: For every $v \in cl(D) \setminus D$ there exists a sequence v_n, $n \geq 1$ in D such that $v_n \to v$. For some j there exists an infinite numbers of indices n such that $f(v_n) = y^j$. Hence for every $v \in cl(D) \setminus D$ there exists j and a sequence $v_n \in D$ such that $v_n \to v$ and $f(v_n) = y^j$ for every $n \geq 1$. Let $f(v) = y^j$ for such arbitrary j. It is easily verified that the extension of f is monotone on $cl(D)$. Therefore it is cyclically monotone on $cl(D)$, and therefore f is cyclically monotone on D.■

We will use a characterization of cyclically monotone functions that can easily be derived from Section 24 in [23].

Theorem 6 (Rockafellar) *Let $D \subseteq R^A$ be a convex and non-empty subset of valuations, and let $f : D \to \bar{Z}(A)$.*

a. f is cyclically monotone on D if and only if there exists a real-valued function U[11] *on D such that*

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle, \quad \forall v_1, v_2 \in D. \tag{9}$$

b. If each of the functions $U_1, U_2 : D \to R$ satisfies (9), then the functions differ by a constant. That is, there exists a real number α such that

$$U_1(v) = U_2(v) + \alpha \quad \forall v \in D. \tag{10}$$

[11] $U(v)$ can be interpreted as the utility function of the buyer when her valuation is v.

c. Suppose that $U : D \to R$ satisfies (9), and let $v_1 \neq v_2 \in D$. Then, the real-valued function

$$\phi(t) = \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle \tag{11}$$

defined for every $t \in [0, 1]$ is non-decreasing, and:

$$U(v_2) - U(v_1) = \int_0^1 \phi(t)dt, \tag{12}$$

where the integral is computed in the sense of Riemann.[12]

The main tool in proving Theorem 2 is the following:

Theorem 7 *Let $D = H_1 \cup H_2$ be a closed convex set, where each H_i is closed convex and non-empty. Let $f : D \to \bar{Z}(A)$ be monotone (not necessary finite-valued). If f is cyclically monotone on every H_i then f is cyclically monotone on D.*

Proof: Because D and the sets H_i are closed, $H_1 \cap H_2 \neq \emptyset$. Let v^* be a fixed valuation in $H_1 \cap H_2$. By Part a of Theorem 6, there exists U_1 on H_1 that satisfies (9) on H_1. By adding a constant, we can choose U_1 such that $U_1(v^*) = 0$. Similarly there exists $U_2 : H_2 \to R$ that satisfies (9) on H_2 and $U_2(v^*) = 0$. By Part b of Theorem 6, $U_1 = U_2$ on $H_1 \cap H_2$. Hence we can define a function U on D by $U(v) = U_i(v)$ for $v \in H_i$. In order to show that f is cyclically monotone on D, it suffices by Part a to show that (9) is satisfied by U on D. Let then $v_1 \neq v_2$ in D. Obviously we can consider only the case $v_1 \in H_1 \setminus H_2$, $v_2 \in H_2 \setminus H_1$. Because H_1, H_2 and D are closed and $v_1 \in H_1 \setminus H_2$ and $v_2 \in H_2 \setminus H_1$, the interval (v_1, v_2) intersects $H_1 \cap H_2$, say $w = v_1 + s(v_2 - v_1)$, $0 < s < 1$ is a valuation at the intersection. By applying Part c of Theorem 6 to v_1, w in H_1, and by a simple change of variables we get:

$$U(w) - U(v_1) = \int_0^s \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt,$$

and similarly

$$U(v_2) - U(w) = \int_s^1 \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt.$$

Therefore:

$$U(v_2) - U(v_1) = \int_0^1 \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt.$$

By the monotonicity of f, the integrand is non-decreasing in t, and therefore the integral is greater or equals the value of the integrand at $t = 0$. Hence,

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle. \blacksquare$$

[12] A non decreasing function is Riemann integrable. It is also Borel measurable and therefore its Riemann integral equals its Lebesgue integral.

3.2 Proof of Theorem 2:

We first show that it suffices to prove that every compact convex set is a proper monotonicity domain. Let D be a set such that $cl(D)$ is convex. By Lemma 5 it suffices to prove that $cl(D)$ is a proper monotonicity domain.

Assume the result holds for every compact convex set, and assume in negation that $f : cl(D) \to \bar{Z}(A)$ is a finite-valued monotone randomized allocation rule, which is not cyclically monotone. Therefore there exist $v_1, v_2, \cdots, v_k$ in $cl(D)$ that contradict (3). Let K be the convex hull of these valuations, then f is finite-valued and monotone on K and it is not cyclically monotone, contradicting our assumption that the assertion holds for compact convex sets.

We prove the theorem for compact convex sets by a double induction process. The first induction is on the number of distinct values, $m(D, f)$ of f on D. If $m(D, f) = 1$ then obviously f is cyclically monotone. Let $m > 1$, and assume we have already proven that for every compact convex D and for every monotone randomized allocation rule $f : D \to \bar{Z}(A)$ with $m(f, D) < m$, f is cyclically monotone on D. We proceed to prove it for every $m(D, f) = m$.

For every (D, f) with $f(D) = \{y^1, \cdots y^m\}$ let $r(D, f)$ be the maximal number r, $1 \leq r \leq m$ for which for every set F of r distinct values in $\{1, \ldots, m\}$, the intersection $\cap_{j \in F} D_j \neq \emptyset$. We prove our result by induction on $r(D, f)$. Let then $r(D, f) = 1$. Since $m > 1$ there exists $j \neq k$ such that $D_j \cap D_k = \emptyset$. Since D_j and D_k are compact and convex we can strongly separate them. That is, there exists $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in D_j, \forall w \in D_k.$$

Denote $H_1 = \{v \in D | \langle v, y \rangle \leq \alpha\}$, $H_2 = \{v \in D | \langle v, y \rangle \geq \alpha\}$. On each H_i the function f takes at most $m - 1$ values, and therefore by the first induction hypothesis f is cyclically monotone on each H_i. By Theorem 7 f is cyclically monotone on D. Suppose the theorem is proved for $1, \ldots, r-1$, $2 \leq r \leq m$. We now prove it for $r(D, f) = r$. If $r = m$ the result follows from Lemma 4. If $r < m$ there exists a set of indices of cardinality $r + 1$, which w.l.o.g. we take to be $\{1, \ldots, r+1\}$, such that $\cap_{j=1}^{r} D_j \neq \emptyset$, and $\cap_{j=1}^{r+1} D_j = \emptyset$. The convex compact sets $\cap_{j=1}^{r} D_j$ and D_{r+1} must be strongly separated. That is, there exists $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in \cap_{j=1}^{r} D_j, \forall w \in D_{r+1}.$$

Let $H_1 = \{v \in D | \langle v, y \rangle \leq \alpha\}$, $H_2 = \{v \in D | \langle v, y \rangle \geq \alpha\}$. On H_1 the function f does not take the value y^{r+1} and therefore by our first induction hypothesis f is cyclically monotone. On H_2, if $m(H_2, f) < m$ then f is implementable on H_2 by the first induction hypothesis.

Suppose $m(H_2, f) = m$. Since $H_2 \subseteq D$, $\delta_{H_2,f}(j,k) \geq \delta_{D,f}(j,k)$ for every $j \neq k$. Therefore, for every j, $H_{2_j} \subseteq D_j$, where $H_{2_j} = \{v \in H_2 | \ \langle v, y^j - y^k \rangle \geq \delta_{H_2,f}(j,k)\}$. Hence, $\cap_{j=1}^r H_{2_j} \subseteq H_2 \cap (\cap_{j=1}^r D_j) = \emptyset$ implying $r(H_2, f) < r$. Therefore by our second induction hypothesis f is cyclically monotone on H_2. Hence f is cyclically monotone on D by Theorem 7.■

3.3 A Note on General Monotone Allocation Rules

The definitions of monotonicity and cyclic monotonicity are not restricted to functions that take only sub-probability values. Hence, every function, $f : D \to R^A$, that satisfies (2) ((3)) is called *monotone* (*cyclically monotone*). Such general functions can be used, e.g., in models with divisible goods. It is therefore interesting to note that without any change in the proofs Theorem 2 holds for such functions. Therefore the following result holds:

Theorem 8 *Let $D \subseteq R^A$ be a domain with a convex closure. Every finite-valued monotone function $f : D \to R^A$ is cyclically monotone.*

4 Proper Monotonicity Domains

In the previous section we showed that convexity is a sufficient condition for a domain to be a proper monotonicity domain, and hence also a monotonicity domain. In the next two sections we study the other direction. In this section we will show that domains of dimension at least 2, whose closure is not convex, are not proper monotonicity domains. In the following section we characterize monotonicity domains via proper monotonicity domains.

For every domain D, let M_D be the linear space generated by all differences $v - w$, where $v, w \in D$. The dimension $d(D)$ of D is defined to be the linear dimension of M_D. It is well-known that D is 0-dimensional if and only if D is a singleton. Let $k \geq 1$. It is also well-known that $d(D) \geq k$ if and only if there exists $k+1$ distinct valuations in D, $v_0, v_1, \ldots, v_k$, such that $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent. In this section we prove:

Theorem 9 *The closure of every proper monotonicity domain of dimension at least 2 is convex.*

In the next two subsections we will prove Theorem 9. The next lemma asserts that proper monotonicity is invariant under rotations and dilations. Hence when assessing whether a set $D \in R^A$ is a proper monotonicity domain, we are free to choose the coordinates in a way that is most convenient to us.

Lemma 10 1. *Let $D \subseteq R^A$. If there exist a monotone finite-valued allocation rule $f : D \to R^A$ which is not cyclically monotone then there also exists such an allocation rule $\tilde{f}$ that outputs only sub-probability allocations, i.e. $\tilde{f} : D \to \bar{Z}(A)$.*

2. A domain $D \subseteq R^A$ is a proper monotonicity domain if and only if $L(D)$ is a proper monotonicity domain, where $L(D)$ is a rotation, affine shift or contraction of D.

Proof: We begin with the first part. Let D be a proper monotonicity domain and let $f : D \to R^A$ be a monotone finite-valued function which is not cyclically monotone. Let $y^1, \ldots, y^m$ be the distinct values of f. There exist $\alpha > 0$ and $y \in M_D$ such that for every $i = 1, \ldots, m$, $\tilde{y}^i = \alpha(y^i + y) \in \bar{Z}(A)$. Let $\tilde{f}$ be the function defined by $\tilde{f}(v) = \tilde{y}^i$ if and only if $f(v) = y^i$. Thus for every $v, w \in D$, $\langle v, \tilde{f}(v) - \tilde{f}(w)\rangle = \frac{1}{\alpha}\langle v, f(v) - f(w)\rangle$. Therefore all inner products in (6) are multiplied by the same positive factor, implying that $\tilde{f}$ is monotone and not cyclically monotone.

To prove the second part we first notice that by the first part we do not need to restrict ourselves to functions that take only sub-probability vectors. Assume that D is not a proper monotonicity domain and let $f : D \to R^A$ be a monotone function which is not cyclically monotone. We show that there exits a monotone function $\tilde{f} : L(D) \to R^A$ which is not cyclically monotone.

Suppose $L(D)$ is a rotation. Thus, there exists a unitary matrix U such that for every $y \in L(D)$ there exists $x \in D$ such that $Ux = y$. For all $x \in L(D)$ let $\tilde{f}(x) = Uf(U^{-1}x)$. For every three points $x, y, z \in D$, we have

$$\langle x - y, f(z)\rangle = \langle Ux - Uy, Uf(z)\rangle = \langle Ux - Uy, \tilde{f}(Uz)\rangle$$

as U is unitary. Since all the monotonicity and cyclic monotonicity constraints are defined via inner products, $\tilde{f}$ is monotone but not cyclic monotone over $L(D)$. Suppose now that $L(D)$ is an affine shift by some fixed vector $\vec{t}$. For every $x \in L(D)$, let $\tilde{f}(x) = f(x - \vec{t})$. Therefore $\langle x-y, f(z)\rangle = \langle (x-\vec{t})-(y-\vec{t}), f(z-\vec{t})\rangle$ which implies the result. Finally, suppose $L(D)$ is a contraction by a constat $c > 0$. For every $x \in L(D)$ let $\tilde{f}(x) = f(cx)$. In this case, all the inner products are multiplied by $c > 0$, and the result follows.■

Part 1 of Lemma 10 provides that a proper monotonicity domain can be equivalent defined using functions that do not output necessarily sub-probabilities, i.e. by replacing $f : D \to \bar{Z}(A)$ with $f : D \to R^A$.

4.1 Domains of dimension $k = 2$

In this section we prove Theorem 9 for domains of dimension $k = 2$.

4.1.1 Preparations

A set $L = \{v, w, z\}$ is called affine independent if its dimension is 2. The convex hull of an affine independent set $L = \{v, w, z\}$ is a simplex denoted by $\Delta(L)$ and its relative interior is denoted by $\Delta^0(L)$.

Let $\alpha > 0, \beta > 0$ be any non-negative reals and let

$$S = \{(0,0), (1,0), (\frac{1}{1+\alpha\beta}, \frac{\alpha}{1+\alpha\beta})\}. \tag{13}$$

Note that S is affine independent. The complement of $\Delta^0(S)$ is the union of the following regions(see Figure 1):

$$U_0 = \{v \in R^2 : v_1 \geq 1 - \beta v_2 \quad \text{and} \quad v_2 \geq 0\},$$

$$U_1 = \{v \in R^2 : v_1 \leq 1 - \beta v_2 \quad \text{and} \quad v_2 \geq \alpha v_1\},$$

and

$$U_2 = \{v \in R^2 : v_2 \leq \alpha v_1 \quad \text{and} \quad v_2 \leq 0\}.$$

For every $0 \leq i, j \leq 2$ let $U_{i,j} = U_i \cap U_j$. In the next proposition we construct a monotone finite-valued function on $R^2 \setminus \Delta^0(S)$. Furthermore this function will violate the cyclic monotonicity condition for every three points v, w, z that are each on a different edge "continuation" of the triangle (see Figure 1). Our parametrization will provide such a construction for any triangle, as we will see later.

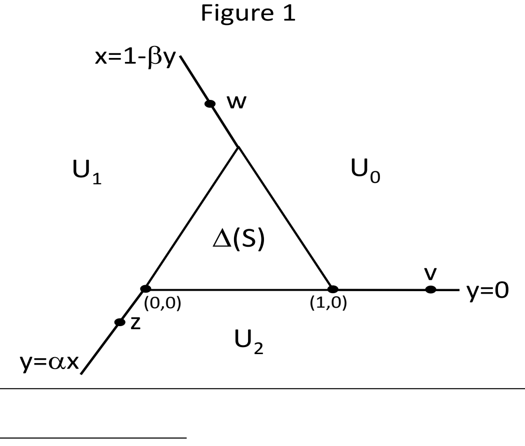
Proposition 11 *There exist a monotone finite valued function $f : R^2 \setminus \Delta^0(S) \to R^2$ which is not cyclically monotone. Furthermore f can be chosen such that its range contains exactly three vectors $y^0 = (0,1)$, $y^1 = (-\frac{\alpha}{1+\alpha\beta}, \frac{1}{1+\alpha\beta})$,$y^2 = (0,0)$ and the following hold:*

1. *For $i = 0, 1, 2$, $f(v) = y^i$ for every $v \in U_i \setminus U_{i,i+1}$.*[13]

2. *For every three vectors v, w, z such that $v \in U_{0,2}$, $w \in U_{0,1}$ and $z \in U_{1,2}$*

$$\langle v - w, f(v)\rangle + \langle w - z, f(w)\rangle + \langle z - v, f(z)\rangle < 0. \tag{14}$$

Figure 1



[13]As usual $U_{2,3} = U_{2,0}$.

Proof: First we show that f is monotone. We need to show that $\langle v-w, f(v)-f(w)\rangle \geq 0$ for every $v, w \in R^2 \setminus \Delta^0(S)$. Three cases should be considered. Assume that $f(v)=y^0$ and $f(w)=y^1$. Thus $v \in U_0$ and $w \in U_1$. Therefore

$$\langle v-w, y^0-y^1\rangle = (v_1-w_1)\frac{\alpha}{\alpha\beta+1} + (v_2-w_2)\frac{\alpha\beta}{\alpha\beta+1},$$

which is non-negative if and only if $v_1-w_1+\beta(v_2-w_2)\geq 0$ since α and β are positive. $v \in U_0$ implies that $v_1+\beta v_2 \geq 1$ and $w \in U_1$ implies that $w_1+\beta w_2 \leq 1$. Therefore $v_1-w_1+\beta(v_2-w_2)\geq 0$.

Next assume that $f(v)=y^1$ and $f(w)=y^2$. Thus $v \in U_1$ and $w \in U_2$. Therefore

$$\langle v-w, y^1-y^2\rangle = (v_1-w_1)\frac{-\alpha}{\alpha\beta+1} + (v_2-w_2)\frac{1}{\alpha\beta+1},$$

which is non-negative if and only if $-\alpha(v_1-w_1)+v_2-w_2\geq 0$. This inequality holds since $v \in U_1$ and $w \in U_2$. Finally assume that $f(v)=y^2$ and $f(w)=y^0$. Thus $v \in U_2$ and $w \in U_0$. Therefore

$$\langle v-w, y^2-y^0\rangle = w_2-v_2 \geq 0,$$

where the last inequality follows since $v \in U_2$ and $w \in U_0$.

To complete the proof we show that part b holds, which in particular shows that f is not cyclically monotone. Let $v \in U_{0,2}$,$w \in U_{0,1}$ and $z \in U_{1,2}$ (see Figure 1). Then $v=(a,0)$ for some $a>0$, $w=(1-\beta b, b)$ for some $b>0$ and $z=(-c,-\alpha c)$ for some $c>0$. Therefore the left-hand side of (14) equals

$$-b-(1-\beta b+c)\frac{\alpha}{\alpha\beta+1} + (b+\alpha c)\frac{1}{\alpha\beta+1} = -\frac{\alpha}{\alpha\beta+1} < 0.\blacksquare$$

4.1.2 Proof of Theorem 9 for $k=2$

Let $D \subseteq R^A$ be a set of dimension $k=2$ whose closure $cl(D)$ is non-convex. By Proposition 11 and Lemma 10, D is not a proper monotonicity domain if there exist affine independent valuations, v, w, z, in D, such that the relative interior of the simplex generated by them does not intersect D. For example, such valuations can be easily detected for the non-convex ring at Figure 2, by choosing the vertices shown in this figure. However, it is not true that such valuations exist for every non-convex set, even not for a closed.

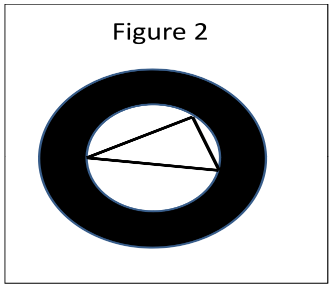


Figure 2

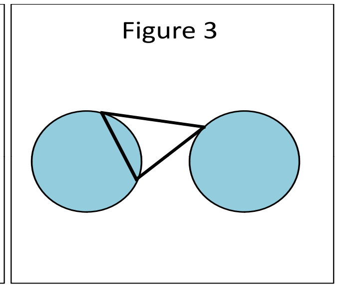


Figure 3

For example, if D is the union of two disjoint closed disks shown in Figure 3, then, as demonstrated in that figure, for every three valuations in D, not on the same line, the relative interior of the triangle generated by them intersects D. Therefore we need a more delicate procedure, which uses the following claim.

Claim 1 *Let D be a set of dimension $k = 2$ whose closure is non-convex. There exist 3 affine independent valuations in D such that the relative interior of the simplex Δ generated by them contains a point, say d for which there exists $\eta > 0$ such that $B(d, \eta) \cap D = \emptyset$, where $B(d, \eta) = \{v \in \Delta^0 | \; ||v - d|| < \eta\}$.*

The proof of Claim 1 is postponed to the end of this proof. Without loss of generality we can assume that $D \subseteq R^2$. Let v_0, v_1, v_2 be affine independent valuations in D such that there exist d and η as in Claim 1. We choose η to be small enough such that $B(d, \eta) \subset \Delta^0(\{v_0, v_1, v_2\})$ (see Figure 4).

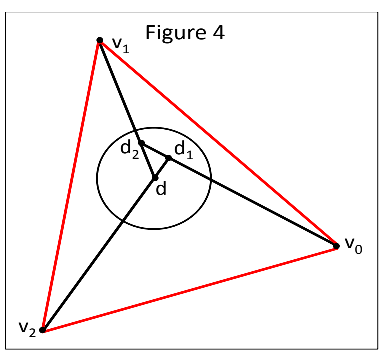


Figure 4

By rotating and shifting the plane we can assume without loss of generality that $v_0 = (0, 0)$ and $d = (0, y)$ for some $y > 0$.[14] Consider the line $v_0 + t(d - v_0 + (\epsilon, 0))$ for $\epsilon > 0$. Let d_1 and d_2 be the points in which this line intersects the lines $v_1 + t(d - v_1)$ and $v_2 + t(d - v_2)$ respectively. There exist small enough ϵ such that $d_1 \in B(d, \eta)$ and $d_2 \in B(d, \eta)$ since for $\epsilon = 0$ all three lines intersect in d^0.

[14]Lemma 10 provides that any shift, rotation or scaling of D preserves the monotonicity domain property. Hence these operations can be done alternatively on the space itself.

To finish the proof note that it is possible to rotate shift and scale the plane such that $S = \{d, d_1, d_2\}$ (see (13)) and v_0, v_1, v_2 are vectors as in part two of Proposition 11. We can now apply Proposition 11 to show that D is not a proper monotonicity domain.■

To complete the proof of the theorem it remains to prove Claim 1.

Proof of Claim 1: Assume in negation that the claim does not hold. Therefore, for every 3 affine independent valuations in D, the interior of the simplex Δ generated by them is contained in $cl(D)$. Therefore the simplex itself is contained in $cl(D)$. As the dimension of D is 2, for every $v_0 \neq v_1$ in D there exists $v_2 \in D$ such that v_0, v_1, v_2 are affine independent, and therefore the simplex generated by these valuation is contained in $cl(D)$, and therefore the interval $[v_0, v_1] \subseteq cl(D)$. Let w_0, w_1 be in $cl(D)$. There exist sequences v_0^n, v_1^n in D such that $v_i^n \to w_i$, $i = 0, 1$. Therefore, every valuation in $[w_0, w_1]$ is a limit of valuations in $cl(D)$, and hence it belongs to $cl(D)$. This implies that $cl(D)$ is convex contradicting the assumption of the claim. Hence, Claim 1 holds, which completes the proof of the Theorem for $k = 2$. ■

4.2 Domains of Dimension $k \geq 3$

In this section we prove Theorem 9 for domains of dimension $k \geq 3$. We will need the following definition. A domain D is called *good*, if for every $v, w \in D$ the projection of D onto $I = [v, w]$ is dense in $[v, w]$.[15] Note that if D is not a good set then its closure is not convex. We first show:

Proposition 12 *If domain D is not good then it is not a proper monotonicity domain.*

Proof: Let $D \subseteq R^A$ be of dimension k where $k \geq 3$. For any closed convex set Q let $\Pi_Q(D)$ denote the projection of D on Q. Since D is not good there exist $v, w \in D$ and an open interval $(a, b) \subseteq [v, w]$ such that $\Pi_{[v,w]}(D) \cap (a, b) = \emptyset$. Let $z \in D$ be a vector in D such that v, w, z are affine independent. Rotate and shift the space such that v, w, z are in the XY plane and $[v, w]$ lies on the X axis. There exist $a < d < b$ and $\epsilon > 0$ such that $B((d, 0), \epsilon) \cap \Pi_{XY}(D) = \emptyset$ where Π_{XY} is the projection to the XY plane. Therefore, by our proof for dimension 2, $\Pi_{XY}(D)$ is not a proper monotonicity domain. In particular there exists a monotone finite-valued function $f : \Pi_{XY}(D) \to R^2$ that is not cyclically monotone. Let $\tilde{f} : D \to R^A$ be the function defined by $\tilde{f}(x_1, \ldots, x_A) = (f_1(x_1, x_2), f_2(x_1, x_2), 0, \ldots, 0)$ for every $x \in D$. Note that $\tilde{f}$ is well defined since for every $x \in D$ either $x_1 \geq b$ or $x_1 \leq a$. Clearly $\tilde{f}$ is monotone finite-valued and not cyclically monotone.■

By Proposition 12 it remains to deal only with good sets. These are studied in the next two subsections. One example for a good set which is not convex is the unit sphere in R^3.

[15] Note that since the interval is closed and convex the projection is well defined.

4.2.1 Preparations

Remark: throughout this section we will denote by v_l the l-th coordinate of v and indices of vectors will be denoted by superscripts.

Let $k \geq 2$ be some integer. Let S^k be the following hyperplane in R^{k+1}:

$$S^k = \{v \in R^{k+1} : \sum_{i=1}^{k} v_i = 0\}. \tag{15}$$

For every $\alpha > 0$, $\varepsilon_1, \varepsilon_2 > 0$, and $\delta > 0$, and $i = 1, \ldots, k$ we define the following regions in S^k:

$$U_i^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, \quad v_i = \max_{j=1}^{k} v_j, \quad v_{k+1} \geq \alpha + \frac{1 - \delta - v_i}{\varepsilon_1} \right\},$$

$$M_i^k(\alpha) = \left\{ v \in S^k : -\alpha \leq v_{k+1} \leq \alpha, \quad v_i = \max_{j=1}^{k} v_j \geq 1 \right\},$$

and

$$D_i^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, \quad v_i = \max_{j=1}^{k} v_j, \quad v_{k+1} \leq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Let

$$P_u^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, \quad \text{for every } i \leq k \quad v_{k+1} \leq \alpha + \frac{(1 - \delta - v_i)}{\varepsilon_1} \right\}$$

and

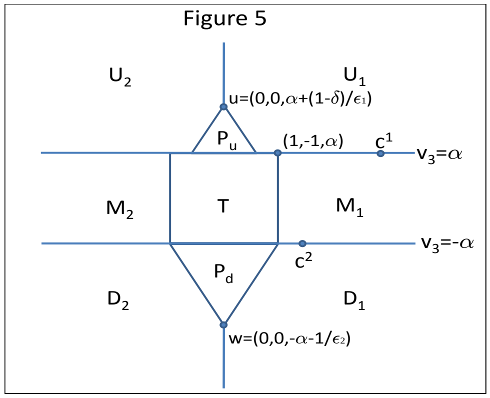
$$P_d^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, \quad \text{for every } i \leq k \quad v_{k+1} \geq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Note that if $v \in P_u$ or $v \in P_d$, then $v_i \leq 1$ for every $i \leq k$. Finally, let

$$T(\alpha) = \{v \in S^k : \text{for every } i \leq k \quad v_i < 1, \quad \text{and } -\alpha < v_{k+1} < \alpha\}. \tag{16}$$

Figure 5 illustrates the hyperplane S^2 and the regions defined above for $k = 2$.

The superscript k and the arguments $\varepsilon_1, \varepsilon_2, \alpha, \delta$ will be dropped whenever these are clear from context. Let $\Omega = \{U_1, \ldots, U_k, M_1, \ldots, M_k, D_1, \ldots, D_k, P_d, P_u\}$, and let $G = \bigcup_{Q \in \Omega} Q$. Observe that $S^k = G \cup T$. For every set L we denote by $ri(L)$ the relative interior of L with respect to S^k.



In the following proposition we show that G is not a proper monotonicity domain. Let u, w be the peaks of P_d and P_u respectively as illustrated in Figure 5. For any $i, 1 \leq i \leq k$, we show how to construct a monotone function on G such that for every 2 points $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$ the sequence of points w, u, c^1, c^2 (see Figure 5) violates the cyclic monotonicity condition (3). This construction will be a key tool in our main proof.

Proposition 13 *Let $\alpha, \varepsilon_1, \varepsilon_2, \delta$ be positive reals such that $2\alpha\varepsilon_1 > \delta$. There exist a monotone and finite-valued function $f : G \to R^{k+1}$, which is not cyclically monotone. Moreover f can be chosen such that its range contains exactly $3k+1$ distinct vectors $y^{U_1}, \ldots, y^{U_k}, y^{M_1}, \ldots, y^{M_k}, y^{D_1}, \ldots, y^{D_k}, y^P$ and such that for some fixed $i, 1 \leq i \leq k$ the following hold:*

1. *For every set $Q \in \Omega$, $f(v) = y^Q$ for all $v \in ri(Q)$ where $y^{P_u} = y^{P_d} = y^P$.*[16]

2. *$f(v) = y^{U_i}$ for all $v \in U_i \cap P_u$, $f(v) = y^{M_i}$ for all $v \in U_i \cap M_i$, $f(v) = y^{D_i}$ for all $v \in M_i \cap D_i$ and $f(v) = y^P$ for all $v \in D_i \cap P_d$.*

3. *For every $v \in G$ other than in 1. and 2., let $f(v) = y^Q$ for an arbitrary Q in which $v \in Q$.*

4. *For every two distinct vectors c^1 and c^2 in which $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$,*

$$\langle w - u, f(w) \rangle + \langle u - c^1, f(u) \rangle + \langle c^1 - c^2, f(c^1) \rangle + \langle c^2 - w, f(c^2) \rangle < 0. \quad (17)$$

where $w = (0, \ldots, 0, -\alpha - \frac{1}{\varepsilon_2})$ and $u = (0, \ldots, 0, \alpha + \frac{1-\delta}{\varepsilon_1})$.

[16]Since all the sets in Ω are defined with equalities we define first the function on the interior of every set in Ω, and then break ties on the boundaries.

Proof: In order to define the range of f we make use of the following notation. Let $e^j(\gamma) \in R^{k+1}$ denote the sum $e^j + (0, \ldots, 0, \gamma)$ where both vectors are in R^{k+1}. The range of f is defined as follows:

$$y^Q = \begin{cases} e^j(\varepsilon_1) & Q = U_j, \\ e^j & Q = M_j, \\ e^j(-\varepsilon_2) & Q = D_j, \\ \bar{0} & Q = P_d \quad \text{or} \quad Q = P_u. \end{cases} \tag{18}$$

We first show that f is not cyclically monotone. To see this it is enough to verify that (17) holds. Let w, v, c^1, c^2 be as in part 4 of the proposition. Since $f(w) = \bar{0}$, $\langle w - u, f(w) \rangle = 0$. Since $c^1 \in U_i \cap M_i$ it has the form $c^1 = (c^1_1, \ldots, c^1_k, \alpha)$. Similarly $c^2 = (c^2_1, \ldots, c^2_k, -\alpha)$. Therefore

$$\langle u - c^1, f(u) \rangle = -c^1_i + \frac{1-\delta}{\varepsilon_1} \cdot \varepsilon_1 = -c^1_i + 1 - \delta,$$

$$\langle c^1 - c^2, f(c^1) \rangle = c^1_i - c^2_i, \quad \text{and} \quad \langle c^2 - w, f(c^2) \rangle = c^2_i + \frac{1}{\varepsilon_2} \cdot (-\varepsilon_2) = c^2_i - 1.$$

Summing up all the terms we obtain that (17)$=-\delta < 0$.

To complete the proof we need to show that f is monotone on G. Let $v = (v_1, \ldots, v_{k+1}), w = (w_1, \ldots, w_{k+1})$ be two vectors. Let e^0 denote the zero vector. We distinguish between the following cases (i will be used now as an arbitrary index):

1. $f(v) = y^{U_i}$ and $f(w) = y^{U_j}$. Thus, $v \in U_i, w \in U_j$. Therefore

   $$\langle v - w, f(v) - f(w) \rangle = \langle v - w, e^i - e^j \rangle = (v_i - w_i) + (v_j - w_j) = (v_i - v_j) + (w_j - w_i) \geq 0$$

   where the last inequality follows since $v_i \geq v_j$ and $w_j \geq w_i$.

2. $f(v) = y^{U_i}$ and $f(w) = y^{M_i}$. Thus, $v \in U_i, w \in M_i$, implying that

   $$\langle v - w, f(v) - f(w) \rangle = \langle v - w, e^0(\varepsilon_1) \rangle = (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0,$$

   since $v_{k+1} \geq \alpha, w_{k+1} \leq \alpha$, and $\varepsilon_1 > 0$.

3. $f(v) = y^{U_i}$ and $f(w) = y^{M_j}$ for $i \neq j$. Thus $v \in U_i, w \in M_j$. Therefore

   $$\langle v - w, f(v) - f(w) \rangle = \langle v - w, e^i(\varepsilon_1) - e^j \rangle = (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0$$

   where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$ and $(v_{k+1} - w_{k+1})\varepsilon_1 \geq 0$.

4. $f(v) = y^{U_i}$ and $f(w) = y^{D_i}$. Thus $v \in U_i, w \in D_i$. Therefore

   $$\langle v - w, f(v) - f(w) \rangle = \langle v - w, e_0(\varepsilon_1) - e^0(-\varepsilon_2) \rangle = (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0$$

   where the last inequality follows since $v_{k+1} \geq \alpha, w_{k+1} \leq -\alpha$, and $\varepsilon_1, \varepsilon_2 > 0$.

5. $f(v) = y^{U_i}$ and $f(w) = y^{D_j}$. $x \in U_i,\ w \in D_j$. Then

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i(\varepsilon_1) - e^j(-\varepsilon_2)\rangle = (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0$$

where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$ and $v_{k+1} \geq w_{k+1}$.

6. $f(v) = y^{M_i}$ and $f(w) = y^{M_j}$. Thus $x \in M_i, y \in M_j$. Therefore

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i - e^j\rangle = (v_i - w_i) + (v_j - w_j) = (v_i - v_j) + (w_j - w_i) \geq 0$$

where the last inequality follows since $v_i \geq v_j$, $w_j \geq w_i$.

7. $f(v) = y^{U_i}$ and $f(w) = y^P$. Thus $v \in U_i, w \in P_u \cup P_d$. Then

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i(\varepsilon_1)\rangle = (v_i - w_i) + \varepsilon_1 \cdot (v_{k+1} - w_{k+1}) \geq$$

$$v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - \varepsilon_1 w_{k+1}. \tag{19}$$

where the last inequality follows since $v_{k+1} \geq \alpha + \frac{1-\delta-v_i}{\varepsilon_1}$. If $w \in P_u$ then $w_{k+1} \leq \alpha + \frac{1-\delta-w_i}{\varepsilon_1}$, and therefore

$$(19) \geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (\alpha \cdot \varepsilon_1 + 1 - \delta - w_i) = 0.$$

If $w \in P_d$ then $w_{k+1} \leq -\alpha$ and $w_i \leq 1$. Therefore, since $2\alpha\varepsilon_1 \geq \delta$,

$$(19) \geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (-\alpha \cdot \varepsilon_1) = 1 - w_i - \delta + 2\alpha\varepsilon_1 \geq 0.$$

8. $f(v) = y^{M_i}$ and $f(w) = y^P$. Thus $v \in M_i$, and $w \in P_u \cup P_d$. Therefore

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i\rangle = v_i - w_i \geq 1 - 1 = 0,$$

where the last inequality follows since $v_i \geq 1, w_i \leq 1$.

9. $f(v) = y^{D_i}$ and $f(w) = y^P$. Thus $v \in D_i, w \in P_u \cup P_d$ Then

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i(-\varepsilon_2)\rangle =$$

$$(v_i - w_i) + \varepsilon_2 \cdot (w_{k+1} - v_{k+1}). \tag{20}$$

If $w \in P_u$ then $w_{k+1} \geq \alpha > -\alpha \geq v_{k+1}$, implying that (20) ≥ 0 because $v_i \geq 1 \geq w_i$. Suppose $w \in P_d$. Therefore $w_{k+1} \geq -\alpha - \frac{1-w_i}{\varepsilon_2}$, and since $v \in D_i$ then $v_{k+1} \leq -\alpha - \frac{1-v_i}{\varepsilon_2}$. Therefore

$$(20) \geq v_i - w_i + (\alpha \cdot \varepsilon_2 + 1 - v_i) - (\alpha \cdot \varepsilon_2 + 1 - w_i) = 0.$$

The other cases in which $f(v) = y^{D_i}$ are very similar to those in which $f(v) = y^{U_i}$. In fact it is easier for monotonicity to hold in these cases since P_d is a larger set than P_u. ■

By Lemma 10 and Proposition 13 we obtain:

Corollary 14 *For any $\alpha, \varepsilon_1, \varepsilon_2, \delta$ such that $2\alpha\varepsilon_1 > \delta$, G is not a proper monotonicity domain.*

4.2.2 Proof of Theorem 9 for good domains with $k \geq 3$

Proposition 15 *Let D be a good domain with $dim(D) = k \geq 3$ with a non-convex closure. Then D is not a proper monotonicity domain.*

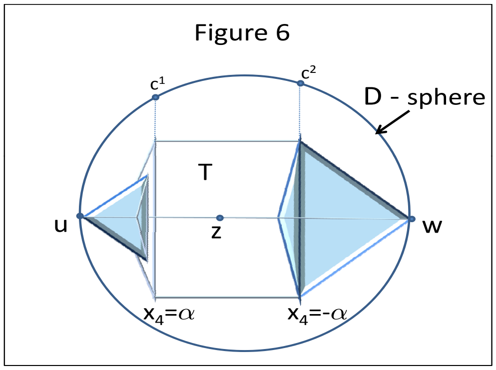
Before the proof we describe our technique. Roughly speaking, the technique resembles the technique in the the proof for $k = 2$, as we embed the structure in the previous section in a way that allows us to apply Proposition 13. In particular we show that for any good domain D there exist parameters α,$\varepsilon_1, \varepsilon_2$ and δ as in Proposition 13 such that the set T and the sets in Ω (see Section 4.2.1) can be embedded in the space so that T is in the relative interior of $ConvexHull(D) \setminus D$ (in the proof of $k = 2$ we located a simplex to be in the relative interior). The peaks of the simplexes $w \in P_d$ and $u \in P_u$ are both in D, and there exist c^1 and c^2 as in Proposition 13 that also belong to D. See Figure 6 for an illustration of this construction when D is a sphere (The colored regions in Figure 6 represent P_u and P_d).

Proof of Proposition 15:

Since $cl(D)$ is not convex there exist $w, u \in D$, $z \in I := [w, u]$ and $r > 0$ such that $B(z, r) \cap cl(D) = \emptyset$, where $B(z, r) = \{v \in R^{k+1} : ||v - z|| < r\}$. We can assume without loss of generality that $D \subseteq R^{k+1}$. Rotate the space so that the positive x_{k+1} axis is from u to w. All other coordinates are parameterized by $x_1, x_2, \ldots, x_k$ such that $\sum_{i=1}^{n} x_i = 0$.

Let I_z be the interval of length $r_1 > 0$ centered in z on I. There exist $r_1 > 0$ such that for every $a = (a_1, \ldots, a_k, a_{k+1})$ with $a_{k+1} \in I_z(r_1)$ and $a_i \leq r_1$ for every $i \leq k$ implies that $a \notin cl(D)$. We scale D by $\frac{1}{r_1}$. Thus, if $a_{k+1} \in I_z$ and $a_i \leq 1$ for every $i \leq k$ then $a \notin cl(D)$.

Let $a^1, a^2, \ldots, a^{k+1}$ be $k+1$ equally spaced vectors in I_z, i.e. there exist $d > 0$ such that for every $i, 1 \leq i \leq k$, $a^{i+1}_{k+1} - a^i_{k+1} = d$.



Let Π_{wu} be the projection operator on the interval $I = [w, u]$. Since D is a good set there exist $k + 1$ distinct vectors $b^1, \ldots, b^k, b^{k+1} \in D$ such that $\Pi_{wu}(b^i) \in (a^{i-1}, a^i)$. For any vector $b = (b_1, \ldots, b_k)$ we define $indmax(b)$ to be some arbitrary index in $\arg\max_{i=1}^k b_i$. That is $b_{indmax(b)} = \max_{i=1}^k b_i$. Note that there exist $i, j \leq k + 1$ such that $i < j$ and $indmax(b^i) = indmax(b^j)$. Define $t := indmax(b^i)$.

Let $c = \frac{\Pi_{wu}(b^j)_{k+1} + \Pi_{wu}(b^i)_{k+1}}{2}$. We now shift the set so that c moves to $(0, 0, \ldots, 0)$. Therefore $\Pi_{wu}(b^j) = -\Pi_{wu}(b^i)$. In particular there exist $\alpha > 0$ such that $\Pi_{wu}(b^j) = (0, \ldots, 0, \alpha)$ and $\Pi_{wu}(b^i) = (0, \ldots, 0, -\alpha)$. We obtained that

$$w = (0, \ldots, 0, w_{k+1}), y = (0, \ldots, 0, -u_{k+1}), b^i = (., \ldots, -\alpha), b^j(., \ldots, \alpha) \quad \text{where} \quad w_{k+1}, u_{k+1} > \alpha.$$

Define ε_1,ε_2 and δ as follows:

$$\varepsilon_2 = \frac{1}{y_{k+1} - \alpha}; \delta < \min\left(\frac{\alpha}{w_{k+1} - \alpha}, \frac{1}{2}\right); \varepsilon_1 = \frac{1 - \delta}{w_{k+1} - \alpha}.$$

Therefore

$$\varepsilon_1 > \frac{1}{2(w_{k+1} - \alpha)}; \quad 2\alpha\varepsilon_1 > \frac{\alpha}{w_{k+1} - \alpha} > \delta.$$

Recall that $T(\alpha)$ is the k dimensional prism (see (16)). Therefore $T(\alpha) \cap D = \emptyset$. We can now apply Proposition 13 with $i = t$, where w, u are as in part 4 of the proposition and $c^1 = b^j$, $c^2 = b^i$. This implies that D is not a proper monotonicity domain.■

5 Monotonicity Domains - Characterization

In this section we complete our characterization of monotonicity domains.

Recall that a domain D is a *monotonicity domain* if every monotone finite-valued randomized allocation rule is also cyclically monotone. Note that every proper monotonicity domain is a monotonicity domain.

Let $H^A = \{v \in R^A : \sum_{a \in A} v_a = 0\}$ be the hyperplane which is orthogonal to the vector $(1, \ldots, 1) \in R^A$. Denote by $\Pi : R^A \to H^A$ the projection onto the hyperplane H^A.

Theorem 16 *Let $D \subseteq R^A$. D is a monotonicity domain if and only if $\Pi(D)$ is a proper monotonicity domain.*

Proof: Assume D is a monotonicity domain and suppose for contradiction that there exists a function $f^0 : \Pi(D) \to \bar{Z}(A)$ which is monotone but not cyclically monotone. Let f^1 be a randomized allocation rule obtained from f^0 by adding an appropriate multiple of $(1, \dots, 1)$ to each value of f^0:

$$f^1(v) := f^0(v) + \frac{1 - \sum_{a \in A} f^0_a(v)}{|A|}(1, \dots, 1).$$

Let f^2 be the natural extension of f^1 to D. That is $f^2(v) = f^1(\Pi(v))$ for every $v \in D$. Thus f^2 is also a finite-valued randomized allocation rule. We claim that f^2 is monotone but not cyclically monotone. To see this it is enough to show that for any $v, w, z \in D$, $\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\rangle$. Let $v, w, z \in D$.

$$\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle + \langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle.$$

Since $f^1(\Pi(w)) - f^1(\Pi(z)) \in H^A$, we have that $\langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle = 0$. Therefore,

$$\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\rangle + \langle \Pi(v), c(1, \dots, 1)\rangle,$$

where c is some real number. Since $\langle \Pi(v), c(1, \dots, 1)\rangle = 0$ we are done.

We proceed to prove the other direction. Assume $\Pi(D)$ is a proper monotonicity domain and suppose that D is not a monotonicity domain, i.e. there exists a randomized allocation rule f^0 on D that is monotone but not cyclically monotone. Let $v_1, \dots, v_k$ be a shortest sequence of valuations which violates the cyclically monotonicity condition:

$$\sum_{i=1}^{k} \langle v_i, f^0(v_i) - f^0(v_{i-1})\rangle < 0. \tag{21}$$

We have that (21)=

$$\sum_{i=1}^{k} \langle \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle + \sum_{i=1}^{k} \langle v_i - \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle = \sum_{i=1}^{k} \langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i)\rangle, \tag{22}$$

where the second equality follows since $v_i - \Pi(v_i)$ is orthogonal to $f^0(v_i) - f^0(v_{i-1})$.

Claim 2 *For any $i \neq j$, $\Pi(v_i) \neq \Pi(v_j)$.*

Proof: Suppose $i < j$ and $\Pi(v_i) = \Pi(v_j)$. Taking all indices modulo k we have $\sum_{l=1}^{k} \langle \Pi(v_l) - \Pi(v_{l+1}), f^0(v_l) \rangle =$

$$\langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i) \rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_j), f^0(v_{j-1}) \rangle +$$

$$\langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j) \rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_i), f^0(v_{i-1}) \rangle =$$

$$\langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i) \rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_i), f^0(v_{j-1}) \rangle + \tag{23}$$

$$\langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j) \rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_j), f^0(v_{i-1}) \rangle] < 0. \tag{24}$$

Clearly at least one of (23) or (24) is negative contradicting the minimality of k. ■

Next we say that f^1 on $\Pi(D)$ is a projection of f^0 if for any $v \in \Pi(D)$, $f^1(v) \in f^0(\Pi^{-1}(v))$.

Claim 3 *Any projection of f^0 is monotone.*

Proof: For any $v, w \in \Pi(D)$ there is $\tilde{v}, \tilde{w} \in D$ such that $\Pi(\tilde{v}) = v$, $\Pi(\tilde{w}) = w$, $f^0(\tilde{v}) = f^1(v)$ and $f^0(\tilde{w}) = f^1(w)$. We have

$$\langle v - w, f^1(v) - f^1(w) \rangle = \langle v - w, f^0(\tilde{v}) - f^0(\tilde{w}) \rangle = \langle \tilde{v} - \tilde{w}, f^0(\tilde{v}) - f^0(\tilde{w}) \rangle. \blacksquare$$

Finally, since by Claim 2 all the $\Pi(v_i)$'s are distinct we can select a projection f^1 of f^0 such that $f^1(\Pi(v_i)) = f^0(v_i)$ for all $i = 1, \ldots, k$. Therefore f^1 is monotone but not cyclically monotone:

$$\sum_{i=1}^{k} \langle \Pi(v_i), f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1})) \rangle = \sum_{i=1}^{k} \langle v_i, f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1})) \rangle =$$

$$\sum_{i=1}^{k} \langle v_i, f^0(v_i) - f^0(v_{i-1}) \rangle < 0.$$

This contradicts that $\Pi(D)$ is a proper monotonicity domain. ■

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