

Monopolistic Competition Between Differentiated Products

With Demand For More Than One Variety

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Abstract

We analyze the existence of pure strategy symmetric price equilibria in a generalized version of Salop (1979)'s circular model of competition between differentiated products - namely, we allow consumers to purchase more than one brand. When consumers purchase all varieties from which they derive non-negative net utility, there is no competition, so that each firm behaves like an unconstrained monopolist. When each consumer is interested in purchasing an exogenously given number (n) of varieties, we show that there is no pure strategy symmetric price equilibrium in general (for $n > 2$ with linear transportation costs). In turn, if the limitation on the number of varieties consumers purchase comes from a budget constraint then we obtain a multiplicity of symmetric price equilibria, which can be indexed by the number of varieties consumers purchase in equilibrium.

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JEL Classifications: L1, L2, L8

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1. Introduction

We are interested in developing a theoretical framework for studying price competition between multiple firms, with the following characteristics: (i) firms offer differentiated products; (ii) consumers have unitary demand for any given product, i.e. every consumer will buy either 0 or 1 units of any brand; (iii) consumer preferences are heterogeneous and "localized" in the sense that they differ in the identity of their ideal brand and each consumer values brands less the further they are in preference space from his ideal brand; (iv) each consumer is potentially interested in buying more than one product. Two examples of markets which fit the description outlined by the four points above are software applications and videogames. There are thousands of software applications as well as games, and different users are interested in different applications/games. A given software or game user's tastes may overlap with another's, yet they may have nothing in common with a third's. Thus, although there is a sense in which competition is localized (any given firm competes only with firms, whose brands are similar to its own), it is not clear how the fact that consumers are generally interested in purchasing multiple products affects the type of competition waged among firms.

We are motivated by the observation that the two main standard models of monopolistic competition do not seem to be well-suited for illustrating this type of competition. Indeed, the spatial framework proposed by Salop (1979) treats competition as a localized phenomenon and satisfies conditions (i) through (iii) above, but not (iv): each consumer is interested in purchasing a single brand. In turn, the representative consumer framework of Spence (1976) and Dixit and Stiglitz (1977) satisfies (iv) (consumers purchase positive quantities of all brands), but not (i), (ii) and (iii) (all products are interchangeable for all consumers). Subsequent papers have generalized each of these models and derived more

solid foundations for their basic assumptions (Sattinger (1984), Perloff and Salop (1985)). There have also been some theoretical attempts to reconcile the two frameworks, the most significant of which is Deneckere and Rotschild (1992). However, none of them has specifically studied price competition in a setting combining the consumer heterogeneity, localized competition and unitary demand per brand features of spatial models with the preference for diversity (demand for multiple products) property of the symmetric aggregate demand formulation.

The framework we propose combines features from both strands of the literature in the following way. We take Salop's circle representation of consumer preferences and firms' locations as the starting point for our model and make the important modifying assumption that each consumer is interested in purchasing several brands, rather than just one.

Therefore all the action in our model depends on what we assume it is that determines how many varieties are purchased. We investigate three possibilities. The first variant of our model postulates that each consumer purchases all brands from which he derives positive net utility: in this case, there is no competition whatsoever among the firms, so that each of them sets its price as an unconstrained monopolist. Second, we assume each consumer is only interested in purchasing an exogenously given number (n) of brands. The striking result in this case is that a symmetric price equilibrium does not exist in general (when transportation costs are linear this is true for all n greater than 2). Third, we assume consumers face a budget constraint. Here, by contrast with the previous case, it turns out that there is a multiplicity of symmetric price equilibria.

The paper is structured as follows: the next section spells out the backbone of our modelling framework. Section 3, 4 and 5 analyze respectively the cases when consumers purchase all varieties offering positive net utility; when they are interested in purchasing an exogenously given number of products and when they face a budget constraint. Section

6 concludes.

2. General framework

In this section, we lay out the common modelling features to the three alternative formulations we explore. There is a continuum of consumers uniformly distributed on the unit circle, with density normalized at 1 and N sellers of differentiated products, each with constant marginal cost equal to c . We denote the N sellers by A_0, A_1, \dots, A_{N-1} and we make the standard assumption that they are equidistantly distributed on the circle, i.e. there is a distance $d \equiv 1/N$ between any two consecutive firms. Let a_0, a_1, \dots, a_{N-1} be their locations, with $a_0 = 0$ by definition and $a_k = kd$.

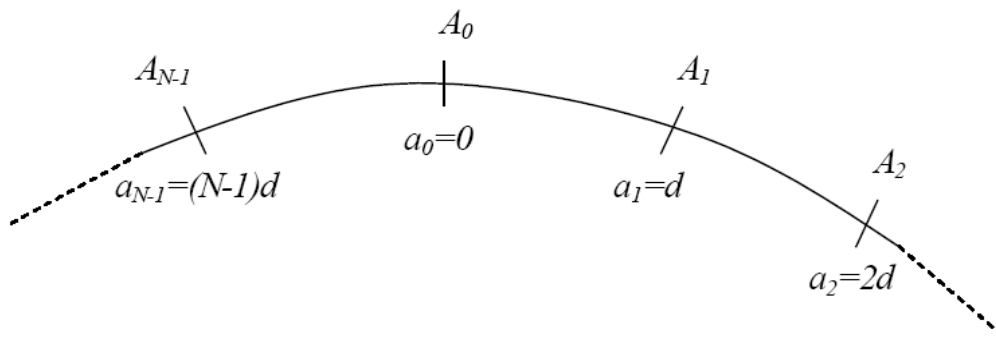


Figure 1

Just like in the Salop-Lerner model, the interpretation of this structure is that a consumer's position on the circle represents his ideal variety and his valuation for any product decreases with its distance to his ideal variety. However, we depart from the standard assumption that consumers are only interested in purchasing one product (variety) and assume that each consumer may purchase more than one variety.

We take the equidistant location of firms as given (we are not interested in studying differentiation incentives here). A simple way of justifying the assumption of equidistant locations is to say that from the point of view of producers the location of any consumer's ideal product has equal chances of being anywhere on the circle.

The gross utility that any consumer derives from her ideal variety is v . In most of the paper we deal with linear transportation costs. Thus, the *net* utility that a consumer located at x on the circle derives from buying product A_i is given by:

$$u(x, A_i) = v - t|x - a_i| - p_{a_i}$$

where $|a - b|$ stand for the distance between points a and b measured on the contour of the circle.

This implies that, in ranking two products A_i and A_j , a consumer compares $t|x - a_i| + p_{a_i}$ with $t|x - a_j| + p_{a_j}$.

Given that we think of "distances" in our model as distances in preference space rather than in physical space (the latter being the common interpretation in spatial differentiation models), it makes sense not to think of "transportation costs" literally. Consequently, t is best interpreted as a taste parameter measuring how quickly the valuation of a consumer for a variety falls with the "distance" to her ideal variety. In other words, t determines how willing each consumer is to "explore" other varieties and thereby how many differentiated products a consumer will end up purchasing.

In each of the three alternative formulations below, our goal will be to find a symmetric price equilibrium, taking as given all the other parameters of the problem.

3. The case when demand is constrained by transportation costs

The first modelling option we explore is to let demand for differentiated products be constrained solely by the positive utility requirement. In other words, we allow each consumer to buy all varieties which offer him non-negative utility, net of price and transportation costs. Thus, the net utility of a consumer located at x on the circle is¹:

$$U(x) = \sum_{\substack{i \in \{1, \dots, N\} \\ v - t|x - a_i| - p_{A_i} \geq 0}} (v - t|x - a_i| - p_{A_i})$$

It is then clear that each firm has market power and since its demand and profits are independent of the prices charged by and the number of other firms present in the market:

$$\begin{aligned} D_{A_i}(p_{A_i}) &= 2 \frac{(v - p_{A_i})}{t} \\ \Pi_{A_i}(p_{A_i}) &= (p_{A_i} - c) D_{A_i}(p_{A_i}) = \frac{2(p_{A_i} - c)(v - p_{A_i})}{t} \end{aligned}$$

Therefore, each firm's profit-maximizing price and resulting profits are:

$$\begin{aligned} p(c) &= \frac{v + c}{2} \\ \Pi(c) &= \frac{(v - c)^2}{2t} \end{aligned}$$

With a more general expression of transportation costs, $T(|x - a_i|)$, $T' > 0$, we would

¹Throughout the paper we assume that when a consumer is indifferent between buying and not buying a variety, he will choose to buy.

obtain:

$$\begin{aligned} p(c) &= \arg \max_p 2(p-c) D(p) \\ \Pi(c) &= 2(p(c)-c) D(p(c)) \end{aligned}$$

where $D(p) \equiv T^{-1}(v-p)$

The two expressions make clear that each firm acts like a monopoly. Moreover, note that we would have obtained the same expressions (modulo a factor 2) if, rather than starting with a circular distribution, we had assumed instead that consumers' valuations for each product were i.i.d. draws from the interval $[0, v]$ according to a distribution $F(x) = 1 - T^{-1}(v-x) = 1 - D(x)$.

Lastly, we can determine consumer total and expected surplus². They are given by:

$$\begin{aligned} Eu(N, c) &= 2N \int_0^{\frac{v-p(c)}{t}} (v-p(c)-tx) dx = \frac{(v-p(c))^2 N}{t} \\ &= \frac{(v-c)^2 N}{4t} \end{aligned}$$

Using the generalized notation above, this expression becomes:

$$Eu(N, c) = N \int_{p(c)}^v D(x) dx$$

Overall, this case is not particularly interesting as all competition is removed when consumers buy all products from which they derive non-negative utility. Thus, in order to make our discussion interesting, we need to impose some kind of constraint on the number of brands consumers can buy.

²They are equal because we normalized consumer mass to 1.

4. The number of varieties demanded is exogenously given

In the second formulation we introduce such a constraint by assuming that there is an exogenously given number of varieties that consumers are interested in buying. Denote this number by n , $n < N$. Then, a consumer located at x solves the following problem:

$$\max_{\{i_1, i_2, \dots, i_n\} \subset \{1, \dots, N\}} \left\{ (v - t|x - a_{i_1}| - p_{A_{i_1}}) + \dots + (v - t|x - a_{i_n}| - p_{A_{i_n}}) \right\}$$

We also assume that v is large enough so that each consumer will in effect buy the n varieties he prefers the most.

4.1. The case $n = 2$

We start by analyzing the case $n = 2$ in order to illustrate the basic mechanisms at work in this case.

First, let us look for a candidate symmetric price equilibrium. In order to do so, suppose all firms except one, say A_0 , charge the same price p , whereas A_0 charges p' . Also, assume p' is close enough to p , more specifically:

$$|p' - p| < dt$$

This assumption ensures that there exists a unique location at which consumers are indifferent between A_0 and A_1 and that it is strictly between a_0 and a_1 (and symmetrically for A_{N-1} and A_0). This location is given by:

$$x_1 = \frac{d}{2} + \frac{p - p'}{2t}$$

Similarly, consumers indifferent between A_0 and A_2 are located at:

$$x_2 = d + \frac{p - p'}{2t}$$

It is easy to see that consumers situated between a_0 and x_1 rank A_0 first and A_1 second. Those located between x_1 and x_2 rank A_1 first and A_0 second. Finally, consumers situated to the right of x_2 prefer both A_1 and A_2 to A_0 . By symmetry, the demand for A_3 is then given by:

$$D_{A_0}(p') = 2x_2 = 2d + \frac{p - p'}{t}$$

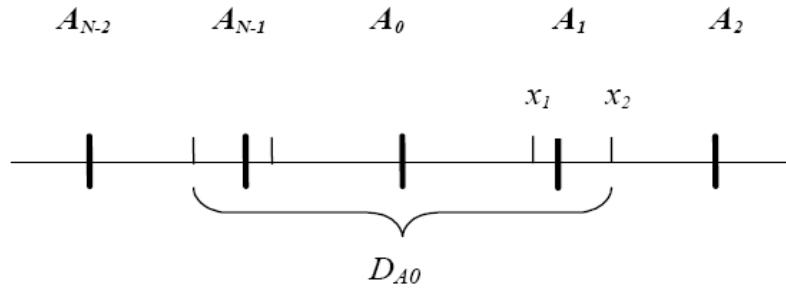


Figure 2

This is the sum of the measures of two types of consumers: those who purchase A_0 as their first choice and those who purchase it as their second choice.

A_0 's profit is then:

$$\Pi_{A_0}(p') = (p' - c) \left(2d + \frac{p - p'}{t} \right)$$

The equilibrium price candidate p^* is then found by maximizing the expression above with respect to p' and imposing the symmetry condition $p' = p$. We obtain:

$$p^* = c + 2dt$$

At this price, all firms earn a profit of:

$$\Pi(p^*) = 4d^2t$$

We now have to make sure that this price is indeed a Nash equilibrium, i.e. to check whether there is no profitable deviation for A_0 , given that everyone else charges p^* . More specifically we have to check whether there are no profitable deviations to p' such that $|p' - p^*| \geq dt$ (we already know that p is better than any p' such that $|p' - p^*| < dt$).

It is precisely the feasibility of these larger deviations that makes the analysis here more complex than the Salop model with demand for a single brand. Indeed, in that framework, the candidate price equilibrium is $c + dt$ and the non-existence of profitable deviations larger than dt is immediate: undercutting by more than dt falls below cost and increasing the price by at least dt clearly loses all demand.

In our case, the effect of introducing demand for more than one variety is to relax price competition: equilibrium candidate prices are higher, which opens the possibility for more numerous undercutting strategies, even though we will prove that they are not profitable in the case $n = 2$.

If A_0 charges $p' > p^* + dt$ then demand for A_0 will be 0: all consumers prefer at least A_{N-1} and A_1 over A_0 . Also, A_0 has no interest in charging $p' \leq p^* - 2dt$, since such a price falls below cost.

Consider now the case $p^* - 2dt < p' < p^* - dt$, or $c < p' < c + dt$. Clearly, all consumers prefer A_0 over A_1 . This is because the difference in prices between the two varieties is too high in order to be compensated by any feasible difference in transportation costs.

The consumer indifferent between A_0 and A_2 is located at $x_2 = d + \frac{p^* - p'}{2t}$. The consumer indifferent between A_0 and A_3 is located at $x_3 = \frac{3d}{2} + \frac{p^* - p'}{2t}$. Consumers located to the right of x_3 prefer both A_2 and A_3 over A_0 ; those located between x_2 and x_3 rank A_2 first and A_0 second and those located between x_2 and A_0 rank A_0 first and A_2 second. By symmetry, demand for A_0 is then:

$$D_{A_0}(p') = 2x_3 = 3d + \frac{p^* - p'}{t}$$

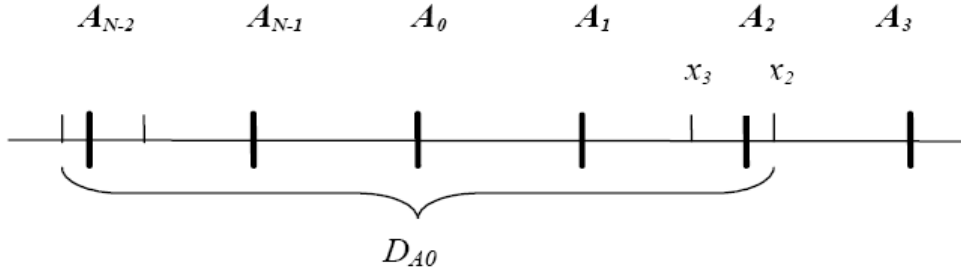


Figure 3

Its profits are:

$$\Pi_{A_0}(p') = (p' - c) \left(3d + \frac{c + 2dt - p'}{t} \right)$$

Differentiating with respect to p' :

$$\frac{d\Pi_{A_0}}{dp'} = \frac{1}{t} (5dt + 2c - 2p') > 3d > 0$$

for $p' \in (c, c + dt)$.

Hence, profit is increasing in p' on the interval $(c, c + dt)$, therefore A_0 will want to

charge p' as close as possible to $p^* - dt = c + dt$:

$$\lim_{p' \rightarrow p^* - dt, p' < p^* - dt} \Pi(p') = 4d^2t$$

and $\Pi(p') < 4d^2t$ for $p' < p^* - dt$.

The reason we have to take a limit above is that demand is discontinuous at $p' = p^* - dt$. Indeed, if A_0 charges $p' = p^* - dt$, then all consumers to the right of A_1 are indifferent between A_0 and A_1 . Consumers located between A_0 and A_1 rank A_0 first and A_1 second. The rest is unchanged from the previous case. Hence, demand for A_0 is now:

$$D_{A_0}(p^* - dt) = 2 \left(\frac{3d}{2} + \frac{1}{2} \times \frac{d}{2} \right) = \frac{7d}{2}$$

and its profit:

$$\Pi_{A_0}(p^* - dt) = \frac{7d^2t}{2} < 4d^2t$$

Lastly, assume A_0 charges $p' = p^* + dt$. Consumers located to the right of A_0 will be indifferent between A_{N-1} and A_0 and prefer A_1 to both. The consumer indifferent between A_0 and A_2 is located at $\frac{d}{2}$. Hence, demand and profit for A_0 are:

$$\begin{aligned} D_{A_0}(p^* + dt) &= 2 \times \frac{1}{2} \times \frac{d}{2} = \frac{d}{2} \\ \Pi_{A_0}(p^* + dt) &= \frac{3d^2t}{2} < 4d^2t \end{aligned}$$

All these properties are illustrated in the following two diagrams, which represent the demand and profit for A_0 as functions of the price p' it charges.

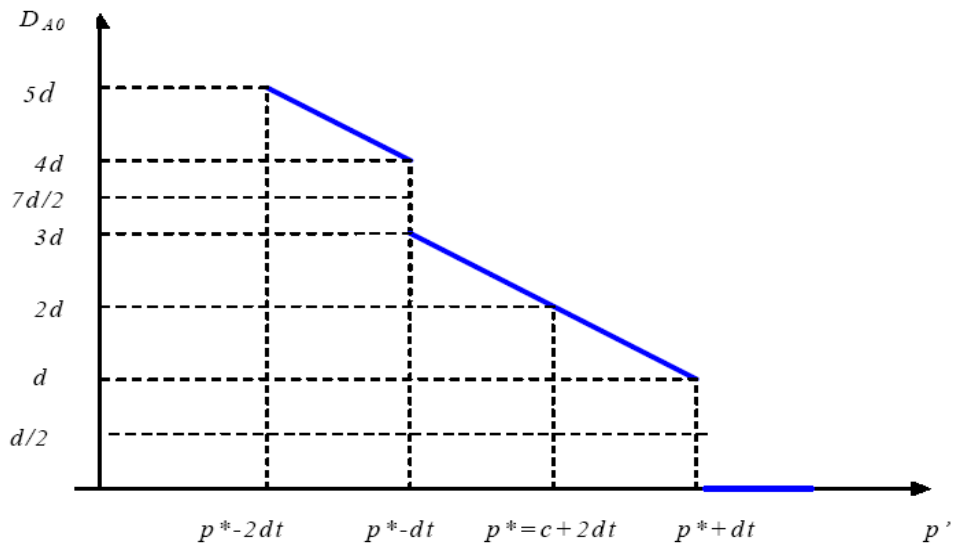


Figure 4

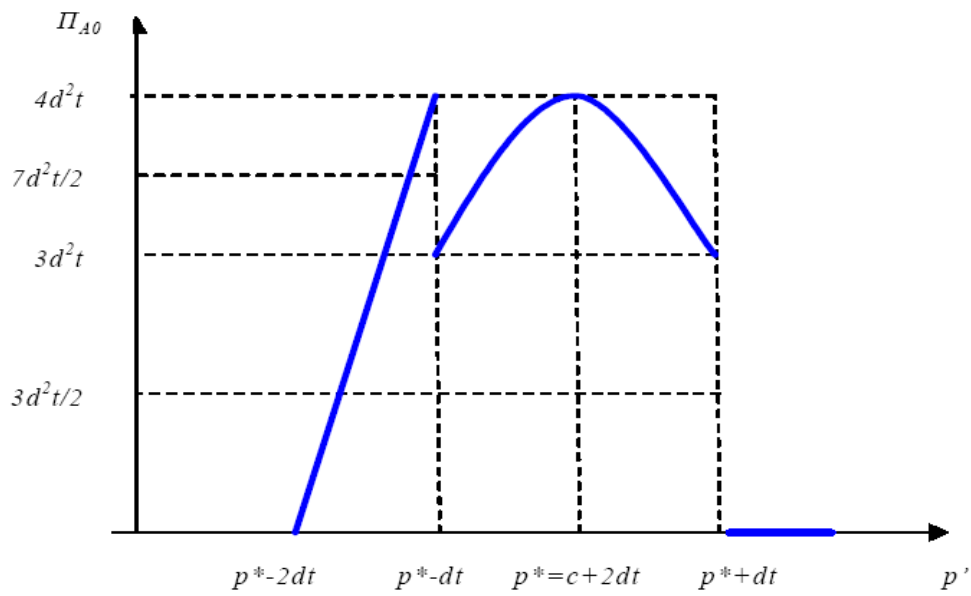


Figure 5

We have thus shown that there is no profitable deviation, which ensures that $p^* = c + 2dt$ is indeed a Nash equilibrium of the price game with differentiated products, when consumers demand exactly 2 varieties. However, as we have seen and as is illustrated in the second diagram above, this is "barely" an equilibrium, in the sense that there is an undercutting strategy which does almost at least as well.

The next section proves the remarkable result that this strategy does strictly better than the one prescribed by the candidate equilibrium and thus precludes the existence of a symmetric equilibrium for $n > 2$.

4.2. The general case

Let now n be any integer strictly bigger than 1. The following proposition contains the central result of this section.

Proposition 1. *With linear transportation costs and equidistant locations, there is no symmetric price equilibrium if the number of varieties demanded by consumers is strictly higher than 2.*

Proof. The determination of the candidate price equilibrium follows the same lines as above and is straightforward. Suppose everyone charges p , except A_0 , who charges p' , with $|p' - p| < dt$. For every A_k with $k > 0$, the consumer indifferent between A_0 and A_k exists and is located at:

$$x_k = \frac{kd}{2} + \frac{p - p'}{2t}$$

It is easy to see that consumers located between x_k and x_{k+1} rank A_0 in $(k + 1)$ th position. Therefore, the last consumer to the right of A_0 who will buy variety A_0 is the

one located at x_n . Demand for A_0 is then:

$$D_{A_0}(p') = 2 \left(\frac{nd}{2} + \frac{p - p'}{2t} \right)$$

The following diagram illustrates the case $n = 4$.

Figure 6

Maximizing profit and imposing that the solution be equal to p yields:

$$p_n^* = ndt + c$$

At this price, each seller has a market share of:

$$x_n^* = nd$$

and makes a profit of:

$$\Pi(p_n^*) = n^2 d^2 t$$

The range of potentially profitable deviations is $(-ndt, +E(\frac{n}{2})dt]$. Indeed, if A_0 charges more than $p_n^* + E(\frac{n}{2})dt$, even the consumer located exactly at a_0 will rank A_0

in $(2E(\frac{n}{2}) + 1)$ th position and will not buy it since $2E(\frac{n}{2}) + 1 \geq n + 1 > n$. Since this consumer is the most likely to buy A_0 , demand for A_0 will be 0.

However, we will prove that *undercutting* strategies at non-discontinuous points are enough to eliminate the symmetric price equilibrium candidate.

Assume that:

$$p' \in (p_n^* - (k + 1)dt, p_n^* - kdt), \text{ for some } k \in [1, n - 1]$$

When A_0 charges such a price, *all* consumers rank A_0 before A_1, A_2, \dots, A_k and the first variety to the right of A_0 for which there exist some consumers that prefer it to A_0 is A_{k+1} . The last consumer to the right of A_0 who will purchase A_0 is then the one who is indifferent between A_0 and A_{k+n} , located at $x_{k+n} = \frac{(k+n)d}{2} + \frac{p_n^* - p'}{2t}$. The following diagram illustrates this for $n = 4$ and $k = 2$:

Figure 7

Demand and profits for A_0 are then:

$$\begin{aligned} D_{A_0}(p') &= 2x_{k+n} = (k+n)d + \frac{p_n^* - p'}{t} \\ \Pi_{A_0}(p') &= (p' - c) \left((k+n)d + \frac{p_n^* - p'}{t} \right) \end{aligned}$$

Differentiating with respect to p' and using the fact that $p' \in (c + (n - (k + 1))dt, c + (n - k)dt)$, we obtain:

$$\begin{aligned} \frac{d\Pi_{A_0}}{dp'} &= \frac{1}{t} ((k + 2n)dt + 2c - 2p') \\ &> \frac{1}{t} ((k + 2n)dt + 2c - 2(c + (n - k)dt)) \\ &= 3kd > 0 \end{aligned}$$

Hence profit is increasing on each interval $(c + (n - (k + 1))dt, c + (n - k)dt)$ and:

$$\lim_{p' \rightarrow (p_n^* - kdt)^-} \Pi_{A_0}(p') = (n - k)(2k + n)d^2t = (n^2 + kn - 2k^2)d^2t$$

Taking $k = 1$, the limit above is equal to:

$$(n^2 + n - 2)d^2t > n^2d^2t = \Pi(p_n^*) \text{ for } n \geq 3$$

This means that for $n \geq 3$, there always exists a profitable undercutting strategy (any price inferior but close enough to $p_n^* - kdt$ will do) and thus there is no symmetric price equilibrium. This concludes the proof. ■

It is interesting to determine demand for A_0 in the remaining cases: $p' = p_n^*$ and $p' \geq p_n^* + dt$.

Let $p' = p_n^* - kdt$, for some $k \in [1, n]$. It is straightforward to prove (same as in the case $n = 2$):

$$\begin{aligned} D_{A_0}(p') &= 2 \left(x_{n+k-1} + \frac{1}{2} (x_{n+k} - x_{n+k-1}) \right) = \left(n + 2k - \frac{1}{2} \right) d \\ \Pi_{A_0}(p') &= (n - k) \left(n + 2k - \frac{1}{2} \right) d^2 t \\ &= \left(n^2 + \left(k - \frac{1}{2} \right) n - k \left(2k - \frac{1}{2} \right) \right) d^2 t < (n^2 + kn - 2k^2) d^2 t \end{aligned}$$

Next, let $p' \in (p_n^* + kdt, p_n^* + (k + 1) dt) = (c + (n + k) dt, c + (n + k + 1) dt)$. Then, the consumer located at $x = a_0$ ranks A_0 in position $2k+1$, i.e. after $A_{N-k}, \dots, A_{N-1}, A_1, \dots, A_k$. The last consumer to the right of A_0 who will purchase A_0 will be the one indifferent between A_0 and A_{n-k} . Therefore, demand and profits for A_0 are:

$$\begin{aligned} D_{A_0}(p') &= 2 \times x_{n-k} = (n - k) d + \frac{p_n^* - p'}{t} \\ \Pi_{A_0}(p') &= (p' - c) \left((n - k) d + \frac{p_n^* - p'}{t} \right) \end{aligned}$$

And:

$$\begin{aligned} \frac{d\Pi_{A_0}}{dp'} &= \frac{1}{t} ((2n - k) dt + 2c - 2p') \\ &< \frac{1}{t} ((2n - k) dt + 2c - 2(c + (n + k) dt)) \\ &= -3kd < 0 \end{aligned}$$

Hence profit is decreasing on each interval $(c + (n + k) dt, c + (n + k + 1) dt)$ and:

$$\begin{aligned} \lim_{p' \rightarrow (p^* + kdt)^+} \Pi_{A_0}(p') &= (n + k)(n - 2k) d^2 t \\ &= (n^2 - kn - 2k^2) d^2 t < n^2 d^2 t \end{aligned}$$

Lastly, for $p' = p^* + kdt$, $k \geq 1$:

$$\begin{aligned} D_{A_0}(p') &= x_{n-k} + \frac{1}{2}(x_{n-k+1} - x_{n-k}) = \left(n - 2k + \frac{1}{2}\right) d \\ \Pi_{A_0}(p') &= (n + k) \left(n - 2k + \frac{1}{2}\right) d^2 t \\ &= \left(n^2 - \left(k - \frac{1}{2}\right) n - k \left(2k - \frac{1}{2}\right)\right) d^2 t < n^2 d^2 t \end{aligned}$$

Thus, no deviation $p' > p^* + dt$ is profitable.

The following diagrams show demand and profits for A_0 as a function of its price p' for $n = 4$.

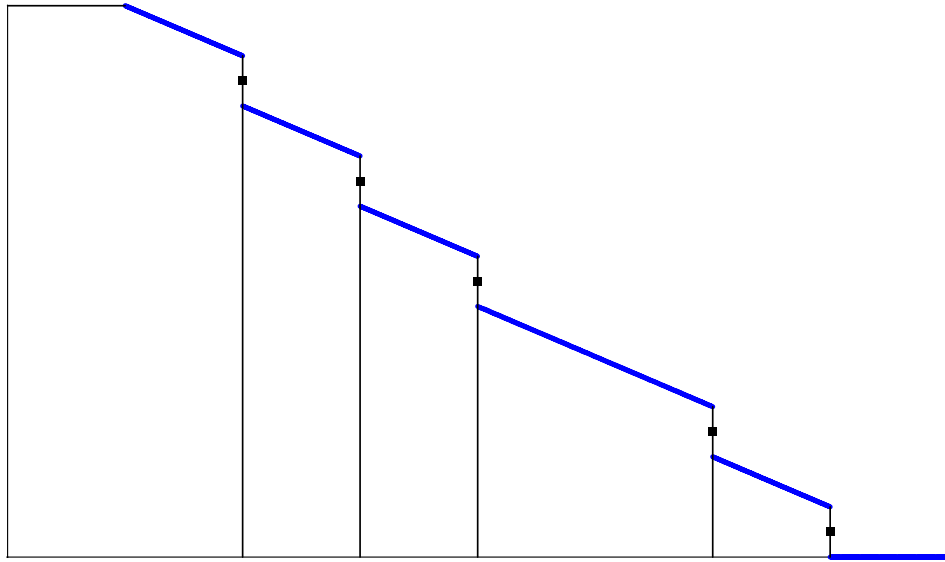


Figure 8

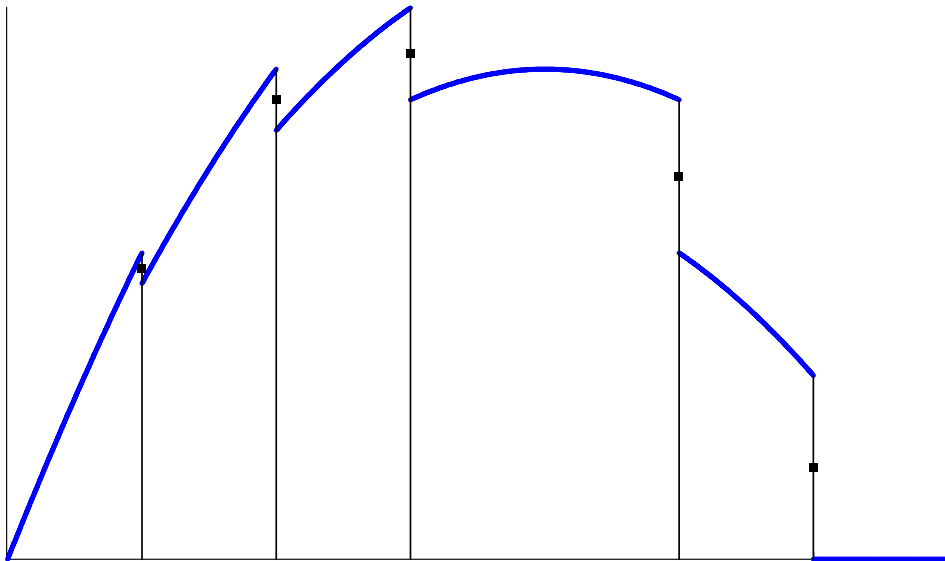


Figure 9

The two diagrams show that the possibility of a profitable deviation arises because demand jumps up sharply at every threshold of the form $p_n^* - kdt$ for $k \geq 1$. This happens because each time A_0 's price p' falls below such a threshold, it "completely" undercuts firm A_k , meaning that the price difference is enough so that no consumer will ever prefer A_k over A_0 . This instantly pushes demand up by d . It is important to note that these marginal consumers of mass d that A_0 gains are those furthest away from A_0 , i.e. those whose "last" (*nth*) purchase (in terms of distance travelled) right before the threshold was A_k . Once p' falls below $p_n^* - kdt$ though, it becomes worthwhile for them to forego A_k and travel the extra distance in order to get A_0 . Note that the reason they do not stop on their way from A_k to A_0 to any of the shops in between (A_i with $1 \leq i \leq k - 1$) is because A_0 was *already* worth travelling the extra mile *conditional* on one having decided to venture past A_k in the direction of A_0 .

Hence, at threshold k , A_0 's profit jumps up by $(p_n^* - kdt - c) d = (n - k) d^2 t$. Clearly, this jump is highest for $k = 1$. However, it is *not* true that the optimal undercutting strategy is near $p_n^* - dt$. To see this, note that profits fall by $(3k + 1) d^2 t$ when A_0 decreases its price from threshold k to threshold $k + 1$, therefore, as long as this is less than what it stands to gain through the jump at threshold $k + 1$, A_0 has every interest in cutting its price further. It will stop when the size of the next jump is no longer worth the loss incurred through the discrete decrease in price needed in order to get to the next threshold. The marginal condition is thus $3k + 1 \geq n - (k + 1)$ or $4k + 2 \geq n$, which is almost the same as the first order condition obtained by maximizing the limit deviation profit at threshold k with respect to k : $k = \frac{n}{4}$. The difference is simply due to integer problems. In our diagram above, the optimal deviation is at the threshold $k = 1$ because $n = 4$. It will be that way for $n \leq 6$; for $n = 7$ for example, the optimal deviation is $p_n^* - 2dt$.

Given the negative result above, one might hope that it is due to the specificity of linear transportation costs, which yield, as is well-known and as we have shown above, discontinuous demand functions. It turns out that the non-existence of a symmetric price equilibrium persists even with strictly convex transportation costs, i.e. td^β with $\beta > 1$, as shown by the theorem below, which we prove in the appendix.

Proposition 2. *If consumers are interested in buying exactly n varieties and transportation costs are convex, i.e. of the form td^β with $\beta > 1$, then there exists $\bar{n}(\beta)$ such that, if $N > n \geq \bar{n}(\beta)$, then there is no symmetric price equilibrium with N equidistantly located firms.*

Proof. See appendix. ■

The two figures below illustrate this result for $\beta = 2$ and $n = 4$. Both demand and profits for A_0 are continuous in p' , however their derivatives are discontinuous, which explains the existence of profitable undercutting strategies (the graphs are drawn only for $p' \leq p^*(n) + td^\beta$).

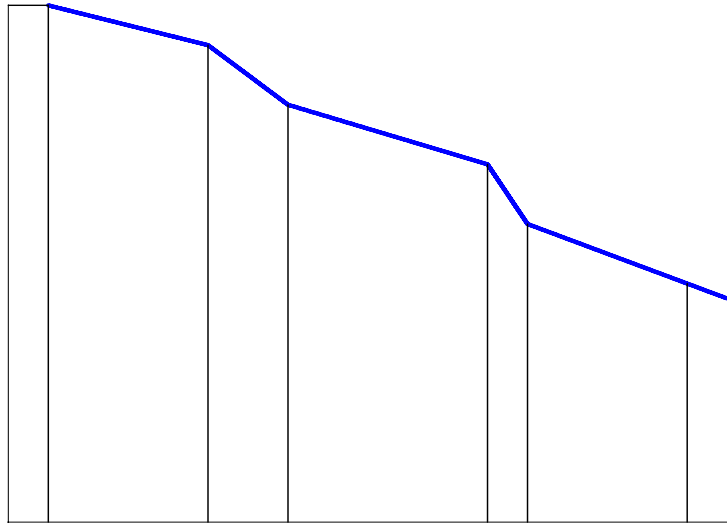


Figure 10

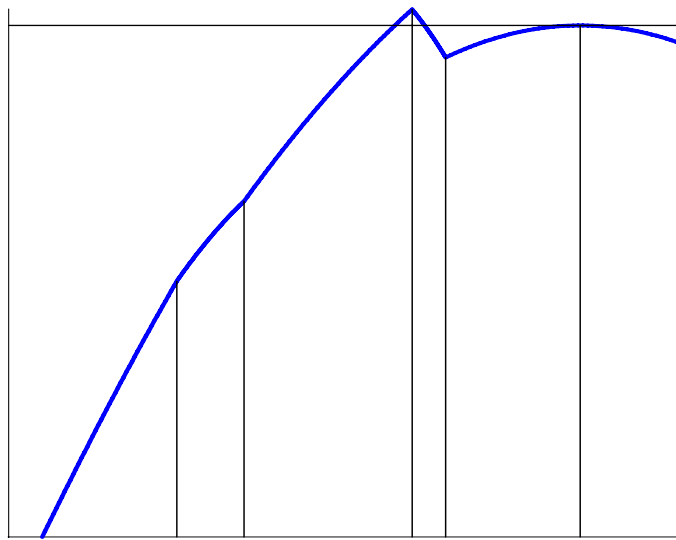


Figure 11

Note that it may well be that there is no symmetric price equilibrium even for $n < \bar{n}(\beta)$. However, in order to prove or disprove this one would have to go through some serious algebra. Our goal was to find a price equilibrium for general n and the propositions above proved that this is impossible. For example, it is straightforward to show that $\bar{n}(2) = 4$, meaning that with quadratic transportation costs there is no symmetric price equilibrium for $n \geq 4$. Therefore, our non existence result is quite robust when we generalize to non-linear transportation costs.

5. The case when consumers face a budget constraint

In the third and last alternative we explore, we assume that the restriction on the number of products that every consumer purchases comes from a budget constraint, rather than from an exogenous limit on the number of varieties purchased like in the previous section. More specifically, we assume that each consumer has a limited budget I : he will purchase varieties by decreasing order of preference, and will stop when the sum of the prices he has paid exceeds I .

Hence, a consumer located at x on the circle solves the following problem:

$$U(x) = \max_{\substack{k \in [1, N] \\ \{i_1, i_2, \dots, i_k\} \subset \{1, \dots, N\} \\ p_{A_{i_1}} + \dots + p_{A_{i_k}} \leq I}} \left\{ (v - t|x - a_{i_1}|) + \dots + (v - t|x - a_{i_k}|) + I - p_{A_{i_1}} - \dots - p_{A_{i_k}} \right\}$$

It is important to note that here consumers choose not only *which* varieties to buy, but also *how many*, as long as they respect the budget constraint. In other words, k in the maximization program above is a choice variable, whereas in our previous model we had imposed an exogenous limit on the number of varieties demanded by consumers, i.e. k was given.

The following proposition characterizes all possible symmetric price equilibria.

Proposition 3. *Assume $v \geq \max\left(\frac{t}{2} + 2dt + c, \frac{I}{2} + t\right)$. Then there are only two possible types of equilibria:*

- *If I can be written as $I = k(c + kdt) + x$, where $0 \leq x < c + kdt$ and $k \in \{1, 2\}$, then $p = c + kdt$ is a symmetric price equilibrium if and only if $x < \max(c, k^2d^2t)$. In this equilibrium, each consumer purchases 2 varieties.*
- *For all integer $n \in [a, \min(\frac{I}{c}, N)]$, where a is the unique positive solution to $\frac{I}{a} = c + (\frac{a}{2} + 1)dt$, there exists a symmetric price equilibrium with $p = \frac{I}{n}$, in which each consumer buys n varieties.*

Proof. Consider a symmetric price equilibrium candidate p . Let $n = E\left(\frac{I}{p}\right)$ and write:

$$I = np + x, \text{ where } 0 \leq x < p$$

In order to characterize p , we assume throughout the proof that all firms charge p , with the exception of A_0 , which charges p' .

Let x_k be the location of the consumer indifferent between A_0 and A_k , irrespective of budget considerations:

$$x_k = \frac{kd}{2} + \frac{p - p'}{2t}, \text{ for } k \geq 1$$

Clearly, x_k exists if and only if $|p - p'| < kdt$.

The following lemma gives a partial (but sufficient for our purposes) characterization of the demand for A_0 :

Lemma 1. *Assume $x < p' \leq p + x$ throughout.*

- If $|p - p'| < dt$, then demand for A_0 is given by: $D(p') = 2x_n = nd + \frac{p-p'}{t}$
- If $p - (k + 1) dt < p' < p - kdt$ for some $k \geq 1$, then: $D(p') = 2x_{n+k} = (n + k) d + \frac{p-p'}{t}$
- If $p' = p - kdt$ for some $k \geq 1$, then:

$$D(p') = x_{n+k} + x_{n+k-1} = \left(n + k - \frac{1}{2}\right) d + \frac{p - p'}{t} = \left(n + 2k - \frac{1}{2}\right) d$$

Proof. If $x < p' \leq p + x$ and given that $I = np + x$, each consumer will purchase exactly n varieties, regardless of whether A_0 is one of them or not (the underlying assumption here is that consumers want to purchase as many varieties as their budget allows; we show below that the condition we imposed on v implies that this assumption is always satisfied). To see this, note that on the one hand A_0 is cheap enough so that purchasing it still allows a consumer to purchase $(n - 1)$ additional varieties priced at p ; on the other hand, it is expensive enough so that one cannot buy it with the extra x dollars left after purchasing n varieties priced at p . Consequently, each consumer will simply purchase the n varieties he likes the most, which means that for p' in this range everything is equivalent to the model with demand for an exogenously given number of varieties fixed at n .

Therefore, if $|p - p'| < dt$, the last consumer to the right of A_0 who will purchase it is the one indifferent between A_0 and A_n , i.e. the one located at x_n . The condition $|p - p'| < dt$ is necessary in ensuring that x_k exists for all $k \geq 1$ and thus that consumers located to the right of x_n rank A_0 in $(n + 1)$ th position, whereas those located between a_0 and x_n rank A_0 in n th position or better. By symmetry we obtain the first expression above.

In the second case, since $p' < p - kdt$, all consumers strictly prefer A_0 over A_1, A_2, \dots and A_k . Therefore, the last consumer to the right of A_0 to rank A_0 in n th position or

better is now located at x_{n+k} instead of x_k .

Lastly, if $p' = p - kdt$, then all consumers strictly prefer A_0 over A_1, A_2, \dots and A_{k-1} ; those to the left of A_k strictly prefer A_0 over A_k and those to the right of A_k are indifferent between A_0 and A_k , meaning that only a proportion of one half of the latter will rank A_0 before A_k . Furthermore, the location of the last consumer to rank A_0 in n th position or better is unchanged from the second case: x_{n+k} . We obtain:

$$\begin{aligned} D(p') &= 2 \left(x_{n+k-1} + \frac{1}{2} (x_{n+k} - x_{n+k-1}) \right) \\ &= x_{n+k} + x_{n+k-1} \\ &= \left(n + k - \frac{1}{2} \right) d + \frac{p - p'}{t} \end{aligned}$$

■

A useful thing to notice, which follows immediately from lemma 1 is that:

$$\lim_{p' \rightarrow (p - kdt)^-} D(p') > D(p - kdt) \text{ for all } k \geq 1$$

implying:

$$\lim_{p' \rightarrow (p - kdt)^-} \Pi(p') > \Pi(p - kdt) \text{ for all } k \geq 1$$

Therefore, when analyzing the profitability of undercutting we can restrict our attention to prices p' such that $p - p' \neq kdt$, for all $k \geq 1$. This is because the profit obtained by A_0 through undercutting at any discontinuous point of its demand is lower than the profit that can be obtained at very close non-discontinuous points.

Assume first that $x > 0$. Then, using the lemma, if p is to be a symmetric price equilibrium, $\Pi(p') = (p' - c) D(p')$ must have a local maximum at $p' = p$. Solving for p ,

this yields:

$$p = c + ndt$$

This cannot be an equilibrium if $n > 2$. Indeed, just like in the model where n is exogenously given, there exist profitable undercutting strategies. For example, A_0 can charge $p' = p - dt - \epsilon dt = c + (n - 1) dt - \epsilon dt$, with ϵ very small, for a profit equal to:

$$\Pi(p') = ((n - 1) - \epsilon) ((n + 2) + \epsilon) d^2t > n^2 d^2t = \Pi(p) \text{ for } n > 2$$

Therefore we must have $n \leq 2$. We still have to check under what conditions there are no profitable deviations.

For $p' > p + dt$, A_{N-1} and A_1 are always strictly preferred to A_0 , hence the latter makes no sales and therefore such a deviation cannot be profitable.

If $n = 1$, charging $p' > p + x = I$ yields 0 sales, hence cannot be profitable. If $n = 2$, $p' > p + x$ makes no sales either. To see this, note that any consumer buying A_0 cannot buy anything else, whereas otherwise he can buy 2 varieties. Consider the consumer located at a_0 . He has the choice between purchasing A_0 or purchasing A_{N-1} and A_1 . He will choose the latter option if and only if:

$$\begin{aligned} 2v - 2p - 2dt &\geq v - p' \\ \iff v &\geq 2p - p' + 2dt \end{aligned}$$

But

$$2p - p' + 2dt < p + 2dt \leq \frac{I}{2} + \frac{2}{N}t \leq \frac{I}{2} + t \leq v \text{ by assumption}$$

Therefore, since not even the consumer whose ideal variety is A_0 will not purchase it,

demand for A_0 is 0 in this case as well.

The only possible deviation we are left with is $p' \leq x$. If $x \leq c$, this cannot be profitable. If $x > c$, such a price ensures that *all* consumers who derive positive utility from the purchase of A_0 will buy it, since it is the only additional variety that can be bought after purchasing n varieties at price p (recall that $I = np + x$). Hence:

$$D(p') = \min\left(\frac{2(v-p')}{t}, 1\right) = 1$$

because $\frac{2(v-p')}{t} > \frac{2(v-p)}{t} \geq \frac{2(v-c-2dt)}{t} \geq 1$

The maximum profit that A_0 can get with such a deviation is then x . This deviation is not profitable if and only if:

$$x < \max(c, n^2 d^2 t)$$

We have thus proven that a symmetric price equilibrium p such that $I = np + x$ with $0 < x < p$ can only exist for $n \leq 2$ and $p = c + ndt$.

Let us now look for symmetric price equilibria such that $I = np$. Lemma 1 still applies for $p' \leq p$ (with $x = 0$). However, the important thing to notice is that now $p' = p$ does not necessarily have to be a local maximum for $\Pi(p') = (p' - c) \left(nd + \frac{p-p'}{t}\right)$. Indeed, lemma 2 below shows that under our assumptions on v , demand for A_0 is 0 for $p' > p$.

Lemma 2. *If A_0 charges p' while all other firms charge $p = \frac{I}{n}$, with $p' > p$ and if $v \geq \frac{I}{2} + t$, then A_0 makes 0 sales.*

Proof. If a consumer does not purchase A_0 then she can purchase n total varieties, whereas if she does, she can only purchase $n - 1$ total varieties. Consider a consumer located at $a_0 + \epsilon$, with $\epsilon > 0$ small. Assume $n = 2k$. She has the choice between purchasing

$A_0, A_1, \dots, A_{k-1}, A_{N-1}, A_{N-2}, \dots, A_{N-k+1}$ or $A_1, \dots, A_k, A_{N-1}, A_{N-2}, \dots, A_{N-k}$ and she prefers the latter if and only if:

$$\begin{aligned} 2v - 2p - t|\epsilon - a_k| - t|\epsilon - a_{N-k}| &\geq v - p' \\ v - 2p + p' - 2kdt &\geq 0 \\ v - 2p + p' - ndt &\geq 0 \end{aligned}$$

One can easily show that this condition is valid for $n = 2k + 1$ as well.

We have:

$$v - 2p + p' - ndt > v - p - ndt = v - \frac{I}{n} - \frac{n}{N}t > v - I - t \geq 0$$

Thus for any small ϵ no consumer located at $a_0 + \epsilon$ will purchase A_0 . Since these are the consumers who like A_0 the most, it is clear that a fortiori no other consumers to the right of A_0 will purchase it. The same is true by symmetry for consumers to the left of A_0 , which completes the proof of the lemma. ■

The result proven in lemma 2 is the crucial difference between the model with demand for an exogenously given number of varieties and the present model with a budget constraint. It is the reason why there exists a symmetric equilibrium in the latter as opposed to the former. Indeed, by assuming that v is large enough, we made sure that a consumer would forego even her ideal variety if she can purchase two varieties instead. This tradeoff is relevant when the budget constraint is binding and the price of the ideal variety is strictly higher than the price of all other varieties. Consequently, since raising p' ever so slightly over p yields 0 profits (this discontinuity in the profit function is entirely due to the budget

constraint), we simply have to make sure that:

$$\Pi(p') \leq \Pi(p) = (p - c)nd \text{ for all } p' \leq p$$

if we want p to be a symmetric price equilibrium immune to undercutting strategies. By contrast, in the previous model p had to be a local maximum for $\Pi(\cdot)$ (i.e. $\frac{d\Pi(p')}{dp'}(p' = p) = 0$), which imposed too high a symmetric price equilibrium candidate.

Given our remark after lemma 1, the necessary and sufficient conditions here are:

$$\begin{aligned} \frac{d\Pi(p')}{dp'}(p' = p) &\geq 0 \\ \text{and for all } k &\geq 1, \sup_{p' \in (p - (k+1)dt, p - kdt)} \Pi(p') \leq \Pi(p) \end{aligned}$$

The first condition is equivalent to:

$$p \leq c + ndt$$

For simplicity, write:

$$p = c + ydt, \text{ with } y \in [0, n]$$

Then:

$$\Pi(p) = ynd^2t$$

For $p' \in (p - (k + 1) dt, p - kdt)$, we have:

$$\begin{aligned} \frac{d\Pi(p')}{dp'} &= (n + k) d + \frac{p - p'}{t} - \frac{p' - c}{t} \\ &= \frac{c + (n + k) dt + p - 2p'}{t} \\ &> \frac{c + (n + k) dt - p + 2kt}{t} > 0 \end{aligned}$$

Thus:

$$\begin{aligned} \sup_{p' \in (p - (k + 1) dt, p - kdt)} \Pi(p') &= \lim_{p' \rightarrow (p - kdt)^-} \Pi(p') = (p - kdt - c) (n + 2k) d \\ &= (y - k) (n + 2k) d^2 t \end{aligned}$$

The second condition above is then satisfied if and only if $(y - k) (n + 2k) \leq yn$ for any integer $k \in [1, y]$, or $2k(y - k - \frac{n}{2}) \leq 0$ for any integer $k \in [1, y]$. This is satisfied if and only if $y - 1 - \frac{n}{2} \leq 0$ or $y \leq \frac{n}{2} + 1$.

We have thus shown that $p = \frac{I}{n}$ is a symmetric price equilibrium if and only if:

$$\begin{aligned} p &\leq c + \left(\frac{n}{2} + 1\right) dt \\ \iff \frac{I}{n} &\leq c + \left(\frac{n}{2} + 1\right) dt \\ \iff n &\geq a \end{aligned}$$

where a is the unique positive solution to:

$$\frac{I}{a} = c + \left(\frac{a}{2} + 1\right) dt$$

Lastly, we have to make sure the interval $[a, \min(\frac{I}{c}, N)]$ is not empty:

$$\frac{I}{(I/c)} = c < c + \left(\frac{I}{2c} + 1\right) dt \implies a < \frac{I}{c}$$

and:

$$N \geq a \iff \frac{I-t}{N} \leq c + \frac{t}{2}$$

Thus, for N large enough, the interval above is well-defined. If N is such that the second inequality above does not hold, then there is no symmetric price equilibrium of the second type.

This completes the proof of the proposition. ■

Clearly, the conditions for existence of the first type of equilibria (there are at most 2, one with $n = 1$ and one with $n = 2$) are quite restrictive and these equilibria will not exist in general. It is the existence of the second type of equilibria which is interesting and which stands in stark contrast with the non-existence result we obtained in the previous section, with demand for an exogenously given number of varieties.

The figures below depict demand and profits around such an equilibrium, with $N = 10$, $c = 1$, $t = 10$, $n = 4$ and $I = 15$ ($a = 3.83$).

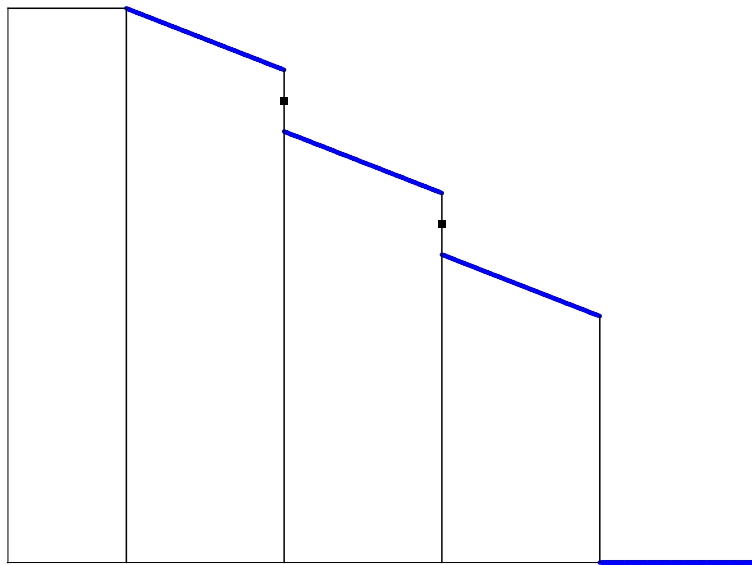


Figure 12

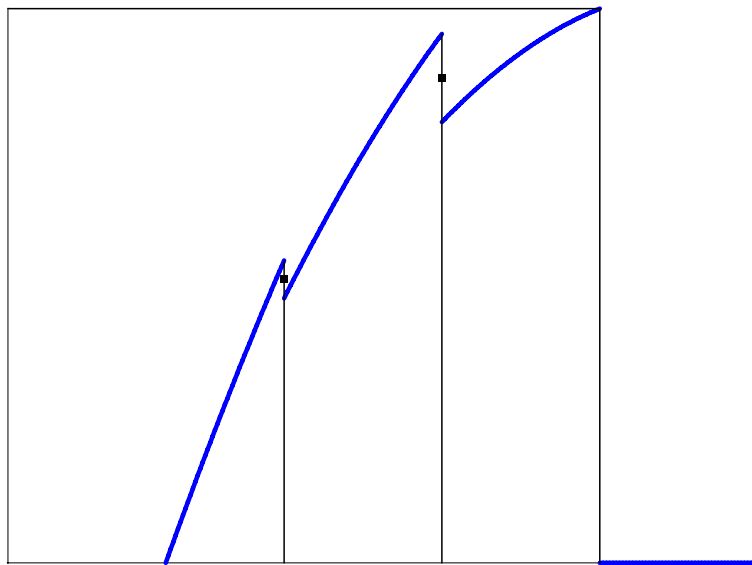


Figure 13

5.1. Welfare analysis

The welfare analysis here (in particular the comparison of the socially optimal number of firms and the one occurring with free entry) is quite different from the Lerner Salop model for two reasons. First, for any given number of firms active in the market there are multiple symmetric equilibria and it is not clear which will occur when an additional firm enters the market. Secondly, before determining the socially optimal number of firms, we need to rank the multiple equilibria that can occur for any given number of active firms.

For any given N , let us index each possible equilibrium that may occur by n , the number of products that consumers buy in that equilibrium. As we have seen above, n lies in the interval $[a, \min(\frac{I}{c}, N)]$. The corresponding symmetric price equilibrium is:

$$p_n = \frac{I}{n}$$

With free entry, the profit for each firm as a function of n and N is:

$$\pi(N, n) = \left(\frac{I}{n} - c\right)nd - f = \frac{I - cn}{N} - f$$

This expression captures the difficulty of analyzing free entry in our context. Indeed, assume as is standard that the zero profit condition holds for $n > 1$. If a potential entrant believes that in case she enters firms will coordinate on an equilibrium n' with $n' \geq n$, then entry is not profitable. However, if she expects firms to coordinate on any equilibrium $n' < n$, entry may be profitable. More precisely, this happens if and only if:

$$\frac{I - cn'}{N + 1} > \frac{I - cn}{N}$$

Thus, in order to ensure that entry is never profitable, we have to require:

$$\frac{I - c}{N} - f \leq 0 \iff N \geq \frac{I - c}{f}$$

This condition says that it is not profitable to enter even if the prevailing equilibrium would be the most favorable to firms, that is $n = 1$ and $p_n = I$.

Remark: this condition might well appear excessively restrictive if one wants to be realistic. Indeed, assume that the current industry equilibrium is $p_n = \frac{I}{n}$ with $n > 1$. It is generally unlikely that the entry of a new firm on the market will lead to higher prices, which here means $n' < n$. Therefore, it is sensible to assume that most often entry on the market will shift the equilibrium to $p_{n'} = \frac{I}{n'}$ with $n' \geq n$.

Consequently, starting with an industry equilibrium at n , entry will occur *at most* until:

$$N = \frac{I - nc}{f}$$

It may in fact stop earlier, depending on the evolution path of price equilibria.

In light of this analysis, the restrictive zero-profit condition above can be interpreted to say that entry should not be profitable even when collusion were possible between firms, enabling them to coordinate on the best equilibrium from their point of view (that is $n = 1$). In other words, we have determined an upper bound on the number of firms that will enter:

$$\overline{N}^{fe} = \frac{I - c}{f}$$

If $c = 0$, the analysis above is simplified, since firm profits are the same in all equilibria,

therefore the free entry number of firms is unambiguously:

$$N^{fe} = \frac{I}{f}$$

In order to determine the expression of total social surplus, we first have to determine total transportation costs as a function of N and n . Simple manipulations (one has to treat the cases n odd and n even separately) yield:

$$T(N, n) = \frac{n^2 t}{4N}$$

Social welfare is then:

$$SW(N, n) = n(v - c) - \frac{n^2 t}{4N} - Nf$$

which we need to maximize subject to $n \leq N$ and $nc + Nf \leq I$. The second constraint simply says that the industry has to be profitable given consumer budget I .

Differentiating with respect to n :

$$\frac{\partial SW}{\partial n} = v - c - \frac{nt}{2N} \geq v - c - \frac{t}{2} > 0$$

Therefore, at least one of the two constraints must be binding at the social optimum.

Assume first that $n = N$. Then $SW(N) = N(v - c - \frac{t}{4} - f)$, which is maximized for $N = \frac{I}{f+c}$, yielding $SW = \frac{v-c-\frac{t}{4}-f}{c+f}$.

Alternatively, assume $nc + Nf = I$. The maximization programme becomes:

$$\max_N \left\{ \frac{I - Nf}{c} v - \frac{t}{4N} \frac{(I - Nf)^2}{c^2} - I \right\}$$

subject to $N \geq \frac{I}{f+c}$.

Straightforward calculations show that the expression above is decreasing in N for $N \geq \frac{I}{f+c}$ (we also use the assumption $v > \frac{t}{2}$ and the natural $c \leq f$):

$$\begin{aligned}
 f(N) &= \frac{I - Nf}{c}v - \frac{t}{4N} \frac{(I - Nf)^2}{c^2} - I \\
 f'(N) &= -\frac{vf}{c} + \frac{tI}{4c^2N^2} (I - fN) \\
 &\leq -\frac{vf}{c} + \frac{tI}{4cN} = \frac{1}{cN} \left(\frac{tI}{4} - Nvf \right) \\
 &\leq \frac{I}{cN} \left(\frac{t}{4} - \frac{vf}{f+c} \right) \\
 &\leq 0
 \end{aligned}$$

Thus the solution is once again $n = N = \frac{I}{f+c}$.

The social optimum is characterized by:

$$n^{so} = N^{so} = \frac{I}{c+f}$$

In the general case when $c > 0$, the socially optimum level of entry is higher than the maximum number of firms under free entry if and only if:

$$\frac{I}{c+f} \geq \frac{I-c}{f} \iff I \leq f+c$$

Thus, free entry always leads to a suboptimal number of firms on the market if this condition is satisfied. Otherwise this number may be exactly optimal, suboptimal or excessive, depending on the price equilibria that prevail.

6. Conclusion

This paper has proposed a theoretical framework for studying competition between differentiated products, when consumers are interested in purchasing more than one brand. Indeed, the two classic frameworks for studying monopolistic competition - based on Salop (1979) and, respectively, on Spence (1976) and Dixit and Stiglitz (1977) - do not seem adequate proxies for markets such as those for software applications and videogames, which combine features of both models, most importantly product differentiation, heterogeneity of consumer tastes and consumer preference for diversity. Accordingly, our model generalizes Salop's circular framework by allowing consumers to purchase more than one variety.

The case in which consumers buy all products offering net positive utility is not very interesting, as there is no competition among firms, so that each behaves like an unconstrained monopolist. Therefore the need to place a restriction on the number of varieties consumers purchase. The surprising result is that when each consumer demands an exogenously fixed number of varieties, a pure strategy symmetric price equilibrium does not exist in general. This is because of two conflicting effects: on the one hand, when consumers buy more products, competition is relaxed so that first order conditions for maximization entail higher prices. But on the other hand higher prices open up the possibility for very aggressive undercutting strategies, which achieve de facto vertical differentiation from rivals offering similar products and thus discrete increases in market shares and profits. By contrast, if we assume instead that the number of varieties consumers purchase is indirectly limited by a budget constraint, then we obtain multiple pure strategy symmetric price equilibria, each corresponding to a different number of varieties bought by consumers. The reason is that in this case the budget constraint eliminates the need for local first order maximization, allowing prices to be lower and therefore immune to the previously described

aggressive undercutting strategies.

7. APPENDIX

Proof. of proposition 2: We start by looking for a symmetric price equilibrium candidate. This stage is similar to the case of linear transportation costs. Assume that all complements producers charge the same price p , with the exception of A_0 , who charges p' .

If $|p - p'| \leq td^\beta$, then for every A_k with $k \geq 1$, x_k , the location of consumers indifferent between A_0 and A_k is given by:

$$p' + tx_k^\beta = p + t(kd - x_k)^\beta$$

A consumer located at x prefers A_0 over A_k if and only if $x \leq x_k$.

We cannot solve explicitly for x_k as a function of p' ; however, we will need the following implicit expression obtained by differentiating both sides above:

$$\frac{dx_k}{dp'} = \frac{-1}{\beta t \left(x_k^{\beta-1} + (kd - x_k)^{\beta-1} \right)}$$

If $|p - p'| \leq td^\beta$, all x_k s $k \geq 1$ are located between A_0 and A_k and $x_1 < x_2 < \dots < x_n < x_{n+1} < \dots$

Consumers located between x_k and x_{k+1} rank A_0 in $(k + 1)$ th position. Therefore, the last consumer to the right of A_0 who will buy variety A_0 is the one located at x_n . Demand and profits for A_0 are then:

$$\begin{aligned} D(p') &= 2 \times x_n \\ \Pi(p') &= 2(p' - c)x_n \end{aligned}$$

Next, we differentiate Π with respect to p' using [] above and we impose that $\frac{d\Pi}{dp'}(p' = p) = 0$, which, given convex transportation costs ensures that $p' = p$ maximizes Π locally. Straightforward calculations yield the following characterization of the symmetric price equilibrium candidate:

$$\begin{aligned} p^*(n) &= c + 2^{1-\beta} t \beta (nd)^\beta \\ D(p^*(n)) &= nd \\ \Pi(p^*(n)) &= 2^{1-\beta} t \beta (nd)^{\beta+1} \end{aligned}$$

We will now prove that we can find $\bar{n}(\beta)$ such that for $n \geq \bar{n}(\beta)$, there always exist profitable undercutting strategies for A_0 .

For simplification purposes, let:

$$\delta = \frac{p^*(n) - p'}{t} \geq 0$$

We have to determine demand for A_0 as a function of δ . Like in the case with linear transportation costs, this depends on the positions of the x_i s. The main difference with strictly convex transportation costs is that now x_i exists for all i and for all δ . However, their relative positions change with δ as we show below.

Denote by $k(\delta)$ the unique, non negative integer such that:

$$\delta \in \left[(k(\delta) d)^\beta, ((k(\delta) + 1) d)^\beta \right)$$

Then:

- if $i \leq k(\delta)$, then x_i uniquely solves:

$$x_i^\beta - (x_i - id)^\beta = \delta$$

and $x_i \geq id$

- if $i \geq k(\delta) + 1$, x_i uniquely solves:

$$x_i^\beta - (id - x_i)^\beta = \delta$$

and $x_i < id$

It is easily seen that $x_i(\delta)$ is continuous and strictly increasing in δ for all i and that:

$$\begin{aligned} x_1(\delta) &> x_2(\delta) > \dots > x_{k(\delta)}(\delta) \geq k(\delta)d \\ x_{k(\delta)+1}(\delta) &< x_{k(\delta)+2}(\delta) < \dots < x_{k(\delta)+l}(\delta) < \dots \end{aligned}$$

Now, for any $i > j$, denote by δ_{ij} the unique value of δ such that $x_i(\delta) = x_j(\delta)$ and let $x_{ij} = x_i(\delta_{ij}) = x_j(\delta_{ij})$. Given the strict inequalities above, it is clear that one must have:

$$\begin{aligned} i &> k(\delta_{ij}) \geq j \\ \delta_{ij} &\in \left[(jd)^\beta, (id)^\beta \right) \end{aligned}$$

Simple calculations yield:

$$\begin{aligned} x_{ij} &= \frac{(i+j)d}{2} \\ \delta_{ij} &= \left(\frac{d}{2}\right)^\beta \left((i+j)^\beta - (i-j)^\beta \right) \end{aligned}$$

Clearly, δ_{ij} is increasing in both i and j . Also, one has the following additional property:

$$x_i(\delta) \geq x_j(\delta) \iff \delta \leq \delta_{ij}$$

We are now in a position to determine the demand for A_0 when it charges p' and everyone else charges p , i.e. when it chooses to undercut by δ .

Consumers will purchase A_0 if and only if they rank it in n th position or better. This means that a consumer to the right of A_0 will purchase it if and only if there are at most $(n-1)$ x_i 's between this consumer's position and A_0 . Hence, by symmetry, demand for A_0 is exactly equal to twice the n th x_i , when one orders them by increasing value. All the difficulty in this case consists in determining which x_i is number n ; indeed, once δ is greater than δ_{n1} it is no longer x_n .

The following lemma characterizes demand for A_0 as a function of δ for $\delta \geq 0$:

Lemma 3. *For all $k \geq 1$:*

- *If $\delta \in [0, \delta_{n1}]$, then $D_{A_0}(\delta) = 2x_n$*
- *if $\delta \in [\delta_{(n+k-1)k}, \delta_{(n+k)k})$ then $D_{A_0}(\delta) = 2x_k$*
- *if $\delta \in [\delta_{(n+k)k}, \delta_{(n+k)(k+1)})$ then $D_{A_0}(\delta) = 2x_{n+k}$.*

Proof. First, for $\delta \leq \delta_{n1}$, $x_1(\delta) \leq x_n(\delta) < x_m(\delta)$ for all $m \geq n+1$. Since for every $i \in (1, n)$, either $x_i(\delta) < x_1(\delta)$ or $x_i(\delta) < x_n(\delta)$, it is clear that the n th x_i to the right of A_0 is x_n , hence demand is $2x_n$.

Figure 14

Take now $\delta \in [\delta_{(n+k-1)k}, \delta_{(n+k)k})$. We have $x_{n+k-1} \leq x_k < x_{n+k} < x_i$ for all $i \geq n+k+1$ and $x_j > x_k$ for all $j < k$. And for any $i \in (k, n+k-1)$, if $\delta \leq (id)^\beta$, then $x_i \leq x_{n+k-1} \leq x_k$ and if $\delta \geq (id)^\beta$, then $x_i \leq x_k$. Hence, the n th x_i is x_k and demand is $2x_k$.

Figure 15

Finally, if $\delta \in [\delta_{(n+k)k}, \delta_{(n+k)(k+1)})$, then $x_{k+1} < x_{n+k} \leq x_k < x_j$ and $x_{n+k} < x_i$ for all $j < k$ and $i \geq n+k+1$. And for $i \in (k+1, n+k)$, if $\delta \leq (id)^\beta$, then $x_i \leq x_{n+k}$ and if $\delta \geq (id)^\beta$, then $x_i \leq x_{k+1} < x_{n+k}$. Hence, the n th x_i is x_{n+k} and demand is $2x_{n+k}$.

■

The resulting profits from undercutting for A_0 are then given by:

- $\Pi(\delta) = 2x_k(p' - c) = 2tx_k \left(\left(\frac{d}{2}\right)^\beta 2\beta n^\beta - \delta \right)$ for $\delta \in [\delta_{(n+k-1)k}, \delta_{(n+k)k})$, where $\delta = x_k^\beta - (x_k - kd)^\beta$

Figure 7.1: Figure 16

- $\Pi(\delta) = 2tx_{n+k} \left(\left(\frac{d}{2}\right)^\beta 2\beta n^\beta - \delta \right)$ for $\delta \in [\delta_{(n+k)k}, \delta_{(n+k)(k+1)})$, where $\delta = x_{n+k}^\beta - ((n+k)d - x_{n+k})^\beta$

We need to compare these profits from undercutting with the profit A_0 obtains by charging $p^*(n)$ like everyone else, i.e. with $\Pi(p^*(n)) = 2^{1-\beta}t\beta(nd)^{\beta+1}$. The difference with the case of linear transportation costs is that here the profit function is continuous in δ , or alternatively in p' .

Consider the following deviation:

$$\begin{aligned}\delta &= \delta_{(n+1)1} = \left(\frac{d}{2}\right)^\beta \left((n+2)^\beta - n^\beta\right) \\ x_1 &= x_n = \frac{(n+2)d}{2}\end{aligned}$$

The resulting profit is:

$$\Pi(\delta_{(n+1)1}) = 2^{-\beta}td^{\beta+1}(n+2) \left(2\beta n^\beta - (n+2)^\beta + n^\beta\right)$$

Comparing it with the equilibrium profit:

$$\begin{aligned}
\Pi(\delta_{(n+1)1}) &\geq \Pi(p^*(n)) \\
&\iff (n+2) \left(2\beta n^\beta - (n+2)^\beta + n^\beta \right) \geq 2\beta n^{\beta+1} \\
&\iff (4\beta+2)n^\beta + n^{\beta+1} - (n+2)^{\beta+1} \geq 0 \\
&\iff \frac{4\beta+2}{n} + 1 - \left(1 + \frac{2}{n} \right)^{\beta+1} \geq 0
\end{aligned}$$

Let:

$$f(n) = \frac{4\beta+2}{n} + 1 - \left(1 + \frac{2}{n} \right)^{\beta+1}$$

It follows that:

$$\begin{aligned}
f'(n) &= \frac{2}{n^2} \left(\left(1 + \frac{2}{n} \right)^\beta (\beta+1) - (2\beta+1) \right) \\
f'(n) \geq 0 &\iff n \geq \frac{2}{\left(\frac{2\beta+1}{\beta+1} \right)^{1/\beta} - 1}
\end{aligned}$$

Therefore $f(\cdot)$ is first increasing, maximum at $n = \frac{2}{\left(\frac{2\beta+1}{\beta+1} \right)^{1/\beta} - 1}$ and decreasing afterwards. Moreover:

$$\begin{aligned}
f(2) &= 2(\beta+1) - 2^{\beta+1} < 0 \\
\lim_{n \rightarrow \infty} f(n) &= 0
\end{aligned}$$

Thus, there exists $\bar{n}(\beta) \in \left[2, \frac{2}{\left(\frac{2\beta+1}{\beta+1} \right)^{1/\beta} - 1} \right)$ such that:

$$n \geq \bar{n}(\beta) \iff f(n) \geq 0$$

We have thus proven that for $n \geq \bar{n}(\beta)$ the following undercutting strategy for A_0 is profitable:

$$p' = p - t\delta_{(n+1)1}$$

Therefore, for all $n \geq \bar{n}(\beta)$ there is no symmetric price equilibrium. ■

8. REFERENCES

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