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DOUBLE-EXPONENTIAL UTILITY FUNCTIONS*  

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Ordinal additivity between two attributes $X$ and $Y$ is a property that permits the utility function $u(x, y)$ to be represented as $\phi(v(x) + w(y))$ for some functions $\phi$, $v$ and $w$ where $\phi$ may be thought of as a single attribute utility function over $v + w$. In applications $\phi$ is usually taken to be either linear (the additive decomposition) or exponential (the multiplicative decomposition). The utility function can be shown to be exactly one of these forms whenever the two attributes are mutually utility independent. In this paper we introduce a relaxation of utility independence and show that it too implies ordinal additivity. We identify the class of utility functions $\phi$ that are consistent with the relaxed assumptions.

One member of this class, $\phi(z) = -\exp(b \exp(-cz))$, which we call double-exponential, seems particularly appealing, for when $b$ and $c$ are positive it is increasing, risk averse and decreasingly risk averse for all $z$. The multiattribute decomposition corresponding to it $-\exp(b \exp(-c(x + w)))$ may be represented as $u(x, y) = c \exp(cu(x)w(y))$ which we call the exponential product.

1. Introduction. Let $u(x, y)$ be a two-attribute utility function on a continuous interval space $X \times Y = [x^0, x^*] \times [y^0, y^*]$, then the conditional utility function $u(x | y)$ is defined by

$$u(x | y) = \frac{u(x, y) - u(x^0, y)}{u(x^*, y) - u(x^0, y)}.$$  

In this paper we will restrict attention to the case where $x$ and $y$ are both scalar, for generalizations to the vector case will normally be obvious. For simplicity, we will also assume that preferences for $X$ increase from $x^0$ to $x^*$ and, for $Y$, increase from $y^0$ to $y^*$. Without loss we will presume $u(x^0, y^0) = 0$ and $u(x^*, y^*) = 1$ and denote $u(x^0, y^0) = a$ and $u(x^0, y^*) = b$. Though the letter $u$ will be used for three functions $u(x, y)$, $u(x | y)$ and $u(y | x)$, no confusion should arise. The equation above may be rearranged to illustrate the assessment problem:

$$u(x, y) = bu(y | x^0) + [a + (1 - a)u(y | x^*) - bu(y | x^0)]u(x | y).$$  

Without simplifying assumptions we need to assess two constants $a$ and $b$, two one-attribute utility functions $u(y | x^0)$ and $u(y | x^*)$ as well as, in principle, an infinite number of one-attribute utility functions, $u(x | y)$.

Keeney and Raiffa (1976) exploited the utility independence assumption that $u(x | y) = u(x | y^0)$ for all $y$. If, in addition, $u(y | x) = u(y | x^0)$, Keeney (1968) showed that

$$u(x, y) = au(x | y^0) + bu(y | x^0) + (1 - a - b)u(x | y^0)u(y | x^0),$$

known as the multiplicative form. The special case in which $b = 1 - a$ is known as the additive form.

Kirkwood (1976) considered the case in which $u(x | y)$ could be assumed always to

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be a member of the same parametric family, for example \( u(x | y) = -\exp(-c(y)x) \). In this case the assessment of \( u(x | y) \) is reduced to determining the one-dimensional function \( c(y) \). Kirkwood described this kind of condition as \( X \text{ parametrically dependent} \) on \( Y \).

Bell (1979) approached the assessment problem by assuming an interpolation condition:

\[
u(x | y) = \alpha(y)u(x | y^0) + (1 - \alpha(y))u(x | y^*).
\]

If \( X \) is interpolation independent of \( Y \) and vice versa, Bell showed that

\[
u(x, y) = au(x | y^0) + bu(y | x^0) + cu(x | y^0)u(y | x^0)
- (c + a)u(x | y^0)u(y | x^*) - (c + b)u(x | y^*)u(y | x^0)
+ (1 + c)u(x | y^*)u(y | x^*),
\]

which requires the assessment of 3 constants and 4 one-attribute utility functions.

While interpolation independence has many advantages for the assessment of complex multiattribute utility functions, it does have a disadvantage not shared by utility independence or parametric dependence. If \( u(x | y^0) \) and \( u(x | y^*) \) are both exponential, one might wish that so too should be \( u(x | y) \) for other values of \( y \). More generally we might ask that local risk aversion properties shared by \( u(x | y^0) \) and \( u(x | y^*) \) would be common in all \( u(x | y) \).

Pratt (1964) defined (local) risk aversion for a utility function \( u(x) \) by the function \( r(x) = -u''(x)/u'(x) \), a quantity that is proportional to the risk premium for small (local) gambles. Define the conditional risk aversion of \( X \) given \( Y \), by the function \( r(x | y) = -u''(x | y)/u'(x | y) \). We note the following properties:

(i) \( r(x | y) \) is independent of \( x \) and \( y \) if and only if \( X \) is utility independent of \( Y \) and \( u(x | y) = cx \) or \( -ce^{-cx} \) for all \( x \) and \( y \), for some constant \( c \),

(ii) \( r(x | y) \) is independent of \( y \) if and only if \( X \) is utility independent of \( Y \), and

(iii) \( r(x | y) \) is independent of \( x \) if and only if \( X \) is parametrically dependent on \( Y \) with the exponential form.

These properties suggest that insight about multiattribute assessment procedures may be gained by considering the problem as one of assessing the conditional risk aversion functions. In this paper we exploit the interpolation idea but apply it to \( r(x | y) \) rather than \( u(x | y) \).

2. Interpolating risk aversion. Suppose that, for some function \( \alpha(y) \),

\[
r(x | y) = \alpha(y)r(x | y^0) + (1 - \alpha(y))r(x | y^*).
\]

For the sake of a name, we will call this condition \textit{risk aversion interpolation independence} with the abbreviation \( X \text{ RAI} II Y \). This property includes utility independence as a special case and, in some cases, it includes parametric dependence as a special case also. For if \( u(x | y^0) \) and \( u(x | y^*) \) are exponential, so too is \( u(x | y) \); if \( u(x | y^0) \) and \( u(x | y^*) \) have constant proportional risk aversion, so too has \( u(x | y) \). However, (4) does not preserve the logarithmic form. For example, if \( u(x | y^0) = \log(x + a_0) \) and \( u(x | y^*) = \log(x + a_1) \), then (4) does not produce the result \( u(x | y) = \log(x + a(y)) \). Nor does it preserve the more general power function family of \( (x + a)^b \). [These forms do satisfy the interpolation equation

\[
\frac{1}{r(x | y)} = \alpha(y)\left(\frac{1}{r(x | y^0)}\right) + (1 - \alpha(y))\left(\frac{1}{r(x | y^*)}\right),
\]

but we leave a study of that relationship for another time.]
An assumption of $X$ RALL $Y$ together with $Y$ RALL $X$ we will denote by $X$ MRAII $Y$, or, equivalently $Y$ MRAII $X$ ($M$ standing for "mutually"). We note that $X$ MRAII $Y$ is a strictly stronger condition than $X$ RALL $Y$, for example $u(x, y) = [x + 1/c(y)] \exp(-xc(y))$ satisfies $X$ RALL $Y$ but not $Y$ RALL $X$.

The use of $r(x | y)$ implicitly requires that $u(x, y)$ be differentiable at least twice. The proofs that accompany the following results require $u(x, y)$ to be differentiable at least three times. Since the proofs are neither elegant nor insightful they have been gathered in a final section of the paper.

**Lemma 1.** $X$ is utility independent of $Y$ and $Y$ RALL $X$ if and only if $u(x, y)$ has an additive or multiplicative decomposition.

Lemma 1 provides slightly weaker assumptions than those required by Keeney for the multiplicative form.

**Lemma 2.** If $X$ MRAII $Y$ then $X$ and $Y$ are ordinally additive.

Our main result will utilize a family of utility functions characterized by a property of the risk aversion function as described by the following lemma.

**Lemma 3.** The risk aversion function $r(x) = -u''(x)/u'(x)$ satisfies the condition $r'(x) = cr(x) + d$ for some constants $c \neq 0$ and $d$, if and only if

$$u(x) = \pm \int_{x_0}^{b} \exp cx \cdot k e^{\theta t} \, dt$$

where $b > 0$, $k$ is any constant, and $\theta$ is $\pm 1$.

The indirect representation of $u$ in this result may mask some of its more useful special cases. For example if $k$ is zero then $u(x) = \exp(b \exp(-cx))$ a function which for all $x$ is increasing, concave and exhibits decreasing risk aversion so long as $b$ and $c$ are each positive. We will refer to this form as the double exponential.

If $k$ is a positive integer and $z = be^{cx}$ then either

$$u(x) = (-1)^{k+1} e^{-z} \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots + (-1)^k \frac{z^k}{k!}\right)$$

or

$$u(x) = (-1)^{k+1} e^{-z} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!}\right)$$

a family that might be useful for small values of $k$.

The case $r'(x) = c^2$ leads to a $u(x)$ which is the cumulative of a normal distribution, the case $r'(x) = -c^2$ to an integral of the form $\int_{x_0}^{x} \exp(t^2) \, dt$. For expository ease it is useful to have a name for the family of utility functions characterized by the equation $r'(x) = cr(x) + d$.

**Definition.** A utility function $u(x)$ with risk aversion function $r(x)$ is one of the extended double exponentials if and only if $r'(x) = cr(x) + d$ for some constants $c$ and $d$ (that might be zero).

Note that another family of utility functions can be characterized by their relationship between $r'(x)$ and $r(x)$. The equation $r'(x) = -kr(x)^2$ characterizes the linear and exponential families ($k = 0$), the logarithmic utility function ($k = 1$), and the power family ($0 \neq k \neq 1$).

**Theorem.** The following conditions are equivalent:

(i) $X$ MRAII $Y$,

(ii) $X$ and $Y$ are ordinally additive and $X$ RALL $Y$.

(iii) $u(x, y) = \phi(v(x) + w(y))$ for some functions $\phi$, $v$ and $w$, where $\phi$ is a member of the extended double exponential family.
3. Discussion. The assessment of utility functions is a task requiring great care. Decision makers are often neither consistent with the axioms of von Neumann and Morgenstern, nor do they always satisfy certain established desiderata such as, in the case of the attribute money, being risk averse and decreasingly so. Although procedures have been devised to construct a single attribute utility function to be consistent with directly assessed points and with declared local properties [Meyer and Pratt 1968, Schlaifer 1971], a common route taken is to find a family of utility functions with the desired local properties and then to select the available parameters in a way that best approximates the direct assessments.

Multiattribute utility assessment is also best thought of as an exercise in the approximation of functions [Fishburn 1977]. The task is made more delicate by the desire to preserve local properties of the true function in the approximation. Even though the interpolation scheme (2) is quite sound as an approximation to the absolute value of the conditional utility function it does not preserve desirable qualitative properties of the generating functions.

The interpolation scheme

\[ r(x | y) = \alpha(y)r(x | y^0) + (1 - \alpha(y))r(x | y^*) \]

appears to permit a great deal of flexibility in approximating the utility function, yet, as Lemma 2 showed, it and the equivalent relation for \( r(y | x) \) require \( X \) and \( Y \) to be ordinally additive. The ordinally additive representation, \( u(x, y) = \phi(v(x) + w(y)) \) when associated, through \( \phi \), with families of single attribute utility functions, provides a range of multiattribute decompositions. If \( \phi \) is linear we have the additive form, if \( \phi \) is exponential we have the multiplicative. If \( \phi \) is the double exponential, \( u(x) = -\exp(b \exp(-cx)) \), then we have \( u(x, y) = -\exp(b \exp(-c(v + w))) \). This may be written equivalently as \( u(x, y) = \exp(bv_1(x)v_2(y)) \) a representation for which the term exponential product seems appropriate.

4. Proofs.

Proof of Lemma 1. If \( X \) UI \( Y \) then \( u(x, y) = f(y) + g(y)h(x) \) for some functions \( f \), \( g \), and \( h \), where \( h(x) = u(x | y^0) \). In this case \( r(y | x) = -(f''(y) + g''(y)h(x)) \)

\[ / (f'(y) + g'(y)h(x)) \]

so that \( Y \) RAII \( X \) only if

\[ \frac{f''(y) + g''(y)h(x)}{f'(y) + g'(y)h(x)} = \alpha(x) \frac{f''(y)}{f'(y)} + (1 - \alpha(x)) \frac{f''(y) + g''(y)}{f'(y) + g'(y)} \]

where I have used the scaling convention \( u(x^0 | y^0) = 0, u(x^* | y^0) = 1 \) to deduce \( h(x^0) = 0, h(x^*) = 1 \). Clearing the denominators in the equation we have

\[(f'' + g''h)f'(f' + g') = \alpha f''(f' + g'h)(f' + g') + (1 - \alpha)(f'' + g'')f'(f' + g'h) \]

or

\[ h\left[g''f'f' + f''g'f' - \alpha f''g'(f' + g') - (1 - \alpha)(f'' + g'')f'f'\right] = \alpha f''f'(f' + g') + (1 - \alpha)f''f'f' + (1 - \alpha)g''f'f' - f''f'(f' + g') \]

or

\[(g''f' - f''g')(\alpha g' + f')h - f'(1 - \alpha)\] = 0.

Now if \( f''g' - g''f' = 0 \) then \( Y \) UI \( X \) immediately since then \( r(y | x^0) = r(y | x^*) \). If \( (\alpha g' + f')h = f'(1 - \alpha) \) then \( f' = kg' \) for some constant \( k \) so that \( f(y) = kg(y) + c \). Hence in this case, \( Y \) UI \( X \) also. Now we use Keeney (1968) to show that \( u \) is multiplicative.
PROOF OF LEMMA 2. The proof is rather lengthy. The reader who wishes merely to be satisfied of the plausibility of the result may be able to do so by consideration of equation (7) in light of Lemma 4 (later in this section).

Integrating (4) we have

\[ -\log u'(x | y) = -\alpha(y)\log u'(x | y^0) - (1 - \alpha(y))\log u'(x | y^*) - s(y) \]

or

\[ u'(x | y) = e^{u(x | y^0)} \begin{bmatrix} u'(x | y)^0(y) \\ u'(x | y^*) \end{bmatrix}. \]

Since \( u(x, y) = f(y) + g(y)u(x | y) \) for some functions \( f, g \) we also have \( \partial u / \partial x = g(y) u'(x | y) \). Therefore we can say that, for arbitrary functions \( a_1, a_2, b_1 \) and \( b_2 \) and a constant \( k_1 \), we have

\[ \frac{\partial u}{\partial x} = \exp(a_1(x) + a_2(y) + k_1 b_1(x) b_2(y)) \tag{5} \]

where \( b_1(x^0) = b_2(y^0) = 0 \) and \( b_1(x^*) = b_2(y^*) = 1 \). By symmetry,

\[ \frac{\partial u}{\partial y} = \exp(c_1(x) + c_2(y) + k_2 d_1(x) d_2(y)) \tag{6} \]

where \( d_1(x^0) = d_2(y^0) = 0 \) and \( d_1(x^*) = d_2(y^*) = 1 \). (In future we will abbreviate \( d_1(x^0) \) as \( d_1^0 \) and \( d_1(x^*) \) as \( d_1^* \) and so on.) To show \( X \sqcap A \sqcap Y \) we must show that \( \log(\partial u / \partial x) - \log(\partial u / \partial y) = v(x) + w(y) \) for some functions \( v \) and \( w \) (Luce and Tukey 1964), that is, we must show \( a_1 + a_2 + k_1 b_1 b_2 - c_1 - c_2 - k_2 d_1 d_2 = v(x) + w(y) \) for some functions \( v \) and \( w \). Equivalently, we must establish that \( k_1 b_1 b_2 - k_2 d_1 d_2 = v_1(x) + w_2(y) \) for some functions \( v_1 \) and \( w_2 \). Since \( b_1^0 = d_2^0 = 0 \) we have \( w_2(y) = 0 \) and similarly \( v_1(x) = 0 \). Hence we must show that \( k_1 b_1 b_2 = k_2 d_1 d_2 \). We will do this by demonstrating, somewhat tediously, that this condition is implied by (5) and (6).

Differentiating (5) with respect to \( y \) we have \( \partial^2 u / \partial x \partial y = (a_1^* + k_1 b_1 b_2)(\partial u / \partial x) \) and, similarly, \( \partial^2 u / \partial y \partial x = (c_1^* + k_2 d_1 d_2)(\partial u / \partial y) \). If these second order derivatives are continuous they are identical so that

\[ (a_1^* + k_1 b_1 b_2)^0 \exp(a_1 + a_2 + k_1 b_1 b_2) = (c_1^* + k_2 d_1 d_2)^0 \exp(c_1 + c_2 + k_2 d_1 d_2) \tag{7} \]

is a necessary and sufficient condition for \( X \sqcap M \sqcap A \sqcap Y \).

Substituting \( y = y^0 \) in (7) we have \( (a_1^0 + k_1 b_1 b_2^0)^0 \exp(a_1 + a_2) = (c_1^0 + k_2 d_1 d_2^0)^0 \exp(c_1 + c_2) \). Similarly,

\[ a_1^0 \exp(a_1^0 + a_2) = (c_1^0 + k_2 d_1 d_2^0) \exp(c_1 + c_2). \]

Substituting \( y = y^* \) in (7) we have

\[ (a_1^* + k_1 b_1 b_2^*)^0 \exp(a_1 + a_2^* + k_1 b_1) = (c_1^* + k_2 d_1 d_2^*)^0 \exp(c_1 + c_2^* + k_2 d_1). \]

Similarly,

\[ (a_1^* + k_1 b_1^*) \exp(a_1^* + a_2 + k_1 b_2) = (c_1^* + k_2 d_1^* d_2^*) \exp(c_1^* + c_2 + k_2 d_2). \]

From these four equations we may identify \( a_1^*, c_1^*, d_1^* \) and \( b_2^* \):

\[ c_1^* = (a_1^0 + k_1 b_1 b_2^0)^0 \exp(a_1 - c_1 + a_2^0 - c_2^0). \tag{8} \]

\[ a_1^* = (c_1^0 + k_2 d_1 d_2^0)^0 \exp(c_2 - a_2 + c_1 - a_1^0). \tag{9} \]

\[ k_2 d_1^* = (a_1^* + k_1 b_1 b_2^*)^0 \exp(a_1 + a_2^* + k_1 b_1 - c_1 - c_2^* - k_2 d_1) - c_1 \quad \text{and} \quad (10) \]

\[ k_1 b_2^* = (c_1^* + k_2 d_1^* d_2^*)^0 \exp(c_1^* + c_2 + k_2 d_2 - a_2 - k_1 b_2) - a_2^*. \tag{11} \]
Also, 
\[ c_i^0 = a_i^0 \exp(a_1^0 - c_i^0 + a_2^0 - c_2^0), \]  
(12) 
\[ c_i^* = (a_1^0 + k_1 b_i^0) \exp(a_i^* - c_i^0 + a_2^0 - c_2^0), \]  
(13) 
\[ k_2 d_i^0 = a_i^* \exp(a_i^0 + a_2^0 - c_i^0 - c_2^0) - c_i^0, \]  
(14) 
\[ k_2 d_i^* = (a_i^* + k_1 b_i^*) \exp(a_i^0 + a_2^0 + k_1 - k_2 - c_i^0 - c_2^0) - c_i^*. \]  
(15) 
Substituting in (7) using (8), (9), (10), (11): 
\[ \exp(a_1 + a_2 + k_1 b_1 b_2) \left[ (1 - b_1)(c_i^0 + k_2 d_i^0 d_2) \exp(c_2 - a_2 + c_i^0 - a_i^0) \right. \] 
\[ + b_1(c_i^* + k_2 d_i^* d_2) \exp(c_i^0 + c_2 + k_2 d_2 - a_i^* - a_2 - k_1 b_2) \bigg] = \] 
\[ (1 - d_2)(a_1^0 + k_1 b_1 b_2^0) \exp(a_1 - c_i^0 + a_2^0 - c_2^0) \] 
\[ + d_2(a_i^* + k_1 b_1 b_2^*) \exp(a_1 + a_2^0 + k_1 - k_2 - c_i^0 - c_2^0) \] 
Now using (14) and (15): 
\[ \exp(k_1 b_1 b_2) \left[ (1 - b_1)(c_i^0 (1 - d_2) + d_2 a_i^0 \exp(a_1^0 + a_2^0 - c_i^0 - c_2^0)) \exp(c_1^0 - a_i^0) \right. \] 
\[ + b_1((1 - d_2)c_i^* + d_2(a_i^* + k_1 b_1 b_2^*) \exp(a_1^0 + a_2^0 + k_1 - k_2 - c_i^0 - c_2^0)) \] 
\[ \times \exp(c_i^0 + k_2 d_2 - a_i^* - k_1 b_2) \bigg] = \] 
\[ \exp(k_2 d_1 d_2) \left[ (1 - d_2)(a_1^0 + k_1 b_1 b_2^0) \exp(a_2^0 - c_i^0) \right. \] 
\[ + d_2(a_i^* + k_1 b_1 b_2^*) \exp(a_2^0 - c_i^0 + k_1 b_1 - k_2 d_1) \bigg]. \] 
Now use (12) and (13): 
\[ \exp(k_1 b_1 b_2) \left[ (1 - b_1)((1 - d_2)a_i^0 \exp(a_2^0 - c_2^0) + d_2 a_i^* \exp(a_2^0 - c_i^0)) \right. \] 
\[ + b_1((1 - d_2)(a_1^0 + k_1 b_2^0) \exp(a_2^0 - c_2^0 + k_2 d_2 - k_1 b_2) \] 
\[ + d_2(a_i^* + k_1 b_1 b_2^*) \exp(a_2^0 - c_i^0 + k_1 - k_2 + k_2 d_2 - k_1 b_2)) \bigg] = \] 
\[ \exp(k_2 d_1 d_2) \left[ (1 - d_2)(a_1^0 + k_1 b_1 b_2^0) \exp(a_2^0 - c_i^0) \right. \] 
\[ + d_2(a_i^* + k_1 b_1 b_2^*) \exp(a_2^0 - c_i^0 + k_1 b_1 - k_2 d_1) \bigg]. \]  
(16) 
Let 
\[ A^0 = a_1^0 \exp(a_2^0 - c_i^0), \quad B^0 = k_1 b_1^0 \exp(a_2^0 - c_i^0), \]  
\[ A^* = a_i^* \exp(a_2^0 - c_i^0) \quad \text{and} \quad B^* = k_1 b_1^* \exp(a_2^0 - c_i^0). \]  
We may assume that not all of these four constants are zero for otherwise we would have \( \frac{\partial^2 u}{\partial x \partial y} = 0 \) which would imply that \( u(x, y) \) is additive. Temporarily, let us use
the notation
\[ f(x, y) = (1 - b_1) \left[ A^0(1 - d_2) + A^*d_2 \right] + b_1 \left[ (A^0 + B^0)(1 - d_2) + (A^* + B^*)d_2 e^{k_1-k_2} \right] \exp(k_2d_2 - k_1b_2) \]
and
\[ g(x, y) = (1 - d_2)(A^0 + b_1B^0) + d_2(A^* + b_1B^*)\exp(k_1b_1 - k_2d_1) \]
so that (16) may be written as
\[ \exp(k_1b_1b_2)f(x, y) = \exp(k_2d_1d_2)g(x, y). \] (17)
Differentiate this with respect to \(x\):
\[ (k_1b_1b_2f + f')\exp(k_1b_1b_2) = (k_2d_1d_2g + g')\exp(k_2d_1d_2) \]
and use (17) to deduce
\[ k_1b_1b_2 - k_2d_1d_2 = g'/g - f'/f. \] (18)
We have
\[ f'(x, y) = b'_1(x)\left[ -\left( A^0(1 - d_2) + d_2A^* \right) + \left( (A^0 + B^0)(1 - d_2) + (A^* + B^*)d_2 e^{k_1-k_2} \right) \exp(k_2d_2 - k_1b_2) \right] \]
and
\[ g'(x, y) = b'_1(x)\left[ B^0(1 - d_2) + d_2\exp(k_1b_1 - k_2d_1) \times (B^* + (A^* + b_1B^*)(k_1 - k_2d_1(x)/b_1(x))) \right]. \]
Note that
\[ f(x^0, y) = g(x^0, y) = A^0(1 - d_2) + A^*d_2, \]
\[ f(x^*, y) = \left[ (A^0 + B^0)(1 - d_2) + (A^* + B^*)d_2 e^{k_1-k_2} \right] \exp(k_2d_2 - k_1b_2) \]
and
\[ g(x^*, y) = \left[ (A^0 + B^0)(1 - d_2) + (A^* + B^*)d_2 e^{k_1-k_2} \right]. \]
Note that \( f(x^*, y) = g(x^*, y)\exp(k_2d_2 - k_1b_2) \).
In what follows we wish to assume that \( b'_1(x) \neq 0 \neq d'_2(y) \). Note that \( b_1(x) = \log u'(x | y^0) - \log u'(x | y^*) \) so that \( b'_1(x) = r(x | y^*) - r(x | y_0). \) If \( b'_1(x) = 0 \) in a subinterval \( I \) of \( [x^0, x^*] \) then \( X \subset Y \) implies \( X \subset Y \) on this subinterval. By Lemma 1 we know that \( b_1(x) = d_1(x) \) and \( b_2(y) = d_2(y) \) on \( I \times [y^0, y^*]. \) Hence it remains to prove that \( b_1 = d_1 \) and \( b_2 = d_2 \) on intervals in which \( b'_1 \neq 0. \) The idea is to partition \([x^0, x^*]\) into intervals \( I_1, \ldots, I_n \) and \([y^0, y^*]\) into \( J_1, \ldots, J_k \) and prove the result on each \( I_i \times J_j \) separately, then rely on the continuity of \( b_1, d_1, b_2 \) and \( d_2 \) to prove the result for the entire space.
Substituting \( \delta = x^0 \) in (18), and letting \( s_0 = d'_1/b'_1, \) we obtain
\[ (k_1b_2 - k_2s_0d_2) \]
\[ = \frac{B^0(1 - d_2) + d_2(\mathcal{B}^* + A^*(k_1 - k_2s_0)) + g(x^0, y) - g(x^*, y)\exp(k_2d_2 - k_1b_2)}{g(x^0, y)}. \] (19)
Similarly, substitution of $x^*$ in (18), with $s^* = d_i^*/b_1$ gives

\[
(k_1b_2 - k_2s^*d_2)
\]

\[
= B^0(1 - d_2) + d_2e^{k_1 - k_2}(B^* + (A^* + B^*)(k_1 - k_2s^*)) + g(x^0, y)\exp(k_1b_2 - k_2d_2) - g(x^*, y)
\]

\[
\frac{g(x^0, y)}{g(x^*, y)}
\]

(20)

Let $z = d_2(y)$ and $w = k_1b_2 - k_2d_2$ so that $k_1b_2 = w + k_2z$. The function $b_2$ may be regarded as a function of $z$ because $d_2(y) \neq 0$. Also let $l_1(z) = g(x^0, y)$, $l_2(z) = g(x^*, y)$, then (19) becomes

\[
l_1(w + k_2z(1 - s^0)) = B^0(1 - z) + z(B^* + A^*(k_1 - k_2s^0)) + l_1 - l_2e^{-w}
\]

(21)

and (20) becomes

\[
l_2(w + k_2z(1 - s^*)) = B^0(1 - z) + ze^{k_1 - k_2}(B^* + (A^* + B^*)(k_1 - k_2s^*)) + l_1e^w - l_2.
\]

(22)

Finally, we may write (21) and (22) as

\[
l_1w + l_2e^{-w} = q_1(z) \quad \text{and}
\]

\[
l_2w + l_1e^w = q_2(z).
\]

(23)

(24)

It will be important later to remember that $l_1$ and $l_2$ are linear in $z$ and that $q_1$ and $q_2$ are quadratic in $z$. The proof will continue by consideration of two cases, the first is when differentiating (23) and (24) and solving for $w'$ produces new information about $w$, the second is when it doesn't.

So, differentiating both (23) and (24) with respect to $z$:

\[
l_1'w + l_1w' + l_2'(-w) - l_2e^{-w} = q_1',
\]

(25)

\[
l_2'w + l_2w' - l_1'e^w - l_1w' - w = q_2'.
\]

(26)

Eliminating $w'$ we have

\[
(l_2 - l_1'e^w)(q_1' - l_1w - l_2'e^{-w}) = (q_2' - l_2'w + l_1'e^w)(l_1 - l_2e^{-w}).
\]

Now use (23) and (24) to clear this equation of $e^w$ and $e^{-w}$:

\[
(l_2 - l_2w + q_2)(q_1' - l_1'w - \frac{l_2'}{l_2}(q_1 - l_1w))
\]

\[
= (l_1 + l_1w - q_1)(q_2' - l_2'w + \frac{l_1'}{l_1}(l_2w - q_2)).
\]

Collecting terms in $w$:

\[
w^2[l_1'l_2 - l_1l_2 + l_2l_1 - l_1'l_2] + w[l_2 + q_2 - l_1' - \frac{l_2'}{l_2}l_1]
\]

\[-l_2(q_1' - \frac{l_2'}{l_2}q_1) - \left(-l_2' + \frac{l_1'l_2}{l_1}\right)(l_1 - q_1) - \left(q_2 - \frac{l_1'q_2}{l_1}\right)(l_1 - q_1)
\]

\[+ \left(q_1' - \frac{l_2'q_1}{l_2}\right)(l_2 + q_2) - \left(q_2' - \frac{l_1'q_2}{l_1}\right)(l_1 - q_1) = 0.
\]
Since the term in \( w^2 \) cancels to zero we conclude that either this equation is an identity or \( w \) is representable as the ratio of two polynomials in \( z \). But now use (24) and Lemma 4 (see below) to see that in this latter case \( w \) must be constant.

If (23) and (24) produce equivalent differential equations in \( w' \) and since \( w(0) = 0 \) is known, we may conclude that (23) and (24) are equivalent. That is, any solution \( w \) to (23) is a solution to (24) and vice versa.

We can show that there cannot be two distinct solutions to (23) and (24) for suppose that \( w = a \) and \( w = b \) were both solutions for some particular choice of \( z \). We would have

\[
 l_1 a + l_2 e^{-a} = q_1, \quad l_2 a + l_1 e^a = q_2, \quad l_1 b + l_2 e^{-b} = q_1 \quad \text{and} \quad l_2 b + l_1 e^b = q_2
\]

from which we may deduce that \( l_2^2 (e^{-a} - e^{-b}) = l_1^2 (e^a - e^b) \), a condition only possible if \( a = b \) or if \( l_1 = l_2 = 0 \). This latter case is ruled out since \( d'_2(y) \neq 0 \).

Equation (23) has exactly one solution only if either \( l_1 l_2 < 0 \) for all \( z \) or if \( q_1 = t_2 l_1 \) for some constant \( t_2 \). For consider the equation \( ax + be^{-z} = c \) where \( a, b \) and \( c \) are constants and \( ab > 0 \). We may as well assume \( b > 0 \). This equation has two solutions whenever \( c > a(\log b - \log a + 1) \), one solution when \( c = a(\log b - \log a + 1) \) and no solution otherwise. By contrast, if \( ab < 0 \) this equation has at most one solution since \( ax + be^{-z} \) is a strictly monotone function of \( x \).

So the only way we could have exactly one solution to (23) and yet have \( l_1 l_2 > 0 \) is if

\[
 q_1 = l_1 (\log l_2 - \log l_1 + 1) \quad \text{or} \quad \frac{l_2}{l_1} = \exp \left( \frac{q_1 - l_1}{l_1} \right) \quad \text{(27)}
\]

If \( l_1 l_2 > 0 \) holds for some \( z \) it must hold for an open interval in which case (27) must hold for all \( z \) (by analytic continuation). By Lemma 4 \( l_2 = t_1 l_1 \) and \( q_1 = t_2 l_1 \) for some constants \( t_1 \) and \( t_2 \).

If \( l_1 l_2 < 0 \) for all \( z \) then \( l_2 = -t_1 l_1 \) for some constant \( t_3 > 0 \). Substitute for \( l_2 \) in (23) and (24) and note that a root of \( l_1(z) = 0 \) must also be a root of \( q_1 \) and \( q_2 \). Hence, for some linear functions \( l_3 \) and \( l_4 \) we may write (23) and (24) as

\[
 w - t_3 e^{-w} = l_3 \quad \text{and} \quad -t_3 w - e^w = l_4 \quad \text{(28)}
\]

Recall that (23) and (24) yield the same equation in \( w' \). From (28) \( w'[(1 + t_3 e^{-w})] = l_3' \) and from (29) \( w'[t_3 + e^{-w}] = -l_4' \) where \( l_3' \) and \( l_4' \) are constants. If \( l_3' = 0 \) then let \( S \) be the subset of the real line on which \( w' \neq 0 \). On \( S \) we must have \( 1 + t_3 e^{-w} = 0 \) so that \( w \) is constant on \( S \). Since \( w \) is continuous, it is constant everywhere. Assuming, therefore, that \( l_3' \neq 0 \) we have

\[
 l_3'(t_3 + e^w) = -l_4'(1 + t_3 e^{-w}) \quad \text{or} \quad l_3' e^w + l_4' t_3 e^{-w} = -l_4' - l_3' t_3 \quad \text{(29)}
\]

Differentiating this twice with respect to \( z \) shows that \( w \) must be constant. The case in which \( q_2 = t_2 l_1 \) is equivalent to that in which \( l_3' = 0 \).

We have shown that (5) and (6) require that \( k_1 b_1 - k_2 d_2 \) be constant, and therefore zero. Since \( b_2^* = d_2^* = 1 \) we have also shown that \( k_1 = k_2 \) so that \( b_2 = d_2 \). By symmetry, \( b_1 = d_1 \). Hence \( k_1 b_1 b_2 = k_2 d_1 d_2 \), hence \( u \) is additive. This ends the proof of Lemma 2.

PROOF OF LEMMA 3. Differentiating the integral gives \( u'(x) = \pm cz^{k+1} e^{\theta z} \) where \( z = be^{cx} \). Differentiating once more gives

\[
 u''(x) = \pm c((k+1)z^b e^{\theta z} + \theta z^{k+1} e^{\theta z})cz = \pm c^2 z^{k+1} e^{\theta z}(k+1 + \theta z).
\]

Hence \( r(x) = -c(k+1 + \theta z) \), so that \( r'(x) = -c^2 \theta z = c(r(x) + c(k+1)) \), from which we conclude that \( d \), the constant in the lemma statement, is equal to \( c^2(k+1) \).
If \( r'(x) = cr(x) + d \) \((c \neq 0)\), then \( \log(c + d)^2 = 2cx + \log a_0^2 \) where \( a_0 \) is a nonzero constant. Hence \( cr + d = a_0 \exp cx \). Integrating once more we have \( -c \log u'(x)^2 + 2dx + k_0 = 2(a_0/c)\exp cx \) which is equivalent to \( u'(x) = Ae^{ax}e^{b\exp cx} \) for constants \( a, b \) and \( A \) so that

\[
u(x) = \int_{x_0}^{x} Ae^{at}e^{b\exp cy} \, dy.
\]

Substitute \( t = \theta be^{cy} \) where \( \theta = \pm 1 \) is chosen so that \( \theta b > 0 \). We have \( dt = c\theta be^{cy} \, dy = ct \, dy \), and \( e^{at} = (t/\theta b)^{a/c} \) so that

\[
u(x) = \int_{t_0}^{t} 1 \, \frac{A}{ct} \left( \frac{t}{\theta b} \right)^{a/c} e^{\theta y} \, dt.
\]

Let \( k = a/c - 1 \) and recognize that \( A/c \) can be any sign and we have

\[
u(x) = \pm \int_{x_0}^{x} b \exp cx \, t^k e^{\theta y} \, dt \quad \text{where} \quad b > 0.
\]

The following result, used repeatedly in the proof of Lemma 2, is undoubtedly proved in some textbook but I have been unable to locate a reference.

**Lemma 4.** If \( p_1, p_2, p_3 \) and \( p_4 \) are polynomials such that \( p_1/p_2 = \exp(p_3/p_4) \) then \( p_1/p_2 \) and \( p_3/p_4 \) are constants.

**Proof.** We may assume that \( p_1 \) and \( p_2 \) have no factors in common otherwise these could be cancelled. Similarly with \( p_3 \) and \( p_4 \). If \( p_2 \) has a root, say \( a \) where \( p_2(a) = 0 \) then \( p_4(a) \) must be zero also. Let \( p_2(x) = (x - a)^kp_5(x) \) for a polynomial \( p_5 \), where \( p_5(a) \neq 0 \) and \( k \) is an integer. We know that

\[
(x - a)^k \frac{p_1}{p_5} = \frac{p_1}{p_5} = (x - a)^k \exp \left( \frac{p_3}{p_4} \right).
\]

Now \( p_1/p_5 \) behaves "sensibly" at \( x = a \) but \( (x - a)^k \exp(p_3/p_4) \) does not. More precisely, \( p_1/p_5 \) is analytic at \( x = a \) but \( (x - a)^k \exp(p_3/p_4) \) has an essential singularity at \( x = a \). (See Apostol 1957.) Hence \( p_2 \) can have no real roots. If \( p_4 \) has a root, say \( p_4(b) = 0 \), then \( p_4(b) < 0 \) is implied since \( p_4 \) is not zero and, therefore, we must have \( p_1(b) = 0 \). Let \( p_1(x) = (x - b)^kp_6(x) \) where \( p_6(b) \neq 0 \). Then

\[
(x - b)^c \frac{p_2}{p_6} = \frac{p_2}{p_6} = (x - b)^c \exp \left( \frac{-p_3}{p_4} \right).
\]

Now \( p_2/p_6 \) is nicely behaved at \( x = b \) but \( (x - b)^c \exp(-p_3/p_4) \) goes to infinity as \( x \to b \). Hence \( p_4 \) has no real roots. Now consider the equation \( p_2/p_1 = \exp(-p_3/p_4) \). We already know that \( p_2 \) and \( p_4 \) have no real roots and by a similar sequence of arguments neither does \( p_1 \).

But, in fact, there is no need to restrict our consideration to real roots. By analytic continuation, if \( p_1/p_2 = \exp(p_3/p_4) \) on the real line it must hold in the complex plane also (Apostol, p. 519). The above arguments now apply for complex roots. (Think of complex roots as factors of the form \( x^2 + k \) where \( k \to 0 \). Now consider behavior as \( x^2 \to -k \).) Hence \( p_1 \) and \( p_2 \) have no real or complex roots and so must be constants.

**Proof of Theorem:** (i) \( \to \) (ii) by Lemma 2. To show (ii) \( \to \) (iii) note that \( X \RAII Y \) is equivalent to \( r(x \mid y) = a(x) + b(x)a(y) \) so that \( r'(x \mid y) = c(x) + d(x)a(y) \) for some functions \( a, b, c \) and \( d \).

\[
r'(x \mid y) = \frac{d}{b} r(x \mid y) - \frac{da}{b} + c = f(x) + g(x)r(x \mid y),
\]

(30)
say. Now \( \partial u / \partial x = v' \phi'(v + w) \) and \( \partial^2 u / \partial x^2 = \phi' + v \phi'' \) so that \( r(x \mid y) = -v''/v' - v' \phi''/\phi' \). Denoting \(-\phi''/\phi' \) by \( r(v + w) \) we have

\[
r(x \mid y) = \frac{-v''}{v'} + v'r(v + w)
\]

so that

\[
r'(x \mid y) = -\frac{v'v''' - v''v'''}{v'v'} + v''r(v + w) + v'v'r'(v + w).
\]

Substituting for \( r(x \mid y) \) and \( r'(x \mid y) \) in (30) and rearranging we get an expression of the form

\[
r'(v + w) = h(x) + j(x)r(v + w)
\]

for some functions \( h \) and \( j \). If \( w(y) \) is not constant, so \( w' \neq 0 \) then differentiating (32) with respect to \( y \) we have \( r''(v + w) = j(x)r'(v + w) \). Either \( r'(v + w) = 0 \) in which case \( r'(v + w) \) is trivially linear in \( r(v + w) \) or \( j(x) \) is constant. If \( j(x) \) is constant then so too is \( h(x) \). Hence \( r'(v + w) \) is linear in \( r(v + w) \) which shows that \( \phi \) is one of the extended double-exponential family.

Now to show (iii) \( \rightarrow \) (i). In this case we know \( r'(v + w) = cr(v + w) + d \). If \( c = 0 \) then \( r(v + w) = d(v + w) \). Substituting in (31) we see that \( r(x \mid y) = -v''/v' + d \phi'(v + w) \) which is of the form \( a(x) + b(x)\alpha(y) \) as required. Hence \( X \RA I I Y \) and, by symmetry \( Y \RA I I X \). If \( c \neq 0 \) then \( \log(cr + d)^2 = 2(c(v + w) + \log a_0^2 \) or \( cr + d = a_0 e^{c(v+w)} \). Substituting once more in (31) gives

\[
r(x \mid y) = \frac{-v''}{v'} + v'\left(a_0 e^{cv + cw} - \frac{d}{c}\right)
\]

which is also of the form \( a(x) + b(x)\alpha(y) \).

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**References**


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