A Theorem Concerning the Integer Lattice

By David E. Bell

This note shows that if an integer linear program in n variables has more than 2^n linear inequality constraints, then either some of the constraints are unnecessary or there is at least one feasible integer point.

The following problem is considered: how many linear constraints can be necessary to bound a region in n-space containing no integer lattice points? It is easy to find 2^n constraints (each one cutting off one corner of the 0–1 cube) that bound a region containing no integer coordinate point, such that removal of any constraint extends that region to include a "lattice" point. In this note we show that if one has 2^n+c, $c>0$, linear constraints that bound a region F containing no lattice point, then at least c of the constraints may be discarded in such a manner that the new feasible region (containing F) still has no lattice points. This question arose in a study of the use of intersecting corner polyhedra to describe integer linear programs [2].

The proof is straightforward if we assume that the linear constraints $C_1, C_2, \ldots, C_m$ are defined using rational coefficients, for then we may suppose without loss that the constraints are all of the strict inequality type (for if $a_i x \leqslant b_i$ defines C_i, then the set of lattice points contained in F is not changed if C_i is replaced by $a_i x < b_i + \epsilon_i$ for some small $\epsilon_i > 0$), though the result is true in general.

Consider each constraint C_j in turn. If C_j excludes no lattice point from F that is not excluded by other constraints, it may be discarded. Otherwise, translate it as far as possible without including a lattice point in the interior of F, and call this constraint C_j'. Necessarily some lattice point x^j lies on C_j' but *not* on $C_1', C_2', \ldots, C_{j-1}', C_{j+1}, \ldots, C_m$. After examining all of the constraints in this manner suppose that we have 2^n+c lattice points x^j; consider their convex hull. By construction, the whole of the convex hull, excepting only the 2^n+c lattice

Published by Elsevier North-Holland, Inc.

points, lies in the interior of F' (the new F). Also by construction, F' contains no lattice point in its interior. But if $c \geqslant 1$, then at least two of the lattice points must differ by an even number in each coordinate, as there are only 2^n possible coordinates modulo 2. Their midpoint, again a lattice point, is necessarily in the interior of F', a contradiction. Hence $c \leqslant 0$.

By an identical argument replacing 2^n by $(1+k)^n$ and counting modulo $1+k$, one can show that if there are more than $(1+k)^n$ necessary constraints, each excluding a lattice point not otherwise excluded, then there are at least k lattice points lying on a straight line in the interior of the region.

In two dimensions this shows that if there are y necessary constraints, then there must be at least on the order of $y^{1/2}$ interior lattice points. In fact one can show (Andrews [1]) that there must be on the order of y^3 such points for large y, so that this bound is far from exact.

Acknowledgements

I would like to thank Professor D. J. Kleitman, to whom the form and simple proof of this paper are due, and also Professor J. Edmonds, Andrew Odlyzko and Carlos Winkler for their interest and suggestions.

References

1. G. E. Andrews, A lower bound for the volume of strictly convex bodies with many boundary lattice points, *Trans. Am. Math. Soc.* **106**, 270–279 (1963).
2. D. E. Bell and M. L. Fisher, Improved integer programming bounds using intersections of corner polyhedra, *Math. Program.* **8**, 345–368 (1975).
3. B. Divis, Lattice points on convex curves, *Monatsh. Math.* **77**, 389–395 (1973).

Cambridge University

(Received November 25, 1975)