Incorporating the Customer’s Perspective into the Newsvendor Problem

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Abstract

The newsvendor problem is a classic in management science partly because selecting an optimal inventory level in the face of uncertain demand is an important problem but also because the solution is so elegant and intuitive: the inventory should be selected so that the probability that the vendor stocks out should be set equal to the ratio of the item’s unit cost to its unit price.

A number of attempts have been made to enrich the factors treated in the analysis, but these usually destroy the elegance of the traditional solution. In this paper we examine the case in which expected demand is related to the expected value anticipated by the customer. We define this value as the consumer surplus (willingness to pay less price charged) times the likelihood that the customer finds the item in stock. With this refinement we show that the optimal inventory is such that the probability of stocking out is equal to the ratio of the item’s unit cost to the customer’s willingness to pay.

We apply the method to cases in which there are multiple products and find that the simple solution is preserved.
1. Introduction

A newsvendor sells one particular newspaper. Each day he can order as many copies of the paper as he likes at 20¢ per copy. He sells them at the suggested price of 50¢ but demand is highly variable, and he frequently has copies left over which he can only throw away. Assuming he wishes to maximize his expected profit, how many newspapers should he order?

If he orders too many he is likely to waste money, if he orders too few he is likely to miss sales. This is the classic newsvendor problem. The solution, as most students learn early in their careers, is that the optimal choice of inventory is such that the chance of at least one lost sale equals 2/5 (20 cents divided by 50 cents), or in general, unit cost divided by unit price. The solution makes sense: if the cost is low, but the price is high, you want a lot of inventory on hand. If the cost is high but the profit margin is low, you’d like to be sure you can sell what you buy.

Let $c$ be the cost per unit of the newspapers, $p$ be the price charged and denote the uncertain demand by $D\xi$ where $D$, a constant, represents the expected (average) demand, and $\xi$ is a non-negative random variable with mean 1, cumulative distribution $F$ and density function $f$. The optimal inventory $I^*$ is defined, implicitly, by

$$1 - F(I^*/D) = c/p . \tag{1}$$

The problem dates from Whitin (1955). An extensive analysis of variations and extensions may be found in Porteus (1990) or Federgruen and Heching (1999). A recent review is given by Petruzzi and Dada (1999).

It is easy to modify the story slightly while still preserving the style of the solution. For example, if we are selling umbrellas not newspapers then any unsold inventory may be carried forward to the next day (period). If $h$ is the holding cost of the inventory for one period (expressed as a fraction of the unit cost) then the new inventory solution is

$$1 - F(I^*/D) = hc/(p - (1-h)c) . \tag{2}$$
The “critical fractile” on the right hand side of (2) is the ratio of the marginal cost of having one unit too many (“overage”) to the sum of that number and the lost marginal revenue of having one too few (“underage”). (See Porteus (1990) or Bell and Schleifer (1995).) The same logic is true in (1).

Another easy refinement is to consider that customer goodwill is lost if a customer finds you out of stock. If the customer has traveled many miles to come to a store for an item and it is out of stock, one can suppose that customer might be reluctant to make the trip again. If $g$ is a dollar amount reflecting the lost goodwill that can be attributed to a customer finding you out of stock then analysis reveals that the critical fractile becomes $c/(p+g)$. But there has been little success in extending the basic model—while preserving its simple solution—much beyond this.

A desirable extension would be to make demand dependent on price, say $D(p)\bar{z}$. As Petruzzi and Dada (1999) demonstrate, while such a formulation can be solved and some general properties of the solution described, the solution is not of a closed form, even implicitly.

In the next section we propose a refinement of the newsvendor problem which does include price as a decision variable that affects both profit and demand, and which includes endogenous feedback on demand of the deleterious effects on the customer of being out of stock. The idea of including stock-out probabilities in the newsvendor seems to be new: the only examples I am aware of are my own paper Bell (1997) and one by Dana and Petruzzi (2001). The latter paper gives some good general results, but does not derive the simple closed form solution that we present here.

2. Customer Added Value

The analysis we have reviewed treats the newsvendor as a rational economic agent carefully weighing costs of underage and overage in selecting inventory levels. But let us also consider the customer as an economic actor. Why is she buying this product? Presumably it is because she feels better off for doing so: the price charged is less than her willingness-to-pay. Her “consumer surplus” can be calculated as $a - p$ where $a$
represents the maximum amount she is willing to pay for an item. (This amount may be thought of as a function, in part, of where else, and at what price, the item is available, or of the cost and value of substitutes.)

Once at the store, the customer’s optimal strategy is, rather simply, to buy the item if \( a > p \) and not to if \( a < p \). In what follows we will assume that the customer also buys if \( a = p \). In light of this we might suppose that the store should set the price at \( p = a \). Indeed many retailers do try to extract the customer’s willingness to pay especially when the customer has no alternatives (beer at the ballpark) or is otherwise committed (two hours into haggling over a new car). But most retailers set prices in advance and clearly label them, they must allow for different levels of willingness to pay among potential customers, and perhaps most importantly, most retailers rely on repeat business and so endeavor to give customers good value.

For the moment we will continue to focus on one store, selling one product, at a price \( p \) and cost \( c \), to customers all of whom have a maximum willingness to pay \( a \). The retailer must select an inventory level \( I \) in advance of observing the demand level. Excess inventory is worthless (though recall that if excess may be carried forward to the next period the algebraic modifications needed are routine).

Let us consider the decision problem faced by the customer. Should she travel to the store to buy the item or not? If she does, and the item is in stock, she benefits by \( a - p \) less her cost of travel. If the store is out of stock, we assume her trip is worthless. Let us denote the probability that the store has the item she seeks as \( q \). Thus her expected value from a store visit may be written

\[
\text{Customer's Expected Value} = (a - p)q .
\]

She should make the trip if this expected value exceeds her travel cost, but not otherwise. With this argument in mind we will assume in our refinement of the newsvendor problem that expected demand is a function of this expected value, namely \( D((a - p)q) \): the higher the expected value, the more people will be attracted to the store. We will assume that \( D \) is an increasing, twice differentiable function; for non-extreme solutions we also require that at the optimal solution, \( DD^* < 2D'D' \). This is a very mild restriction; all
concave functions satisfy it, as do all power functions. In Bell (1997) I suggest that
\[ D((a - p)q) = k(a - p)^2 q^2 \] is a plausible function; if travel costs are proportional to
distance from the store, and if potential customers are distributed evenly around the store,
then the number of potential customers whose travel cost is less than the expected value
added will grow as the square of that value added.

Our formulation assumes an equilibrium between the actions of the store and its
customers. Customers know what price the store charges, and how often it is in stock
(but not, of course, whether it is in stock on a particular occasion until arriving at the
store). The store, based on experience, can estimate a demand function \( D \bar{z} \) where \( \bar{z} \) is
an error distribution, with mean 1, independent of \( D \).

In what follows it is important to distinguish between the probability that the store
stocks out during an inventory cycle, and the probability that a customer finds the store to
be out of stock. If the newsvendor selects an inventory \( I \) he will stock out (lose at least
one sale) if \( D \bar{z} > I \). This is a different probability from the quantity \( 1 - q \) which
represents the fraction of customers who are disappointed, or, equivalently, the
probability that any particular customer will be disappointed. For example, if demand is
equally likely to be five or six and the vendor stocks five items, then the probability of a
stockout is one half. However, the probability that any given customer is disappointed is
only 1/11. More generally \( q = \text{Expected Sales} / \text{Expected Demand} \).

To see this relationship another way, note that the total consumer surplus that the
store provides is equal to \( (a - p)\text{Expected Sales} \). Thus the average consumer surplus
among those who wish to be served is \( (a - p)\text{Expected Sales}/\text{Expected Demand} \) or
\( (a - p)q \). This definition supposes that customers arrive at the store independently of the
stocking cycle, and of fluctuations in demand. If a store always restocks over the
weekend, and certain shoppers go to the store only on Mondays, or only on Saturdays,
their likelihood of finding the store in stock will differ. Or if the variation \( \bar{z} \) is due to the
weather, \( \bar{z} \) being high when the weather is fine, say, then shoppers who only shop when
the weather is fine will experience a different level of stockouts. In this paper we confine
ourselves to the average scenario.
Assumption Each prospective customer anticipates the same expected surplus, equal to 
\[(a - p)\text{Expected Sales} / \text{Expected Demand} = (a - p)q.\]

To underline the complexity of our formulation, note that the average demand \(D\) depends on \(q, D((a – p)q)\), and \(q\), in turn, depends on average demand. Instead of dealing with the quantity \(I\) it will be more convenient to express our calculations in terms of the proportional inventory \(i\) defined by \(I = Di\).

The vendor’s expected profit 
\[= p \text{ Expected Sales} – cI\]
\[= pq \text{ Expected Demand} – c i \text{ Expected Demand}\]
\[= (pq – ci) \text{ Expected Demand}\]
\[= (pq – ci) D((a – p)q) .\]

It is this quantity that is to be maximized by the newsvendor, by appropriate selection of \(p\) and \(i\). The quantity \(q\) is, of course, itself a function of \(i\). We have

\[qD = \text{ Expected Sales} \]
\[= \int_{0}^{\infty} Dz f(z) dz + \int_{0}^{\infty} I f(z) dz \]
\[= D \int_{0}^{j} zf(z) dz + \int_{i}^{\infty} f(z) dz \]

or
\[q = \int_{0}^{j} zf(z) dz + \int_{i}^{\infty} f(z) dz . \] (3)

Note that \(dq/di = \int_{0}^{\infty} f(z) dz = 1 - F(i)\), so \(q\) is the probability a customer is able to obtain the product and \(dq/di\) is the probability that the store stocks out.

**Theorem 1** A newsvendor, facing an uncertain demand \(D((a – p)q)\), where \(p\) is the sale price of an item costing \(c\), \(q\) is the probability a customer finds the item in stock, \(a\) is a constant and \(\bar{z}\) is an error distribution that is independent of \(p\) and \(q\), should select an inventory level so that the probability that at least one customer finds the item to be out of stock equals \(c/a\).
**Proof**  The classic newsvendor problem is a special case in which \( D \) is constant and \( p \) is fixed. Though it is easy to prove Theorem 1 directly, it is insightful to do it via the following lemma.

**Lemma 1** Suppose the newsvendor desires to maximize his profits subject to customers having a *fixed* expected consumer surplus \( v = (a - p)q \). Then, for any \( v \), the optimal inventory level is such that the probability of a stockout is \( c/a \).

**Proof of Lemma 1** Now we are to maximize \( (pq - ci)D((a - p)q) \) subject to the constraint \( (a - p)q = v \) where \( v \) is a chosen constant. Since \( pq = aq - v \) we may regard the problem as maximizing

\[
(aq - ci - v)D(v) \tag{4}
\]

subject to (3). But (4) may be decomposed as \((aq - ci)D(v) - v D(v)\). The second term is constant, so our problem is to maximize \((aq - ci)D(v)\). But this is *exactly the classic problem*. The expected demand \( D(v) \) is constant, and \( p \) (which is constant in the classic problem) is replaced by \( a \). Hence the inventory solution is to stock out with probability \( c/a \). This proves Lemma 1.

**Theorem 1** is a corollary to Lemma 1: since the inventory result is true for all choices of \( v \), it is also true for the particular choice of \( v \) that maximizes the newsvendor’s total profit. This ends the proof of Theorem 1.

In the next section we consider extensions of the newsvendor problem to cases where the vendor has more than one product to offer.

It will help later if we note that the inventory solution \( dq/\text{di} = c/a \) is unchanged if we generalize the objective function from \((pq - ci)D((a - p)q)\) to \((pq - ci + k)D((a - p)q)\) where \( k \) is any constant. For the purposes of the current formulation \( k \) (if negative) could represent an expense for each customer that visits the store, whether they buy or not; for example it could represent the cost of sales assistance or wear and tear on the store. The constant \( k \) could also be positive, for example if visiting customers could be expected to spend money on other (unspecified) impulse items. It will also be useful to note the following result.
Lemma 2  If the vendor is to select, from a set of possible products with varying values of $c$ and $a$, the most profitable product to sell then

(i) Among those with the same ratio $c/a$ he will prefer the product with the highest value of $a - c$.

(ii) Among those with the same difference $a - c$ he will prefer the product with the lowest value of $c/a$.

In particular, one product is more profitable than another if it has both a higher $a - c$ and a lower $c/a$.

Proof  It is easy to see that profits increase with $a-c$. If $c/a$ is constant then the optimal values of $i$ and $q$ are unchanged (the optimal $i$ depends only on $c/a$ and the optimal $q$ depends only on $i$). If $a$ and $c$ each increase by a multiple $k$ then suppose we increase $p$ by a factor $k$ also. This may not be the optimal $p$ for the new values of $a$ and $c$, but it certainly gives higher profits than before: the profits were $(pq-ci)D((a-p)q)$ and are now at least $(kpq-kci)D((ka-kp)q)$, clearly higher. Similarly if $c/a$ gets smaller while $a-c$ is constant then we may assume $a$ and $c$ have each been lowered by some amount $k$.

Suppose we lower $p$ by $k$ also. Profits used to be $(pq-ci)D((a-p)q)$ but are now $((p-k)q-(c-k)i)D((a-p)q)$ or $(pq-ci)D((a-p)q)+k(i-q)D((a-p)q)$. Since $i > q$ (see (3)) profits have increased even before we re-optimize for $p$ and $q$.

In the remainder of the paper we extend the model to the case of multiple products. The inventory solution still has the simple style of solution of the classic newsvendor problem.

3. Multiple Products

Suppose the vendor (store) offers two products. Each customer desires either or both of the products so long as the price is below respective reservation values of $a_1$ and $a_2$. If the vendor sets prices of $p_1$ and $p_2$ and an inventory policy such that the customer finds the products in stock a fraction $q_1$ and $q_2$ of the time then the expected added value to the customer of a trip to the store is
Customer's Expected Value = \((a_1 - p_1)q_1 + (a_2 - p_2)q_2\).

We assume therefore that the uncertain demand is \(D((a_1 - p_1)q_1 + (a_2 - p_2)q_2)\). The unit costs to the vendor are \(c_1\) and \(c_2\) respectively. If \(I_1\) and \(I_2\) are the vendor’s chosen inventory levels then, as before, we define the “inventory policy” variables \(i_1\) and \(i_2\) by \(I_1 = D_i_1, I_2 = D_i_2\). The vendor’s goal is to maximize

\[
(p_i q_i - c_i i_i + p_2 q_2 - c_2 i_2)D((a_1 - p_1)q_1 + (a_2 - p_2)q_2)
\]

and the optimal inventory occurs when

\[
\frac{dq_1}{di_1} = \frac{c_1}{a_1} \quad \text{and} \quad \frac{dq_2}{di_2} = \frac{c_2}{a_2}
\]

that is, the single product solution carries over to each product independently.

To see this, consider the vendor’s problem in stocking product 1, with product 2’s policy assumed fixed (perhaps optimally, perhaps not). The problem is to maximize \((p_i q_i - c_i i_i + k)D((a_1 - p_1)q_1 + b_2)\) where \(k\) is a constant (equal to \(p_2 q_2 - c_2 i_2\)) and \(b_2\) is a constant (equal to \((a_2 - p_2)q_2\)). We know the solution to this is \(\frac{dq_1}{di_1} = \frac{c_1}{a_1}\). That is, no matter how we go about pricing, and stocking, product 2, the inventory policy for product 1 is the same. The presence of product 2 increases overall demand for product 1 but does not change the inventory policy of product 1, though it might well influence the optimal price.

Our conclusion obviously generalizes to any number of products. It also may be adapted to reflect the real world concern that total inventory must be less than some budget constraint. In this case inventory policies are related to the ratios \(\frac{\lambda c_i}{a_i}\) where \(\lambda\) is selected so that the budget is met. As the budget gets tighter, eventually the product with the highest ratio \(\frac{c_i}{a_i}\) will no longer be carried.

A more complicated multiple product case occurs when the customer wants at most one of the products (that is they are substitutes). This might well be the case if the vendor is selling multiple kinds of newspapers.
**Theorem 2** Suppose the vendor may carry either or both of two products for which \( a_1 - c_1 > a_2 - c_2 > 0 \) and \( c_1 / a_1 > c_2 / a_2 > 0 \). If customers will buy one, but not both of the products, then the vendor should stock product 1 so that there is only a \((c_1 - c_2)/(a_1 - a_2)\) probability that at least one customer will find it stocked out, and should stock product 2 so that the probability that at least one customer will find both products stocked out is \( c_2 / a_2 \).

Note that the restrictions on \( c \) and \( a \) in the statement of the theorem merely rule out cases in which one item is not carried at all in the optimal solution. The proof of the theorem is in Bell (1997).

4. **Discussion**

The paper has concerned the methodological implications of endogenizing the customer as a rational actor into the newsvendor problem. Perhaps more important are the intuitive implications. Traditionally, overage (the cost of having stocked one too many) and underage (the opportunity cost of stocking one too few) have been measured from the vendor’s (short term) perspective. In our formulation however, we see that these should more properly be considered relative to the combined vendor-customer value chain: overage remains \( c \) but underage is \( a - c \). It may be that such a perspective will prove beneficial in more complex problems where the system solution is less obvious.
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